

Surface bundles over surfaces with multiple fiberings

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- The ultimate inspiration for this work comes from the Thurston norm.

Quick recap: Thurston norm is a (semi) norm on $H_2(M_3; \mathbb{R})$.

$\|x\|_T$ for $x \in H_2(M_3; \mathbb{Z})$ measures the "minimal complexity" of an embedded surface Σ with $[\Sigma] = x$.

One aspect of the theory: significant interaction between $\|\cdot\|_T$ and the collection of maps $f: M \rightarrow S^1$ realizing M as a surface bundle $\Sigma \rightarrow S^1$.

In particular, if $b_1 M \geq 2$ and $\Sigma \rightarrow S^1$ is a fibering, then there are infinitely many distinct $p_i: M \rightarrow S^1$, realizing M as a surf. bundle over S^1 in inf. many ways, fin. many for each genus g .

- There is not an obvious way to extend $\|\cdot\|_T$ directly to 4-mfds and higher. Still interesting to wonder what parallels exist for surf. bundles over S^1 and surface bundles over surfaces (STBS).
- On the face of it, this should not get far. $\|\cdot\|_T$ is ultimately a ~~theory~~ theory of manifolds fibered over S^1 , that fibers are surfaces is almost accidental.

- There are significant differences. But:
 - (1) There are some surprising commonalities
 - (2) There are rich and interesting new phenomena.

Vis à vis (1):

Thm 1 (S-) Let $\Sigma_g \xrightarrow{\downarrow_p} E$ be SBS with monodromy $p: \pi_1 \Sigma_h \rightarrow \text{Mod } \Sigma_g$.

Suppose that $H^1(\Sigma_g; \mathbb{Q})^p := H^1(\Sigma_g; \mathbb{Q}) / \langle v = p(x)v, x \in \pi_1 \Sigma_h \rangle$ is trivial. Then $E \xrightarrow{p} \Sigma_h$ is the unique way that E fibers as SBS.

(Remark) For $\Sigma_g \xrightarrow{\downarrow_p} M^3$ a fibered 3-mfd with monodromy ϕ ,

$$H^1(M^3; \mathbb{Q}) \approx p^* H^1(S^1) \oplus H^1 \Sigma_g^\phi, \text{ so condition } b_1(M) = 1$$

says that $H^1 \Sigma_g^\phi = \{0\}$. Thus Thm 1 is a direct counterpart, although the converse is no longer true for SBS.

Proof is totally different from the 3-mfd case.

Let's turn to item (2), and discuss SBS with multiple fiberings.

The major difference: F.E.A. Johnson showed every SBS admits finitely many fiberings only.

Reasonable question #1: Are there any 4-mfds with 2 structures?

Yes: eg $\Sigma_g \times \Sigma_h$.

Reasonable question #2: Are there any nontrivial examples?

(2)

Answer: Yes! The quintessential example is due to Atiyah & Kodaira.

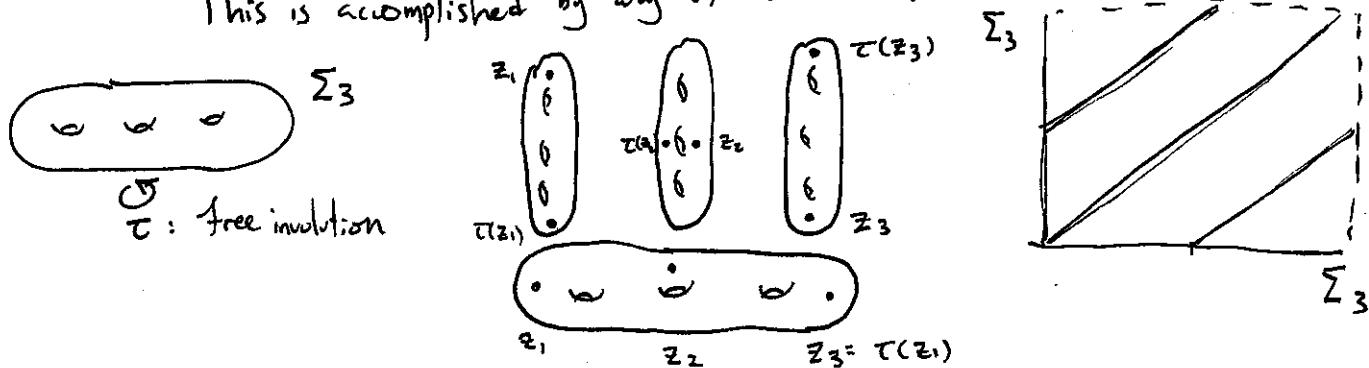
They weren't trying to do this per se.^(*) The observation that their construction double-fibers is due to Le Brun.

