

## A prehistory of the Johnson homomorphism.

This is ultimately a story about the Torelli group, but may not seem like it. Nevertheless, I want to say a few words about it before we get started.

SES:  $1 \rightarrow \mathcal{I}_g \rightarrow \text{Mod } \Sigma_g \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1$ .

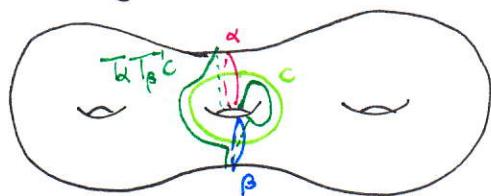
Also define  $\mathcal{I}_{g,*}$ ,  $\mathcal{I}_{g,1}$  as the kernels of  $\text{Mod } \Sigma_{g,*} \rightarrow \text{Sp}(2g, \mathbb{Z})$ .

$\text{Mod } \Sigma_{g,1} \nearrow$

What are some elements in this ~~group~~?

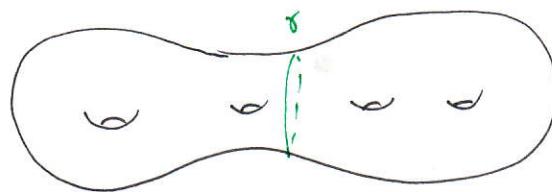
Rmk: We will see how  $\mathcal{I}_{g,1}, \mathcal{I}_{g,*}$  are easier to study.

Bounding Pair (BP) map:



$\alpha, \beta$  bound subsurface; BP is  $T_\alpha T_\beta^{-1}$

Separating Twist:



$T_\gamma$  is a separating twist.

Both of these lie in  $\mathcal{I}_g$ , as can be seen from the formula for the action of a Dehn twist in homology.

An old (pre-Johnson) question: Let  $\mathcal{K}_g$  denote the group generated by all sep. twists. Is  $\mathcal{K}_g = \mathcal{I}_g$ ? Is it finite-index?

References: - Johnson survey article ('83)

- Sullivan: On the intersection ring of compact 3-manifolds ('74)

- Chillingworth: Winding Numbers on Surfaces, I, II. ('72)

We will discuss this question, and explain how Johnson's homomorphism blows it out of the water, as well as doing so much more. I'd like to do this by investigating some of the historical predecessors to Johnson's work, which he cites in his survey article.

Ultimately, we will see how Johnson hom. is the basic ingredient in the theory of the (co)-homology of  $\Gamma_g$ , or in other words, the theory of characteristic classes of Torelli surface bundles.



### Sullivan's Question

$M^3$  closed, oriented. Map  $H^1 M \xrightarrow{\otimes^3} H^3 M \cong \mathbb{Z}$ ,  $a \otimes b \otimes c \mapsto abc[M]$ .

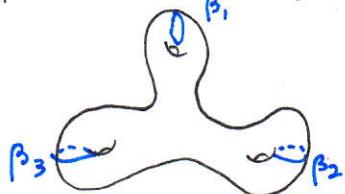
$$\downarrow \quad \quad \quad \nearrow$$

$$\Lambda^3 H^1 M$$

This gives us an element of  $\text{Hom}(\Lambda^3 H^1 M, \mathbb{Z}) \cong \Lambda^3 H_1 M$ . Call it  $\mu$ .

Q. (Sullivan). Suppose  $b, M = \beta$ . Can we get any  $\mu \in \Lambda^3 \mathbb{Z}^\beta = \mathbb{Z}^{(\beta)}$ ?

A. (-""). Yes. Here's how. Fix a handlebody  $S$  of genus  $\beta$ :



There is the inclusion  $i: \partial S \rightarrow S$ , giving  $H_1(\partial S) \rightarrow H_1(S)$ .

