Plane curve singularities and mapping class groups

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Plane curve singularities

Topologists tend to like manifolds. But
singular spaces have very rich topology too!

An arena for algebraic geometry and topology to interact

Plane curve singularity: $f : \mathbb{C}^2 \to \mathbb{C}$
with $f_x = f_y = 0$ at origin.

e.g. $f(x, y) = x^3 + y^4$

Now study the family of *smooth* level surfaces $f^{-1}(t)$ (Milnor fibers)

Versal deformation spaces

Basic theme: *deform* singularities to simpler ones

There exists a *finite-dimensional* "universal" deformation space $V_f \cong \mathbb{C}^\mu$

For $f(x, y) = x^3 + y^4$:

$$V_f = \{f_\lambda(x, y) = x^3 + y^4 + \lambda_1 xy^2 + \lambda_2 y^2 + \lambda_3 xy + \lambda_4 y + \lambda_5 x + \lambda_6\}$$

For "bad" choices of λ, $f_\lambda^{-1}(0)$ is singular:

$$Disc_f = \{\lambda \in \mathbb{C}^\mu \mid f_\lambda^{-1}(0) \text{ singular}\}$$

Versal discriminant complement $V_f \backslash Disc_f$ supports the family of surfaces $\lambda \mapsto f_\lambda^{-1}(0)$

Topology of versal discriminant complements

$Disc_f$ is a very singular hypersurface

Deligne: For "ADE" singularities, $V_f \backslash Disc_f$ is a $K(\pi,1)$ for the associated Artin group

Conjecture (Arnol'd-Thom): For an *arbitrary* plane curve singularity, $V_f \backslash Disc_f$ should be a $K(\pi,1)$

But what is π?

Lönne: computation of π for *Brieskorn-Pham* singularities

Artin group plus extra "triangle relations"

In general, we have no idea!

Monodromy

One approach: find a quotient

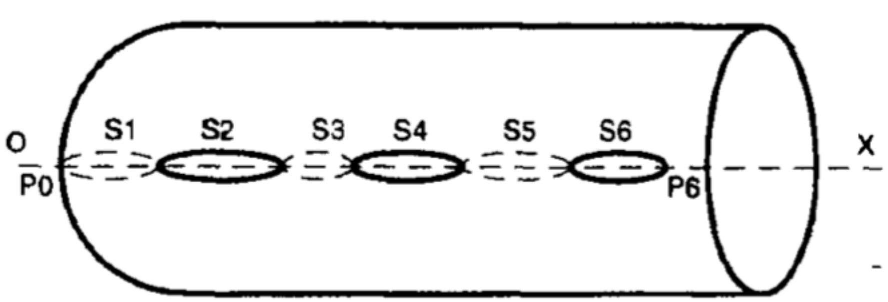
Family of surfaces: *monodromy* homomorphism

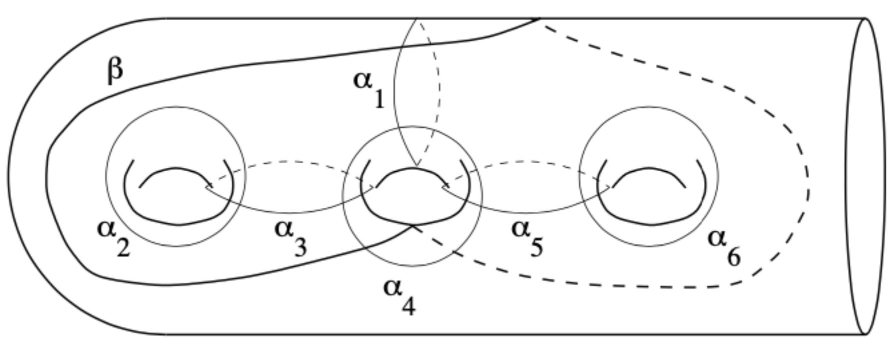
$$\rho_f : \pi_1(V_f \setminus Disc_f) \to \mathrm{Mod}(\Sigma)$$

Question (Sullivan, 1970's): For which singularities is ρ_f injective?

Perron-Vannier: Injective for type A,D. (Type A: braid group)

Wajnryb: Not injective for E_6, E_7, E_8





Nothing further known since 1995!

Monodromy groups of singularities

Theorem (Portilla Cuadrado — S.):

For any isolated plane curve singularity of genus g≥5 other than types A/D, the monodromy group is a *framed mapping class group.*

We answer Sullivan's question as a corollary.

Corollary:

For *any* isolated plane curve singularity of genus g≥7 other than types A/D, the monodromy homomorphism is not injective.

Proof:

Easy to show $H^1(V_f \backslash Disc_f; \mathbb{Q}) \neq 0$. But Randal-Williams showed that framed mapping class groups have $b^1_{\mathbb{Q}} = 0$ for g≥7.

Framed mapping class groups

$\mathrm{Mod}(\Sigma)$ acts on set of isotopy classes of framings

$\mathrm{Mod}(\Sigma)[\phi]$: stabilizer of ϕ

Nowhere-vanishing 1-form—> invariant framing ϕ

The form $\dfrac{dx}{(f_\lambda)_y}$ is holomorphic and nonvanishing on each fiber $f_\lambda^{-1}(0)$.