(x): (A&K were looking for a SBS with nonzero signature)

Construction: Take an "interesting" fiberwise branched covering of a product.

Need to specify where in each fiber to branch.

This is accomplished by way of a (multi)-section.



We'd like to take a double branched cover over $z, \tau(z)$.

There is an obstruction to doing this, but by passing to a cover $\Sigma_{129} \rightarrow \Sigma_3$, we can make it vanish. We obtain:

$$\begin{array}{ccc} E & \supseteq & \mathbb{Z}/2 \\ & \downarrow & \\ \Sigma_{129} \times \Sigma_3 & \longrightarrow & \Sigma_3 \end{array}$$

(NB: One can also view this as a way of finding a smoothly-varying family of Riemann surface structures on the fiber, or as a complete curve in moduli space.)

This E has an array of interesting properties, aside from being doubly-fibered:

- $\sigma(E) = 256$. This can be seen in a variety of ways, eg by the Hirzebruch G-signature theorem, or by a computation involving "vertical vector fields" (thanks to D. Margalit!).
- Thm. of Thurston asserts all SBS admit symplectic structures. LeBrun showed that the symplectic structures on E coming from the two fiberings are inequivalent in the sense that one cannot be deformed into another thru symplectic forms, even up to the action of $\text{Diff } E$.

(3)

Open question: Can either monodromy rep. $\pi_1, \Sigma_{g,q} \rightarrow \text{Mod } \Sigma_g$ or $\pi_1, \Sigma_3 \rightarrow \text{Mod } \Sigma_{3,1}$ be lifted to $\text{Diff}^+ \Sigma_g$? i.e. does there exist $\tilde{\rho}$ as shown?

$$\begin{array}{c} \tilde{\rho}: \text{Diff}^+ \Sigma_g \\ \downarrow \\ \pi_1, \Sigma_n \xrightarrow{\rho} \text{Mod } \Sigma_g \end{array}$$

(In other words, does E admit a flat structure?)

Sadly I don't have anything to say about that. Instead, here's a second open question that I think is pretty fundamental:

Fundamental question: For $g, h \geq 2$, does there exist E^4 admitting at least three fiberings $p_i: E^4 \rightarrow \Sigma_{h_i}$?

Problem: What can you say about $N(g,h) = \max_{\text{SBS}} \# \text{fiberings among all } \Sigma_g \rightarrow E \rightarrow \Sigma_h$?

How does N grow in g ? In h ? Is it independent of either variable? Both? Does $N(g,h) \equiv 2$?

Theorem 2 (S-) Under the additional assumption that the monodromy $\rho: \pi_1, \Sigma_n \rightarrow \text{Mod } \Sigma_g$ lies in $\mathcal{X}_g = \langle T_\gamma, \gamma \text{ separating} \rangle$, then Fund. Q/n has a negative answer.

Even stronger, if $\Sigma_g \rightarrow E$ with $\rho: \pi_1, \Sigma_n \rightarrow \mathcal{X}_g$ has two fiberings, then $E \cong \Sigma_g \times \Sigma_h$.