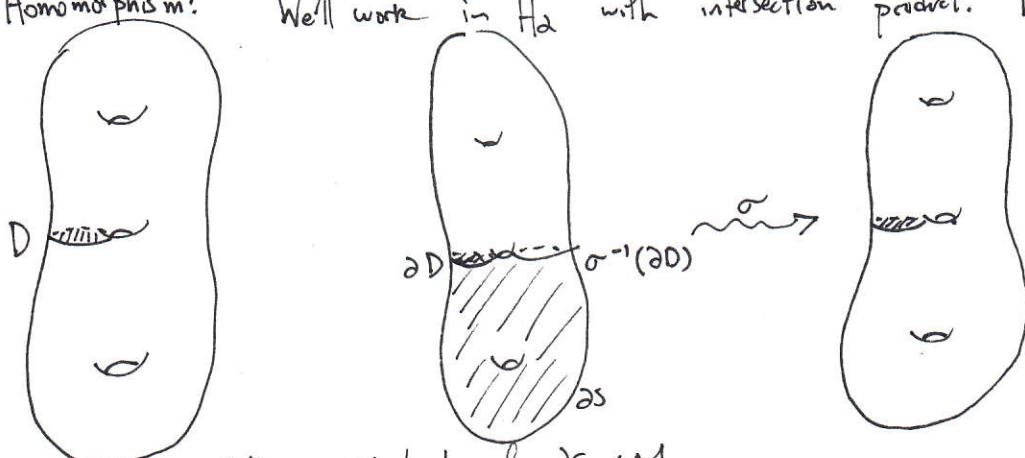
Identify  $\ker(i_*) \cong \mathbb{Z}^\beta$  (as shown)

Consider the group  $\Gamma \leq \text{Diff}(\partial S)$  acting trivially on  $\ker(i_*)$ .

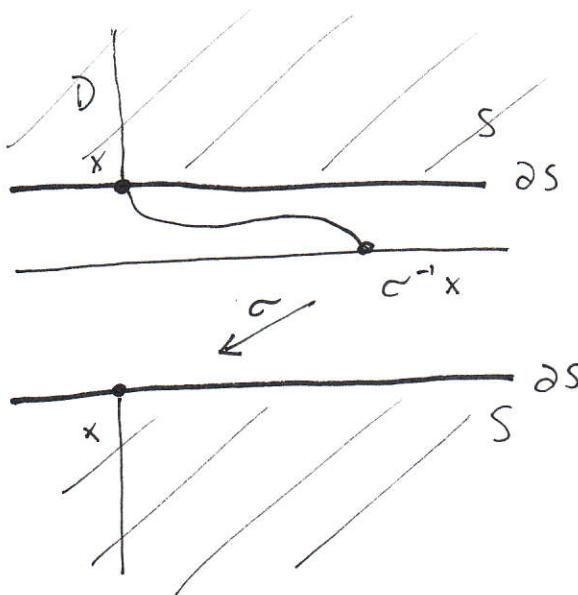
Any  $\gamma \in \Gamma$  determines a Heegaard decomposition  $M_\gamma = S \cup_\gamma S$ . Assumption  $\gamma \in \Gamma$  tells us that  $H_1 M \cong \ker(i_*) \cong \mathbb{Z}^\beta$ .

Prop.:  $\Gamma \rightarrow \Lambda^3 H$ ,  $\sigma \mapsto \mu_\sigma$  is a surjective homomorphism.

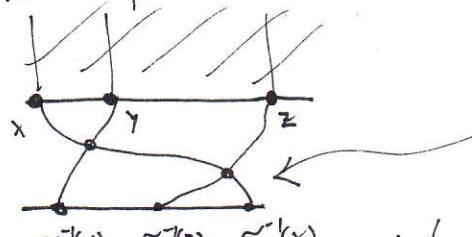
Pf Homomorphism? We'll work in  $H_2$  with intersection product. What does  $H_2$  consist of?



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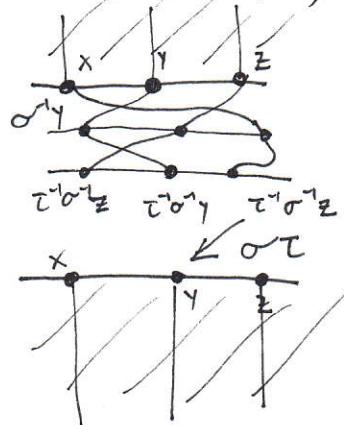
To compute the intersection we take three such surfaces:



Points of triple intersection give rise to the invariant  $\mu$ .

