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§0: Recap

Last time, Weiqun discussed the notion of a \mathbb{Z} -linear central filtration.

Let's recap: F : group, $G = \text{Aut}(F)$. $F = F_1 \supset F_2 \supset F_3 \supset \dots$

a filtration, This gives rise to a filtration on G :

$$G_d = \bigcap_{n \geq 1} \left(G \xrightarrow{P_{n,d}} \text{Aut}\left(\frac{F_n}{F_{n+d}}\right) \right), \quad G = G_0 \supset G_1 \supset \dots$$

We saw how $G_0/G_1 \ll GL(F^{ab})$ is a \mathbb{Z} -linear group.

We also saw that for $d \geq 1$, G_d/G_{d+1} is abelian: know that

$$[G_m, G_n] \ll G_{m+n}, \text{ so that when } d \geq 1, [G_d, G_d] \ll G_{2d} \text{ and}$$

so $[G_d/G_{2d}, G_d/G_{2d}]$ is trivial.

Today, we will examine these structures in depth for the case $F = \pi_1 \Sigma_g$,

with the filtration $F_i = F^{[i]} = [F^{[i-1]}, F]$ the lower central series.

We will see deep connections to topology, indicating that perhaps pure group theory is not the end of the story here.

§1 Basics of the Torelli group and the Johnson filtration.

$$G_0 = \text{Aut}(\pi_1 \Sigma_g) = \text{Mod}(\Sigma_g, *). \quad G_n = \ker(G \rightarrow \text{Aut}(F^{ab})).$$

This is commonly known as the Torelli group, written $\mathcal{I}_{g,*} \triangleleft \text{Mod}(\Sigma_g, *)$.

$F_1/F_2 = \mathbb{Z}^{2g}$, so $G_0/G_1 \triangleleft \text{GL}_{2g}(\mathbb{Z})$. But it is a strict subgroup.

In fact,

Prop.

$$1 \rightarrow \mathcal{I}_{g,*} \rightarrow \text{Mod}(\Sigma_g, *) \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1 \text{ is exact.}$$

- There is the algebraic intersection pairing $i(\cdot, \cdot)$ on $H_1 \Sigma_g$ which is preserved by any diffeo of Σ_g . This is a symplectic form.

- Alternatively, we can see this purely from the group theory.

Fact: $\mathcal{L}(F_n)$ is the free Lie algebra on $H_1 F_n \cong \mathbb{Z}^n$. In particular,

$$\mathcal{L}_2(F_n) \cong \overset{\text{free group}}{F_n} \overset{[2]}{\bigg/} \overset{[3]}{F_n} \cong \bigwedge^2 H_1 F_n.$$

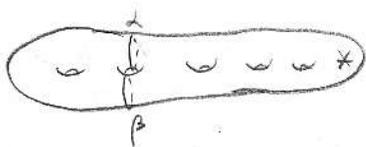
An automorphism α of $\pi_1 \Sigma_g$ is an automorphism of F_{2g} which preserves the relator $[x_1, y_1] \dots [x_g, y_g]$ (up to conjugacy). α can then descend to an automorphism of $\mathcal{L}(F_{2g})$. The condition of preserving $[x_1, y_1] \dots [x_g, y_g]$ translates to preserving $\text{im}([x_1, y_1] \dots [x_g, y_g])$ in $\mathcal{L}_2(F_{2g}) = \bigwedge^2 H_1 F_{2g}$.

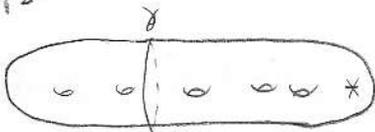
But the img is $x_1 \wedge y_1 + \dots + x_g \wedge y_g$; i.e. a symplectic form.

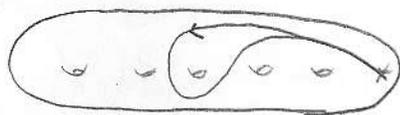
- Surjectivity is true, but not relevant for our purposes here.

Some basic facts about Torelli:

• Elements include the following:

• BP maps:  $T_\alpha T_\beta^{-1}$
(discuss)

• BSCC maps: T_γ 

• Push maps: 

(In fact, $\text{Push} \cong \pi_1 \Sigma_g \cong \text{Inn } \pi_1 \Sigma_g \triangleleft \text{Aut } \pi_1 \Sigma_g$).