Basic obstacle: how do you detect the presence of alternative fiberings in the absence of some strong organizing principle like the Thurston norm accomplishes for 3-manifolds?

One answer lies in the cohomology ring.

• (Observation 1) If $F \xrightarrow{p} E \xrightarrow{q} B$ is a fiber bundle of closed oriented manifolds, and $\chi(F) \neq 0$, then $p^*: H^* B \rightarrow H^* E$ is an injection.
(Gysin homomorphism)

• (Observation 2) If $M^n \xrightarrow{f} N^m$ is a map of closed oriented $n-m$ flds, and $\deg(f) \neq 0$, then $f^*: H^* N \hookrightarrow H^* M$ is an injection.

Want to apply this to a double-fibered $E \xrightarrow{p_1 \times p_2} \Sigma_h \times \Sigma_{h_2}$.

Question: Is $\deg(p_1 \times p_2)$ always nonzero? It is for the Atiyah-Kodaira examples.

Obs. 1 (which always holds in our setting) leads to the following principle:

Principle: To each surface bundle structure on E , we can associate a subring $R \leq H^* E$ with $R \approx H^* \Sigma_h$ (the pullback of $H^*(\text{Base})$).

(This rests on Lemma (S-): P_1, P_2 distinct fibers $\Rightarrow p_1^* H^* \Sigma_h \cap p_2^* H^* \Sigma_{h_2} = H_0 E$ (i.e. they have trivial intersection in pos. degree). This Lemma is how you prove Thm. 1.)

We completely understand $H^* \Sigma_g$ as a ring. This leads to the following:

Problem: Describe $H^*(E; \mathbb{Z})$ for a SBS $\Sigma_b \xrightarrow{\downarrow} \Sigma_h \xrightarrow{E}$, in terms of the monodromy rep. $\rho: \pi_1 \Sigma_h \rightarrow \text{Mod } \Sigma_g$.

One of the major obstacles I had to overcome was how to address this problem. Aside from being (I hope) intrinsically satisfying, this is a key ingredient in Thm 2, and is also intimately related to the cohomology of the Torelli group.

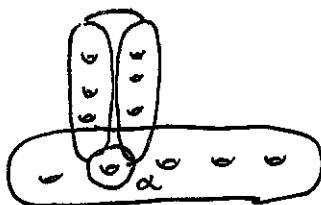
Historical roots:

- D. Sullivan, 1974: Any form $\Lambda^3 \mathbb{Z}^b \rightarrow \mathbb{Z}$ arises as ^{from} triple cup product on M^3 with $b, M^3 = b$.
- D. Johnson used this in his theory of the Torelli group (1980)

I want to give some indication of what H_3^* of a SBS with torsion-free monodromy looks like, in such a way that the intersection form is computable.