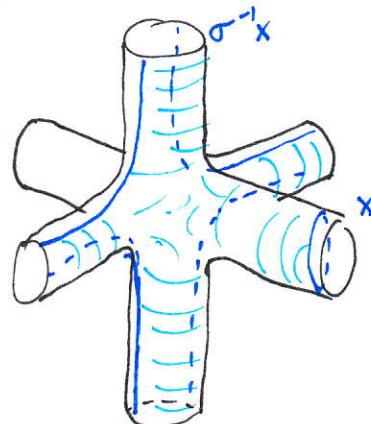
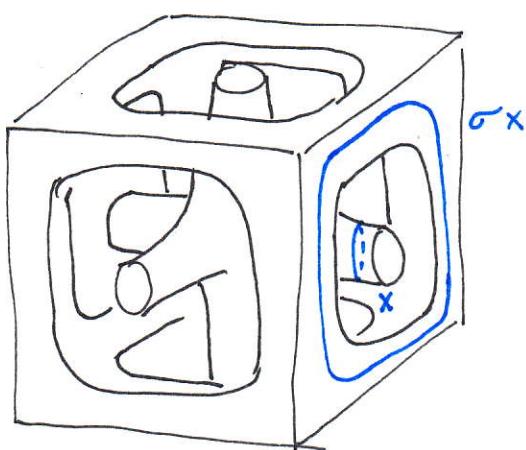
Working with this homology basis is convenient insofar as all intersections occur only in the collar neighborhoods. If we have

two homs  $\sigma, \tau$ , we make a collar for  $\sigma \circ \tau$  by concatenation:



Hence it is a hom. Therefore, to show surjectivity, it suffices to explain how to obtain any  $x_1 \wedge x_2 \wedge x_3$ . Start with the following observation: if  $H^3 = \langle x, y, z \rangle$ , the corresponding  $\mu$  is  $x \wedge y \wedge z$ . So we'd like to exhibit  $T^3$  as  $M\sigma$  for some  $\sigma$ , and we know we'll need to take 5 genus 3.

This is how:



Squint / iso tote and you will see this is the "standard" surface  $\Sigma_x$

(3)

For general  $\beta$ , just do the  $\sigma$  map on three handles at a time.

A special case of the above construction is if we take  $\sigma \in \mathcal{I}_g$  (fixing all the homology). Then  $\sigma \mapsto \mu_\sigma$  gives a homomorphism  $\mathcal{I}_g \rightarrow \mathbb{Z}^{(\binom{g}{3})}$ .

(N.B.: This really is defined on the closed Torelli group. More on this later)

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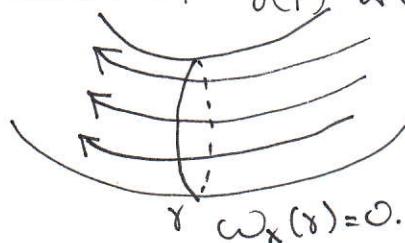
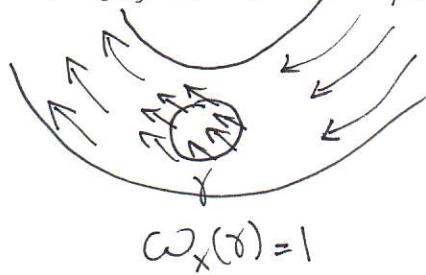
Let's describe another "accidental" procedure for obtaining a homomorphism out of  $\mathcal{I}_{g,*}$  (now the puncture is important!). This idea is due to Chillingworth, and ~~Johnson observes that~~ it is in fact powerful enough to answer the above Question.

In fact, the Sullivan invariant can do this too. ~~But~~ It's easy to see that  $\mu_\phi = 0$  for any  $\phi \in \mathcal{X}_g$ , but finding a  $\psi \in \mathcal{I}_g$  with  $\mu_\psi \neq 0$  is not totally trivial, although it can be done.