• \mathcal{I}_g is normally generated by genus-1 BP maps.
(D. Johnson)

• \mathcal{I}_g is finitely generated! (D. Johnson)
($g \geq 3$)

• \mathcal{I}_2 is an infinite-rank free group (G. Mess).

The image of τ_1

This is the problem with perhaps the most immediate relevance:

τ_1 is a map from $I_{g,x}$ to an abelian group, hence gives a lower bound on the abelianization of $I_{g,x}$.

• First observation: $Sp_{2g}(\mathbb{Z})$ acts on $I_{g,x}$ by conjugation, and on $H_1 \Sigma_g$, hence on $\text{Hom}(H, \mathbb{Z}_r(F))$ for any r . One can easily check that $\tau_n(g \times g^{-1}) = g \cdot \tau_n(x)$ for all n , so that the image of τ_n is an $Sp(2g, \mathbb{Z})$ -submodule. This adds a vital amount of structure to the problem: once we have one vector in the image, we have its entire orbit.

• Second observation: $\Lambda^3 H$ can be embedded in $\text{Hom}(H, \Lambda^2 H)$.

How? Can put $\Lambda^3 H$ in $H \otimes \Lambda^2 H$ by
 $abc \mapsto a \otimes bc + b \otimes ca + c \otimes ab$.

Now $\text{Hom}(H, \Lambda^2 H) \cong H^* \otimes \Lambda^2 H$, and $H^* \cong H$ by way of sympl. pairing.

Prop: $\text{Im}(\tau_2) = \Lambda^3 H$.

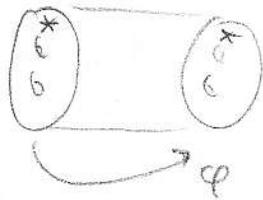
Pf (Have to complete some BP's, use the sympl. structure).

Question: Why $\Lambda^3 H$?

Answer: There is a alt. definition of τ that makes this obvious.

- Alternate construction: Poincaré Products and the LCS.

Take $\varphi \in \mathcal{I}_g^*$, and build the surface bundle M_φ .

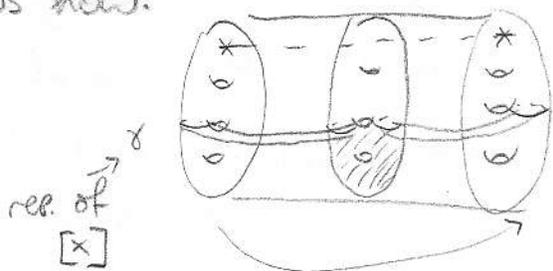


Generally, for any $\varphi \in \text{Mod}(\Sigma_g, *)$ given a section (i.e. choice of marked pt on each fiber) there is an embedding

$$H_1 \Sigma_g^{\varphi} \longrightarrow H_2 M_\varphi$$

invariant homol. classes

Here's how:



$\varphi(x)$. Necessarily in the same homol. class.

Can find embedded surface in $S_{g, \varphi}$

$\Sigma_g \times I$ with $\partial S_{g, \varphi} = \gamma \times \{0\} \cup \varphi(\gamma) \times \{1\}$.

Moreover, this misses the marked pt.

Ply this in; connect with a tube. This is the class $\Sigma_{g, \varphi} \in H_2 M_\varphi$.

For any 3-mfd, there is the cap product $\Lambda^3 H^1 M \rightarrow \mathbb{Z}$.

Dually, represent classes in H_2 by embedded submflds, and then take intersection form: $\Lambda^3 H_2 M \rightarrow \mathbb{Z}$.

Def τ can be described by

$\tau: \mathcal{I}_g^* \rightarrow \text{Hom}(\Lambda^3 H_1 \Sigma_g, \mathbb{Z})$ by sending φ to the intersection form restricted to $\Lambda^3 H_1 \Sigma_g \subseteq \Lambda^3 H_2 M_\varphi$.

This mirrors the fact that the target is $\Lambda^3 H$ obvious.