

# Massey products, the Lower Central Series, and the Poincaré Conjecture

Nick Salter, FFSS, April 24, 2014

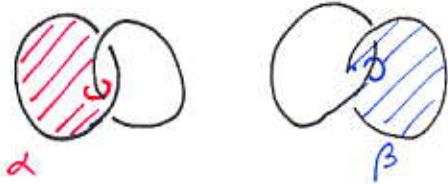
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Let's start with the Hopf link:



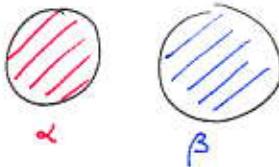
One way to detect linkedness is to use the cup product in  $H^*(S^3 - L)$ .

Lefschetz duality:  $H^1(S^3 - L, \mathbb{Z}) \cong H_2(S^3 - L, L; \mathbb{Z}) = \langle \alpha, \beta \rangle$ :

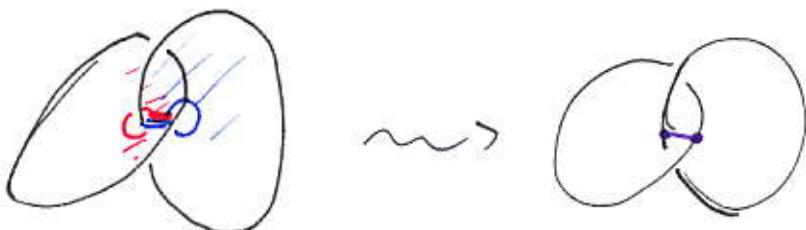


For an unlinked pair of circles,

$$\alpha \cup \beta = 0.$$

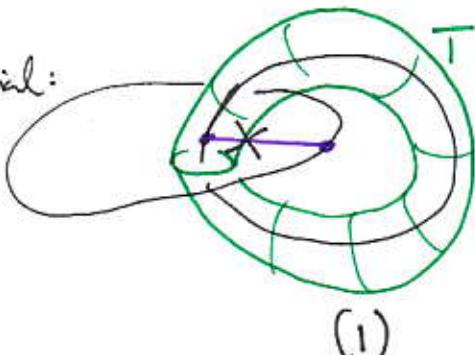


But in  $L$ :



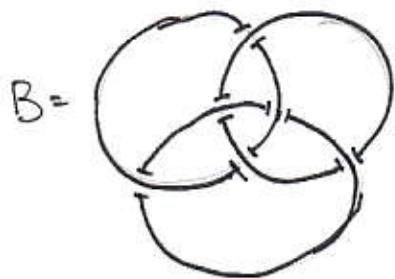
$$\gamma = \alpha \cup \beta \in H_1(S^3 - L, L, \mathbb{Z})$$

And  $\gamma$  is nontrivial:



$$\gamma \cup T = \pm 1, \text{ where } T \in H_2(S^3 - L, \mathbb{Z}).$$

Next consider the Borromean link:



$B =$

A similar analysis as in the case of  $L$  shows that the cup product pairing in  $H^*(S^3 - B; \mathbb{Z})$  is trivial.

But it turns out that "higher multiplicative structures" on  $H^*(S^3 - B; \mathbb{Z})$  do not vanish. This is the notion of a Massey Product. For simplicity, we'll only define the special case that we need.

Def

Let  $(C^\bullet, \delta)$  be a cochain complex with homology  $H^\bullet$ .

Suppose  $\alpha, \beta, \gamma \in H^1$  satisfy  $\alpha \cup \beta = \beta \cup \gamma = 0$ .

Pick  $a, b, c \in C^1$  representing  $\alpha, \beta, \gamma$ . Then by assumption,  $\alpha \cup b = \delta x$ ,  $b \cup c = \delta y$  for  $x, y \in C^1$ .

