

[4-manifolds can be surface bundles over surfaces
in many ways]

Nick Sitter - Georgia Tech, Sept. 15, 2014.

• Preliminaries

- A surface bundle is a fiber bundle $\Sigma_g \rightarrow E$
 \downarrow
 B

Typically we like $g \geq 2$ and for B to be a manifold ($\Rightarrow E$ is as well)

- The theory of surface bundles has deep interactions with topology in dimensions 1, 2, 3, 4, as well as with algebraic, complex, symplectic, hyperbolic geometry, and geometric group theory.

• A (very) brief tour of fibered 3-manifolds

Recipe for 3-manifolds: Take $\phi \in \text{Diff}(\Sigma_g)$, form mapping torus

$$M_\phi^3 = \Sigma_g \times I / (x, 1) = (\phi(x), 0).$$

After Agol, every closed hyperbolic 3-manifold has a finite-sheeted cover of this form!

One basic feature of the theory: it often is the case that ϕ is far from unique: distinct $\phi \in \text{Diff } \Sigma_g$, $\psi \in \text{Diff } \Sigma_h$ can make diffeomorphic $M_\phi \cong M_\psi$.

Thm (Thurston) $H_2(M^3)$ admits a ^{canonical} decomposition into finitely many rational cones. ~~At some~~ $\rightarrow \{E_k\}$. If $x \in E_k \cap H_2(M, \mathbb{Z})$ corresponds to the fiber of a fibration over S^1 , then every $y \in E_k \cap H_2(M, \mathbb{Z})$ also does. Thus M is realized as a surface bundle over S^1 with fiber of genus g for infinitely many g , but fin. many for each genus.

◦ Background on SBS with multiple fiberings.

- Move up a dimension, and consider surface bundles over surfaces (SBS). $\Sigma_g \rightarrow E$
 \downarrow
 Σ_h . Theory is richest when $h \geq 2$.

- Here there's no obvious generalization of the Thurston norm.

Theorem (FEA Johnson) Every 4-manifold E has only finitely many distinct $\Sigma_g \rightarrow E$ with $g, h \geq 2$.

$$\begin{array}{c} \Sigma_g \rightarrow E \\ \downarrow \\ \Sigma_h \end{array}$$

Here "distinct" means one of two equiv. things:

• Distinct $\pi_1 \Sigma_g \triangleleft \pi_1 E$ with $\pi_1 E / \pi_1 \Sigma_g \cong \pi_1 \Sigma_h$

• $\sqrt{\text{Fibers}}$ $F_1 \rightarrow E$ $F_2 \rightarrow E$ have nonempty transverse intersection.
 Generic $\downarrow p_1$ $\downarrow p_2$
 Σ_h Σ_h

The obvious question: How many fiberings can a single E^4 admit?

More precise version: Define

$$N(d) = \max \left\{ \text{Number of distinct fiberings of } E^4, \chi(E) \leq 4d \right\}$$

FEAJ's argument shows $N(d) \leq \sigma_0(d) (d+1)^{2d+6}$
 ($\sigma_0(d)$ = sum of divisors function)

Problem: Find lower bounds on $N(d)$

- Products $\Sigma_g \times \Sigma_h$, as well as the Atiyah-Kodaira construction to be described shortly, show $Z \leq N(d)$.

For a long time, $2 \leq N(d) \leq \sigma_0(d)(d+1)^{2d+6}$ were the best-known bounds.

-I spent a long time trying to prove this was sharp: $N(d) = 2$.

Culminating theorem:

Thm (S-): Let $\Sigma_g \rightarrow E \rightarrow \Sigma_h$ be SBS, with monodromy $\rho: \pi_1 \Sigma_h \rightarrow \mathcal{K}_g$ contained in the subgroup $\mathcal{K}_g \leq \mathcal{I}_g$ ^{generated by} ~~consisting of~~ all separating Dehn twists. Then either $E \cong \Sigma_g \times \Sigma_h$ is the trivial bundle, or else E has a unique surface bundle structure.

I presented this theorem at GTC in May (Thars, Dan!)