Chillingworth's idea is to define a "winding number" of a curve on a surface. To do this, we choose a vector field  $X$  on  $\Sigma_{g,*}$  punctured surface, and choose  $X$  so that it does not vanish on  $\Sigma_{g,*}$ .

(How? Pick any  $X$  on  $\Sigma_g$ , and cut out a disk  $n$ th of all the punctures.)

If  $\gamma$  is a regular ( $\gamma \neq 0$ ) curve, the winding number  $\omega_X(\gamma)$  is defined to be the winding number of  $\gamma(p)$  around  $X(p)$ .



- Does this depend on  $X$ ? Oh yeah. But this is a feature, not a bug.

Facts:

- 1) If  $\phi \simeq \text{id}$ , then  $\omega_{\phi_* X} = \omega_X$ . (Just flow the vector field back!)
- 2) If  $X_1, X_2$  are two vflds, define  $d(X_1, X_2): \{\text{regular curves } \gamma\} \rightarrow \mathbb{Z}$ ,  
 $d(X_1, X_2)(\gamma) = \omega_{X_1}(\gamma) - \omega_{X_2}(\gamma)$ . Then  $d(X_1, X_2)$  is a cohomology class.

Def (Chillingworth invariant)

$$t: \mathcal{I}_{g,*} \longrightarrow \text{Hom}(H_1\Sigma, \mathbb{Z})$$

$$t(\phi)(\gamma) = \omega_{\phi_* X}(\gamma) - \omega_X(\gamma) = d(\phi_* X, X)(\gamma)$$

- This is independent of  $X$ :

$$[d(\phi_* X, X) - d(\phi_* Y, Y)](\gamma) = \cancel{d(\phi_* X, \phi_* Y)} [d(\phi_* X, \phi_* Y) - d(X, Y)](\gamma)$$

$$= d(X, Y)[\phi_* \gamma - \gamma] = 0, \quad \text{since } [\phi_* \gamma - \gamma] = 0 \quad (\phi \in \mathcal{I}_{g,*}).$$

- This is a homomorphism:

$$t(\phi\psi)(\gamma) = d(\phi_* \psi_* X, X)(\gamma) = [d(\phi_* \psi_* X, \psi_* X) + d(\psi_* X, X)](\gamma)$$

$$= [d(\phi_* X, X) + d(\psi_* X, X)](\gamma) \quad \text{by above.}$$

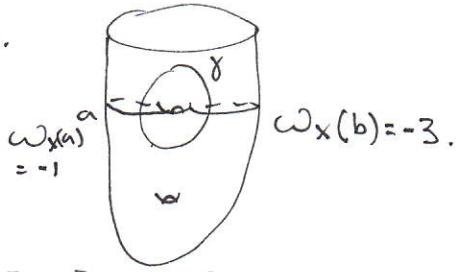
- After Poincaré duality, Chillingworth gives us a map  $t: \mathcal{I}_{g,*} \rightarrow H_1\Sigma \cong \mathbb{Z}^{2g}$

Definition:  $\mathcal{C}_{g,*} = \text{ker } t$ . These are the mapping classes where for all curves  $\gamma$ ,  $\gamma$  and  $\phi_* \gamma$  have the same winding numbers (rel. any  $X$ ).

So to show that  $\mathcal{I}_{g,*}$  is in fact infinite-index in  $\mathcal{I}_{g,*}$ , we need to find a  $\phi$  with  $t(\phi) \neq 0$ . ~~Nonvanishing~~ In other words, we need to find a  $\phi$  and a curve  $\gamma$  with  $\omega_X \gamma \neq \omega_X \phi_* \gamma$ .

This is slightly hard to draw, but here's an idea.

Select  $X$  on  $\Sigma_{2,1}$  s.t.



Fact: Winding numbers "shear" under Dehn twists:

$$\omega_X(\phi_* \gamma) = \omega_X(\gamma) \pm i(a, \gamma) \omega_X(a).$$

$$\text{Then } \omega_X(T_a T_b^{-1} \gamma) = \underline{\omega_X \gamma \pm 2}.$$

Finally, the Johnson homomorphism!