Observe that  $\delta(x \cup \gamma + a \cup y) = \delta x \cup \gamma - x \cup \delta \gamma + a \cup y - a \cup y$   
 $= (\alpha \cup b) \cup c - a \cup (b \cup c) = 0$ .

So  $x \cup c + a \cup y$  determines a cohomology class.

As always, we must be cognizant of the choices made.

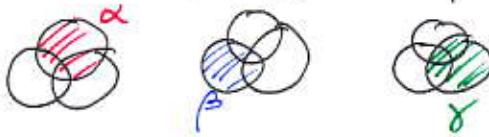
It turns out  $[x \cup c + a \cup y]$  does depend on the choices of  $x, y$ . (But not on choices of  $a, b, c, \dots$ ).

The Massey triple product  $\langle \alpha, \beta, \gamma \rangle = \{[x \cup c + a \cup y]\}$ ,

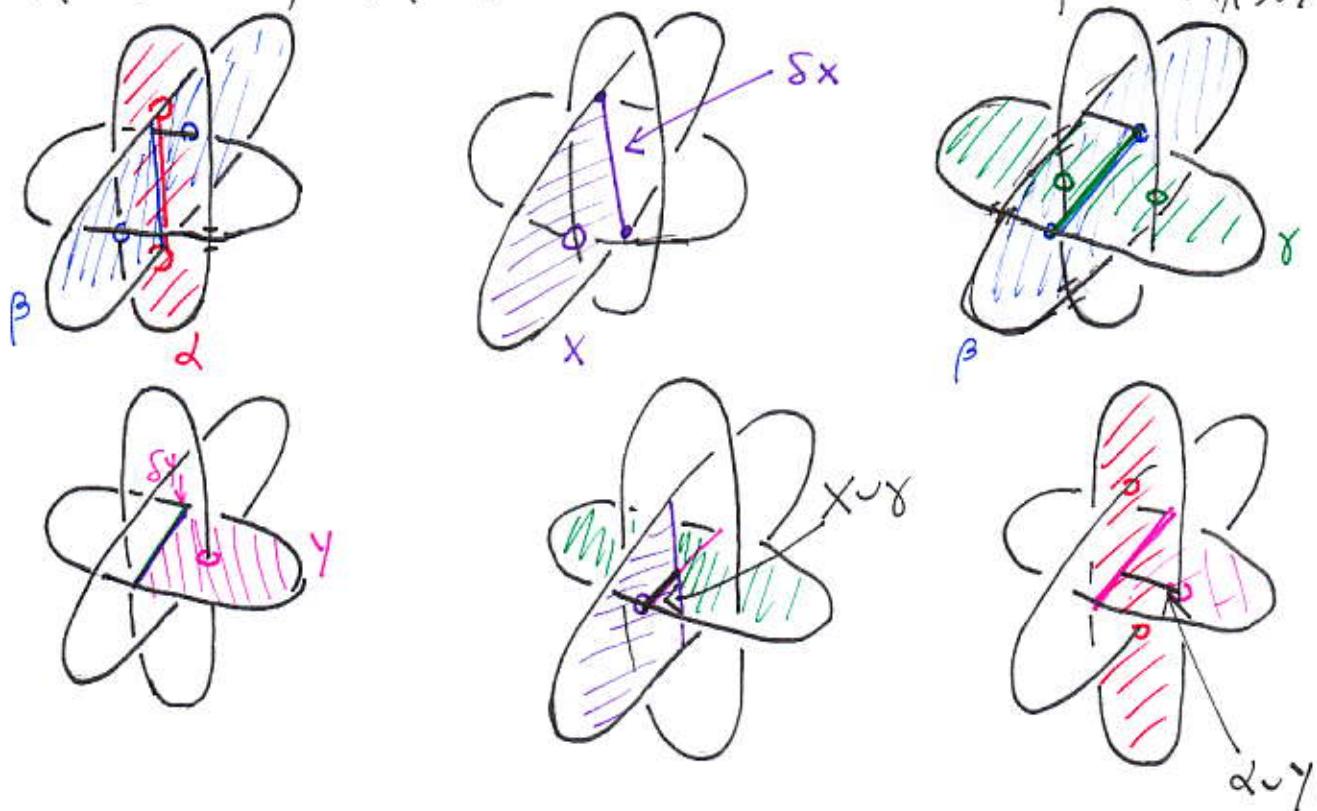
as  $x, y$  range over all such choices.