- Spectral sequence computation: $H_3 E \approx P^! H_1 B \oplus H_1 F^P$

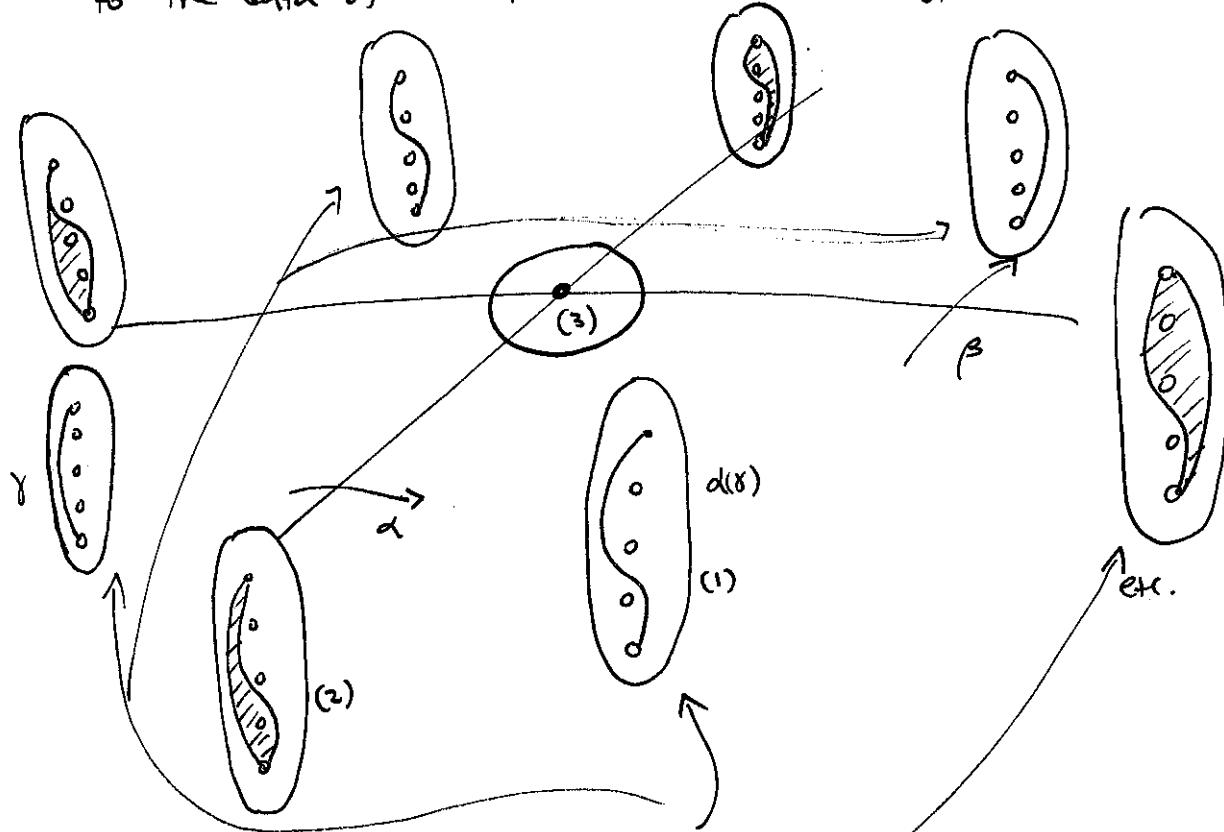
- $P^! H_1 B$:



These classes correspond to subbundles over submanifolds in B (this is a concrete realization of the Gysin hom. in homology).

- $H_1 F^P$ is considerably more subtle.

Recall the goal here is to construct $M \subseteq E$ a closed 3-mfd, assoc. to the data of $c \in H_1 F^P$ (an invariant homology class).



(1) First build this ($\cong D^2 \times S^1$)

(2) Connect up over the 1-skel.

(3) Fill in near the 0-skel.

Two essential features:

- (1) Completely determinable from the monodromy data.
- (2) Intersection pairing is completely computable. (Just need to put the relevant submanifolds in general position).

The formulas end up involving the Johnson homomorphism (in the case of Σ_g), or more generally Morton's extension to $\text{Mod } \Sigma_g$, as well as the higher Johnson invariants $H_i(\Sigma_g) \rightarrow \Lambda^{i+2} H$.

With this in place, can describe Thm 2:

- The assumption $\text{im}(p) \leq \Sigma_g \Rightarrow H^* E \approx H^* B_1 \otimes H^* F$. ($F_i \rightarrow E$ the Σ_g -fib.)
- Now use the principle: constrain places in $H^* E$ where $p_i^* H^* B_2$ can live.
- In $H^* \Sigma_g$, there are lots of x, y s.t. $x \cup y = 0$. But this interacts very poorly with $H^* B_1 \otimes H^* F$. In fact, $p_i^* H^* B_2$ must be contained in $H^* F_i$.

This shows that E is at most double-fibered. (by Lemma)

- By studying annihilators of $p_i^* H^* B_2$, can improve this to the full statement.