That afternoon, I found the first example of a SBS with > 2 fiberings. The culmination of this is the following improvement in the lower bound on $N(d)$:

Thm (S-): $2^{(d+2)/6} \leq N(d) \leq \sigma_0(d)(d+1)^{2d+6}$

For the experts, the monodromy of ^{some of} these examples live in \mathcal{I}_g .

Corollary: my theorem above is sharp wrt the Johnson filtration

◦ How do you make interesting surface bundles?

- For the remainder of the talk, I want to show you some methods for building SBS.

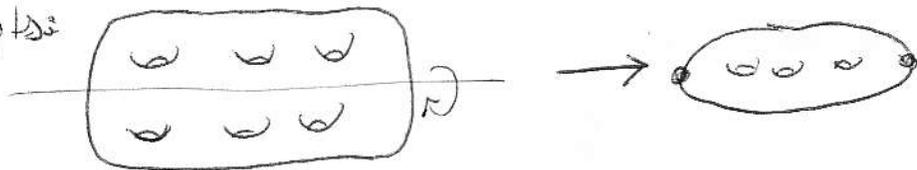
- The Atiyah-Kodaira construction:

Based around the branched cover construction:

A branched cover of surfaces is a map $f: \Sigma_g \rightarrow \Sigma_n$ such that

f is a covering map when finitely many pts are deleted from Σ_g .

Better to think of f as a quotient by a group action $G \curvearrowright \Sigma_g$ with fin. many fixed pts:



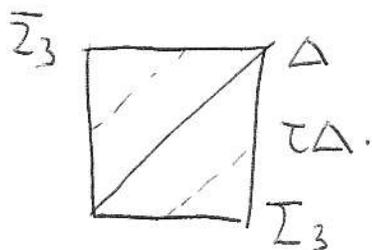
A-K construction proceeds by taking an interesting fiber-wise branched cover of a product.

- To specify branched cover, need to specify the branch points. "Interesting" here means that branch points move round as fibers change.

- Simplest example requires two distinct points in each fiber. One pt: Pick out z in the fiber over z . This is the diagonal section $\Delta: \Sigma \rightarrow \Sigma \times \Sigma$.

- If Σ is equipped with a free involution $\tau: \Sigma \rightarrow \Sigma$, then $\tau(z)$ and z are always different points.

Simpliest $\Sigma = \Sigma_3$. In summary:



Want to take branched covers here. There is a finite obstruction, which we remove by enlarging the base:

$$\begin{array}{ccc}
 E & & \\
 \downarrow & & \\
 \Sigma_{12g} \times \Sigma_3 & \longrightarrow & \bar{\Sigma}_3 \times \bar{\Sigma}_3 \\
 \downarrow & & \downarrow \\
 \Sigma_{12g} & \longrightarrow & \bar{\Sigma}_3
 \end{array}$$

Fiber of $E \rightarrow \Sigma_{12g} \times \Sigma_3 \rightarrow \Sigma_{12g}$ has genus 6, since we took a fiberwise double-branched cover.

But also! E is a fibering, with fiber genus 321.

$$\begin{array}{ccc}
 E & & \\
 \downarrow & & \\
 \Sigma_{12g} \times \Sigma_3 & \longrightarrow & \bar{\Sigma}_3
 \end{array}$$

E has amazing properties which have been important in 4-mfld top, alg. geom, complex geom, symplectic geom.

Now I'll explain my construction.

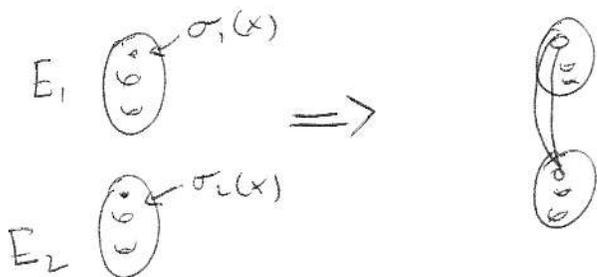
Section sum:

Suppose $\Sigma_{g_1} \rightarrow E_1$ $\downarrow \nearrow \sigma_1$ B , $\Sigma_{g_2} \rightarrow E_2$ $\downarrow \nearrow \sigma_2$ B