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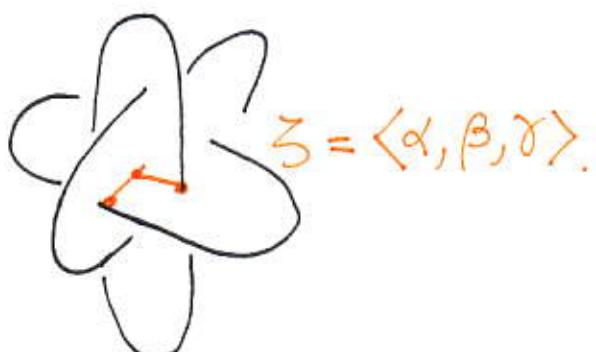
Now let  $\alpha, \beta, \gamma \in H_2(S^3 - B, B) \cong H^1(S^3 - B)$  be spanning disks for the components of  $B$ :



As remarked,  $\alpha \cup \beta = \beta \cup \gamma = \gamma \cup \alpha = 0$ . So let's compute  $\langle \alpha, \beta, \gamma \rangle$ :



Voilà:



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So far, the LCS has been absent from our discussion.

To incorporate it, we'll have to take a foray into group cohomology.

~ Crash Course on  $H^1(G, \mathbb{Z}), H^2(G, \mathbb{Z})$  ~

Def  $H^k(G, \mathbb{Z}) := H^k(K(G, 1), \mathbb{Z})$ .

• So  $H^1(G, \mathbb{Z}) = H^1(K(G, 1), \mathbb{Z}) = \text{Hom}(G, \mathbb{Z})$ .

Recall  $H^2(X, \mathbb{Z}) = [X, K(\mathbb{Z}, 2)] = [X, \mathbb{C}P^\infty]$ .

And  $\mathbb{C}P^\infty = \text{Gr}(1, \mathbb{C})$  parametrizes  $\mathbb{C}$ -line bundles.

Let  $\alpha \in H^2(X, \mathbb{Z})$  be ~~such a bundle~~<sup>a class</sup>, giving rise to  $S^1 \rightarrow \tilde{X}_\alpha \downarrow X$  by taking unit circle bundle of associated  $\mathbb{C}$ -bundle.

This gives rise to SES of groups  $1 \rightarrow \mathbb{Z} \rightarrow \tilde{G}_\alpha \rightarrow G \rightarrow 1$  (when  $X = K(G, 1)$ ). And  $\mathbb{Z} = \pi_1 S^1$  is central.

• Fact: This is a correspondence:

$$H^2(G, \mathbb{Z}) \longleftrightarrow \left\{ \begin{array}{l} \text{Equivalence classes of} \\ \text{central } \mathbb{Z}\text{-extensions} \\ 1 \rightarrow \mathbb{Z} \rightarrow \tilde{G}_\alpha \rightarrow G \rightarrow 1 \end{array} \right\}.$$

Our goal now will be to attempt to understand cup 2 Massey products in this setting.

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Start with the cup product.

I want to define a map  $H^1(G, \mathbb{Z}) \otimes H^1(G, \mathbb{Z}) \rightarrow H^2(G, \mathbb{Z})$ .

Let  $\alpha, \beta \in H^1(G, \mathbb{Z})$ , i.e.  $\alpha: G \rightarrow \mathbb{Z}$ ,  $\beta: G \rightarrow \mathbb{Z}$  are homs.

Then  $\alpha * \beta: G \rightarrow \mathbb{Z}^2$  is given.