Conclude: monodromy *contained* in $\mathrm{Mod}(\Sigma)[\phi]$

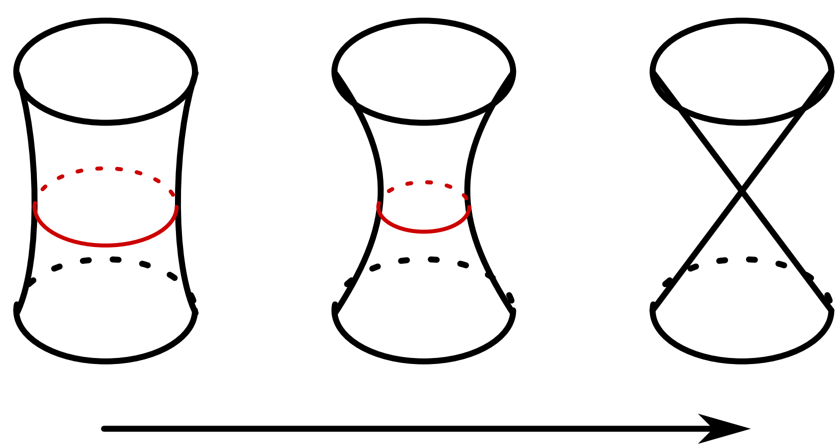
Philosophical point: Milnor fibers are translation surfaces!
There must be a rich story here, but no one has told it yet.

Idea of proof

Need to show that ρ_f *surjects* onto $\mathrm{Mod}(\Sigma)[\phi]$.

Work of Calderon—S. provides technology for doing this: we find criteria under which a set of Dehn twists generate a framed mapping class group.

Dehn twists arise in monodromy via *vanishing cycles.*



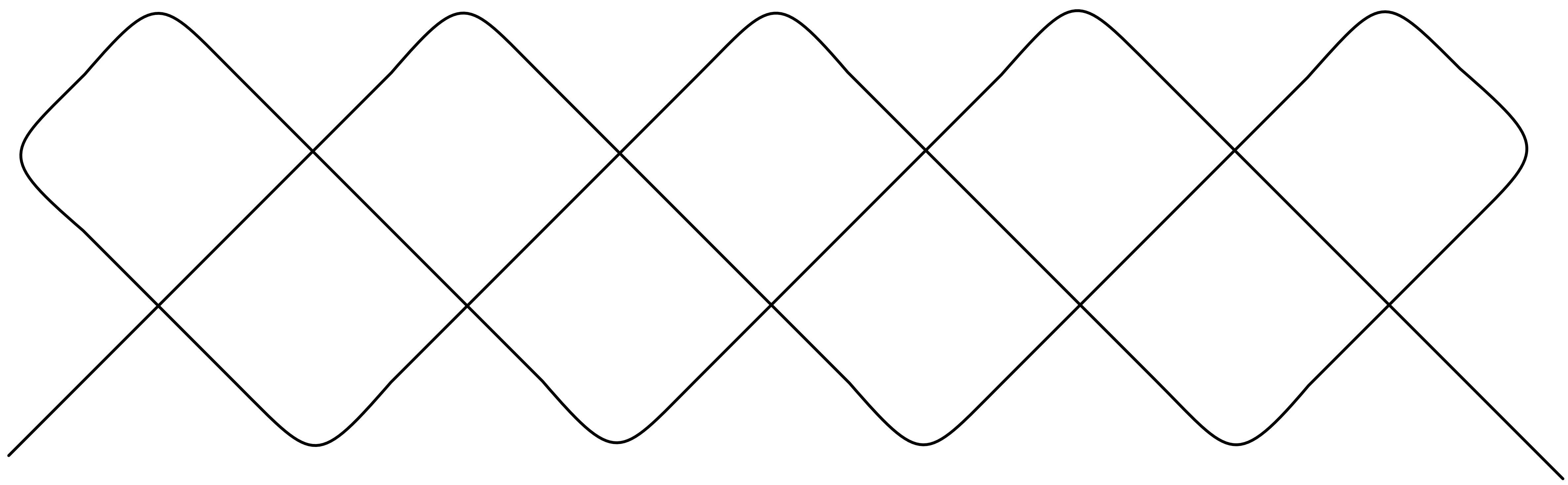
Theory of *divides* gives a picture of Milnor fiber endowed with a distinguished finite set of vanishing cycles.

Ultimate idea: show that every singularity has a divide that satisfies the hypotheses for CS.