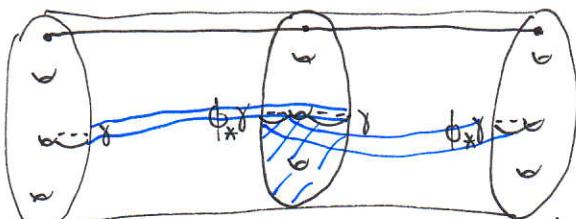
My favorite definition (there are three that I know of):

This follows Sullivan. Instead of using  $\phi \in \mathcal{I}_{g,*}$  to make a Heegaard splitting, use it to make a surface bundle instead, called  $M_\phi$  again.

The marked pt memo that  $H^1 M_\phi$  has a canonical splitting:  $H^1 M_\phi \cong_{\text{perco}}$

$M_\phi$  has a section  $\pi$ ; which endows  $H^1 M_\phi$  with a splitting  $H^1 M_\phi \cong \text{im } \pi^* \oplus_{\text{perco}} \ker \phi^*$ .

↓  
More concretely, in  $H_2$  we have the following. :



$\phi_* \gamma, \gamma$  are in the same homology class; i.e. together they bound a subsurface. In fact, they bound two. We decide which one to take, by picking the one missing the marked pt.

This gives an injection  $H_1 \Sigma \hookrightarrow H_2 M_\phi \cong H^1 M_\phi$ , and now we look at the intersection pairing restricted to this space; an element of  $\text{Hom}(\Lambda^3 H_1 \Sigma_g, \mathbb{Z}) \cong \Lambda^3 H$ .

Define  $\tau(\phi)$  the Johnson homomorphism by

$$T: \mathcal{I}_{g,*} \rightarrow \Lambda^3 H ; \quad \tau(\phi) \text{ the up product form.}$$

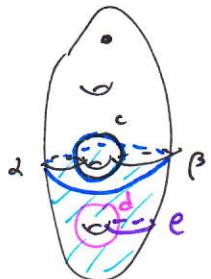
(6)

The advantage is now that computing is transparent.

For a sep. twist:



The entire homology basis is left invariant, so the corresponding surfaces in  $H_2 M/\phi$  are tori, for which triple intersections vanish. So  $\tau(\phi) = 0$ .



For  $\phi = T_a T_b^{-1}$ , the surface  $I_c$  is as shown:

$\Sigma_d, \Sigma_e$  are tori, and you can see that  $\Sigma_c \cap \Sigma_d \cap \Sigma_e = 1$ .

In fact  $\tau(\phi) = c \wedge d \wedge e$ . Generally, if  $[a] = [b]$ , and  $[c]$  is the symplectic dual to  $[a]$  (rel some fixed  $\omega$ ), then  $\tau(\phi) = c \wedge \omega|_{\Sigma_c}$ . In particular this is nonzero.

Johnson in fact showed that  $\mathcal{I}_g$  is the entire kernel of  $\tau$ .

(NB:  $\Lambda^3 \mathbb{Z}^{2g}$  is the torsion-free abelianization of  $\mathcal{I}_{g,*}$ .

Johnson computed  $\mathcal{I}_{g,*}^{ab}$  and showed there was 2-torsion "coming from 3-manifold topology" in the sense that it is defined using the Rochlin invariant. There is no such 2-torsion for  $\mathcal{I}_{g,*}$ )

Lastly I should explain how the work of Sullivan and Chillingworth is incorporated. Being  $\mathbb{Z}$ -abelian quotients of  $\mathcal{I}_g$ , they must factor through  $\tau$ . Here's how:

$$\begin{array}{ccc} \mathcal{I}_g & \xrightarrow{\tau} & \Lambda^3 H / H \\ & \searrow \text{Sullivan} & \downarrow \\ & & \Lambda^3 W \end{array}$$

( $W$  an Lagrangian subspace of  $H$ )

$$\begin{array}{ccc} \mathcal{I}_{g,*} & \xrightarrow{\tau} & \Lambda^3 H \\ & \searrow t & \downarrow \text{Tensor contraction} \\ & & H \end{array}$$

(Chillingworth)