Thinking cohomologically, I can use this to pull back the "fundamental class" (i.e. generator) of  $H^2(\mathbb{Z}^2, \mathbb{Z})$ .

As a group extension:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathbb{Z} & \rightarrow & H & \rightarrow & \mathbb{Z}^2 \rightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \alpha * \beta \\ 1 & \rightarrow & \mathbb{Z} & \rightarrow & \tilde{G}_{\alpha * \beta} & \rightarrow & G \rightarrow 1 \end{array}$$

What group extension is this? It's the Heisenberg group!

Now let's consider Massey products. Suppose  $\alpha, \beta \in H^1(G, \mathbb{Z})$  satisfy  $\alpha * \beta = 0$ .

This means the following:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathbb{Z} & \rightarrow & H & \rightarrow & \mathbb{Z}^2 \rightarrow 1 \\ & & \uparrow f & & \uparrow \alpha * \beta & & \\ 1 & \rightarrow & \mathbb{Z} & \rightarrow & \tilde{G}_{\alpha * \beta} & \rightarrow & G \rightarrow 1 \end{array}$$

There exists a section!  $\uparrow$ .

So we obtain a map

$$f \circ \sigma: G \rightarrow H, g \mapsto \begin{pmatrix} 1 & \alpha(g) & \xi(g) \\ 0 & 1 & \beta(g) \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, if  $\beta * \gamma = 0$ , there is a map  $G \rightarrow H, g \mapsto \begin{pmatrix} 1 & \beta(g) & \omega(g) \\ 0 & 1 & \gamma(g) \\ 0 & 0 & 1 \end{pmatrix}$ .

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Now let  $\tilde{N}$  be the nilpotent group  $\left\{ \begin{pmatrix} 1 & a & d & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, a, \dots, f, e \in \mathbb{Z} \right\}$ .

$$\mathbb{Z}(\tilde{N}) \cong \mathbb{Z} = \left\{ \begin{pmatrix} 1 & 0 & 0 & f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \text{ and } N = \tilde{N}/\mathbb{Z}(\tilde{N}) = \left\{ \begin{pmatrix} 1 & a & d & * \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

There is a central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \tilde{N} \rightarrow N \rightarrow 1.$$

Now note that the maps  $g \mapsto \begin{pmatrix} 1 & \alpha(g) & \xi(g) \\ 0 & 1 & \beta(g) \\ 0 & 0 & 1 \end{pmatrix}$ ,  $g \mapsto \begin{pmatrix} 1 & \beta(g) & \omega(g) \\ 0 & 1 & \gamma(g) \\ 0 & 0 & 1 \end{pmatrix}$

can be combined to give a map  $G \rightarrow N$ ,

$$g \mapsto \begin{pmatrix} 1 & \alpha(g) & \xi(g) & * \\ 0 & 1 & \beta(g) & \omega(g) \\ 0 & 0 & 1 & \gamma(g) \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \text{Thus we obtain a class}$$

$\langle \alpha, \beta, \gamma \rangle \in H^2(G, \mathbb{Z})$   
of associated to the pullback  
~~of~~  $1 \rightarrow \mathbb{Z} \rightarrow \tilde{N} \rightarrow N \rightarrow 1$ .

This is the Massey product in group cohomology!

This also gives an indication of how to define Massey products of higher orders, and how this is related to  $n$ -step nilpotent quotients of  $G$ .

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## Poincaré Conjecture:

Glib statement: Any simply-connected homology sphere is  $S^3$  itself.

So let's study the fundamental groups of homology spheres.

Key fact: • Every homology sphere is constructed by taking the standard <sup>a</sup> ~~an~~ Heegaard splitting for  $S^3$ , and pre-composing the gluing map  $i_g: \Sigma_g \rightarrow \Sigma_g$  by some  $\phi \in \mathcal{I}_g$ .