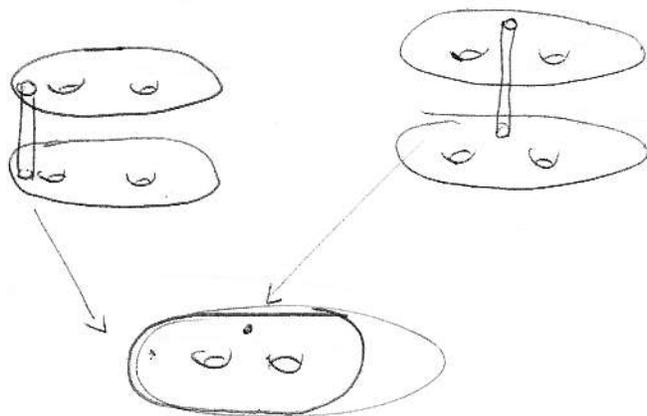
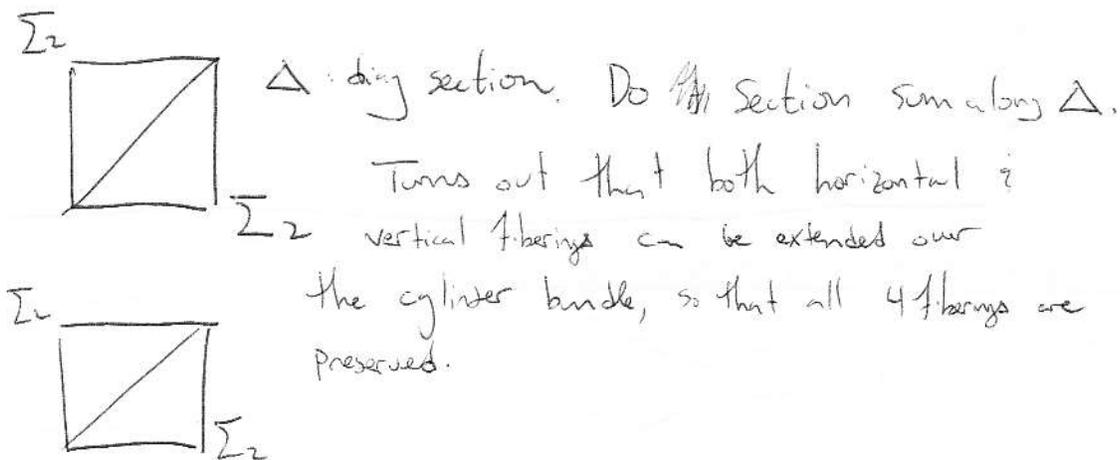
with $2(\chi(\sigma_1)) \equiv 2(\chi(\sigma_2))$. Then $E_1 \setminus \mathcal{V}(\sigma_1)$ can be glued to $E_2 \setminus \mathcal{V}(\sigma_2)$.
 (\Leftrightarrow Euler numbers equal)

The result is a fiberwise connect-sum.

Fibers The point of attachment in each fiber is specified by σ_1, σ_2 .



Example 1:



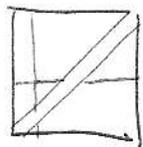
Monodromy is a "cylinder drag".

Explain common philosophy:

Perform surface operations fiberwise, using an interesting section to

(7) Parameterize necessary data.

Four fiberings:



(Appeal to Π_1 condition)



Pf of theorem: We'll name E_n^4 with $\chi(E_n^4) = 24n - 8$,
admitting 2^n distinct fiberings



Stack! Need two distinct plans to attach: $\tau \in \Sigma_3$ shows
up again. 2 choices (H or V) in n components
 $\Rightarrow 2^n$ maps to Σ_3 .

And $\chi(E_n) = \chi(\text{Base}) \chi(\text{Fiber})$
 $\quad \quad \quad -4 \quad \quad 2 - 6n$

Examples over bases of different genera:

Take covers $\Sigma \xrightarrow{f_1} \Sigma_1$
 $\quad \quad \quad \Sigma \xrightarrow{f_2} \Sigma_2$