•  $\phi$  and  $\psi \in \mathcal{I}_{g,1}$  give rise to the same  $M = \overset{S_\phi = S_\psi}{\underset{i_g \circ \psi = \xi_1, i_g \circ \phi \circ \xi_2}{\text{iff}}}$  iff  $\xi_1, \xi_2 \in \mathcal{N}_{g,1}$  (handlebody group).

•  $\pi_1(\mathbb{M}_\phi) = \langle a_1, \dots, a_g | P(\phi(a_i)) \rangle$ , with

$$\pi_1(\Sigma_g) = \langle a_1, \dots, a_g, b_1, \dots, b_g | \dots \rangle \xrightarrow{P} F_g = \langle a_1, \dots, a_g \rangle.$$

This suggests an approach to P.C.: Show that if  $\pi_1(\mathbb{M}_\phi) = \{1\}$ , then  $\phi \in \mathcal{N}_{g,1}$ .

To do this, we need to get a handle on when  $\{P(\phi(a_i))\}$  give rise to a set of relations that don't generate the trivial group.

This is where the Johnson filtration comes in.

$$\text{Mod } \Sigma_{g,1} \cdot \mathcal{I}_{g,1}(0) \supset \mathcal{I}_{g,1}(2) \supset \mathcal{I}_{g,1}(3) \supset \mathcal{I}_{g,1}(4) \supset \dots$$

There are homomorphisms  $\tau_n: \mathcal{I}_{g,1}(n) \rightarrow \text{Hom}(H, \frac{F_g^{(n)}}{F_g^{(n+1)}})$ ,

which measure the action of  $\phi \in \mathcal{I}_{g,1}(n)$  on the  $n^{\text{th}}$  nilpotent truncation of  $\pi_1(\Sigma_g)$ .

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These  $\tau_k$ 's can also be defined in terms of Massey products in the mapping torus  $M\phi$ .

So here's how a proof of PC would go in this framework:

- I identify, for all  $k$ ,  $\text{im}(\tau_k|_{(N_{j,1} \cap I_{j,1}(k))}) = W_k \leq \text{Hom}(H, F_2^{(k)} / F_2^{(k+1)})$
- I identify the entire image  $V_k = \text{im}(\tau_k)$
- Show that if  $\phi \in I_{j,1}(k)$  satisfies  $\tau_k(\phi) \notin W_k$ , then  $\pi_1(S_\phi) \neq \{1\}$ .

Together this shows that any counterexample to P.C.

lies in  $N_{j,1} \cdot \left( \bigcap_{k \geq 1} I_{j,1}(k) \right) = N_{j,1}$  and so isn't a counterexample at all.

Example: Step 1. ( $k=2$ ) (Johnson)

Fact 1:  $F^{(2)} / F^{(3)} \cong \Lambda^2 H$ , and  $\text{im } \tau_2 \leq \text{Hom}(H, \Lambda^2 H) \cong \Lambda^3 H$ .

Fact 2: (Morita)  $\text{im}(\tau_2|_{(N_{j,1} \cap I_{j,1})}) = W_j = \langle x_i \wedge x_j \wedge y_k, x_i \wedge y_j \wedge y_k, y_i \wedge y_j \wedge y_k \rangle$

So if  $\tau(\phi) \notin W_j$ , it contains a summand of the form  $x_i \wedge x_j \wedge x_k$ .

Now by untangling the definition of  $\tau_2$ , it is possible to show that in these circumstances,  $\pi_1(S_\phi) \neq \{1\}$ .

Sketch:  $x_i \wedge x_j \wedge x_k \rightsquigarrow p(\phi(x_i)) = x_i(x_j, x_k) \pmod{F_2^{(3)}}$ .  
(also lie)

Now work in  $F_{\langle a_1, \dots, a_3 \rangle} / F_2^{(3)}$  to see that it is impossible to write  $x_i$  as a word in  $\{p(\phi(x_j))\}$ .