Example

$$f(X, Y) = X^3 + Y^{10}$$

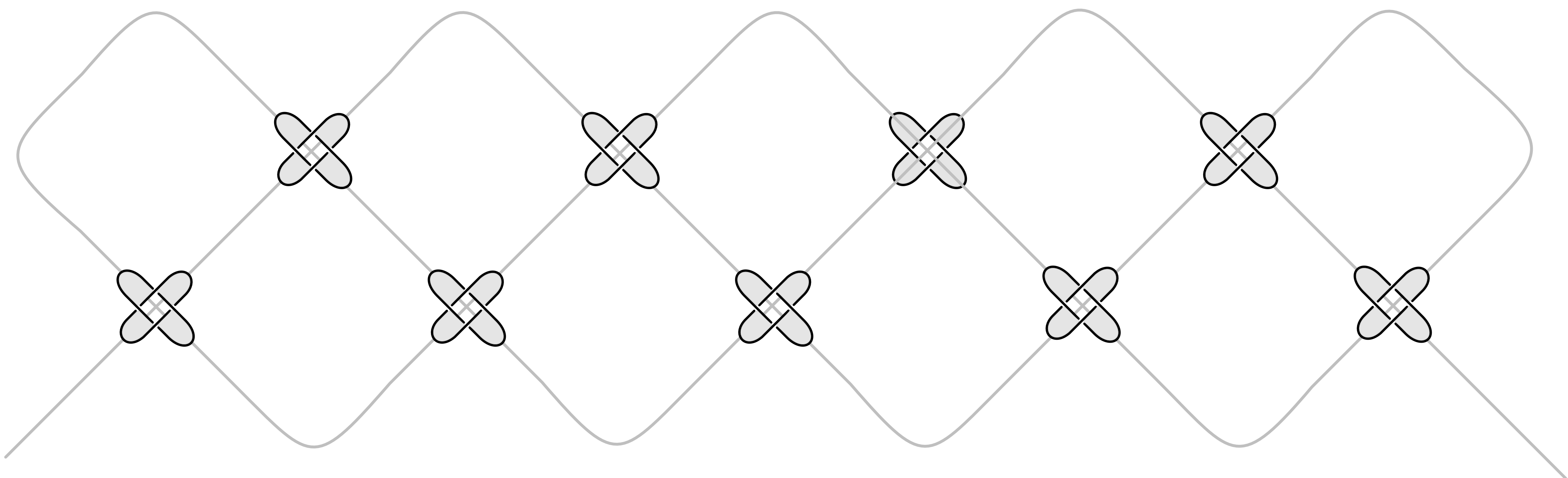
Step 1: Draw real points of a suitable perturbation



Example

$$f(X, Y) = X^3 + Y^{10}$$

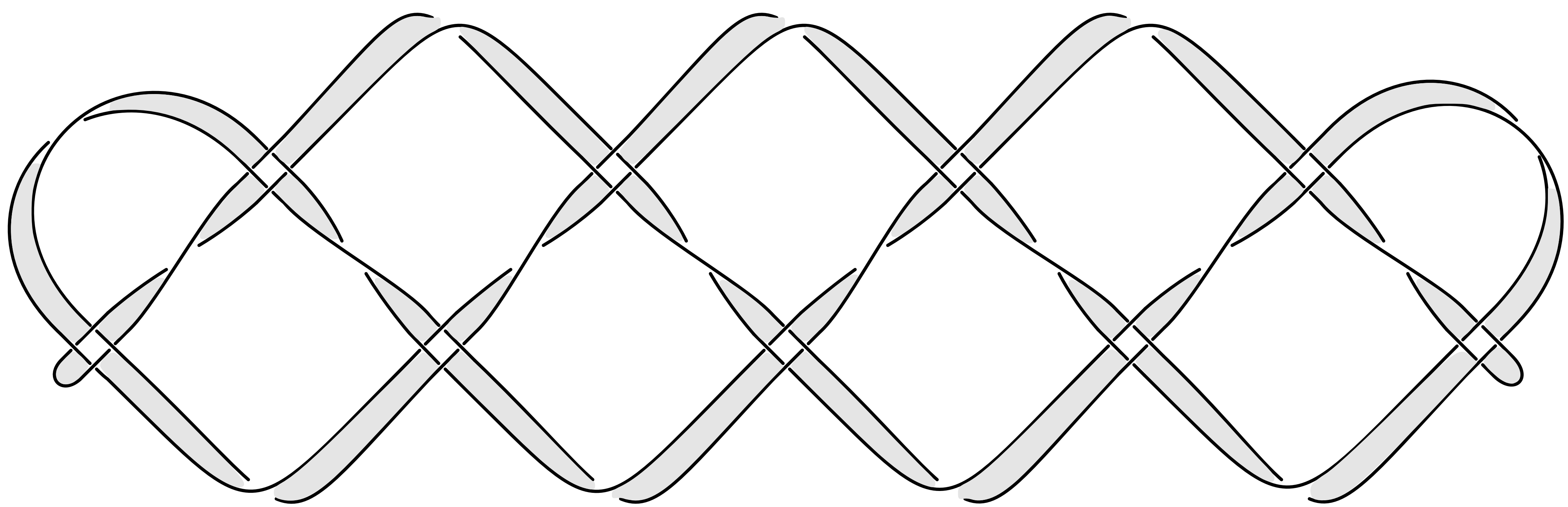
Step 2: Convert crossings to knotted annuli



Example

$$f(X, Y) = X^3 + Y^{10}$$

Step 3: Convert arcs to twisted bands

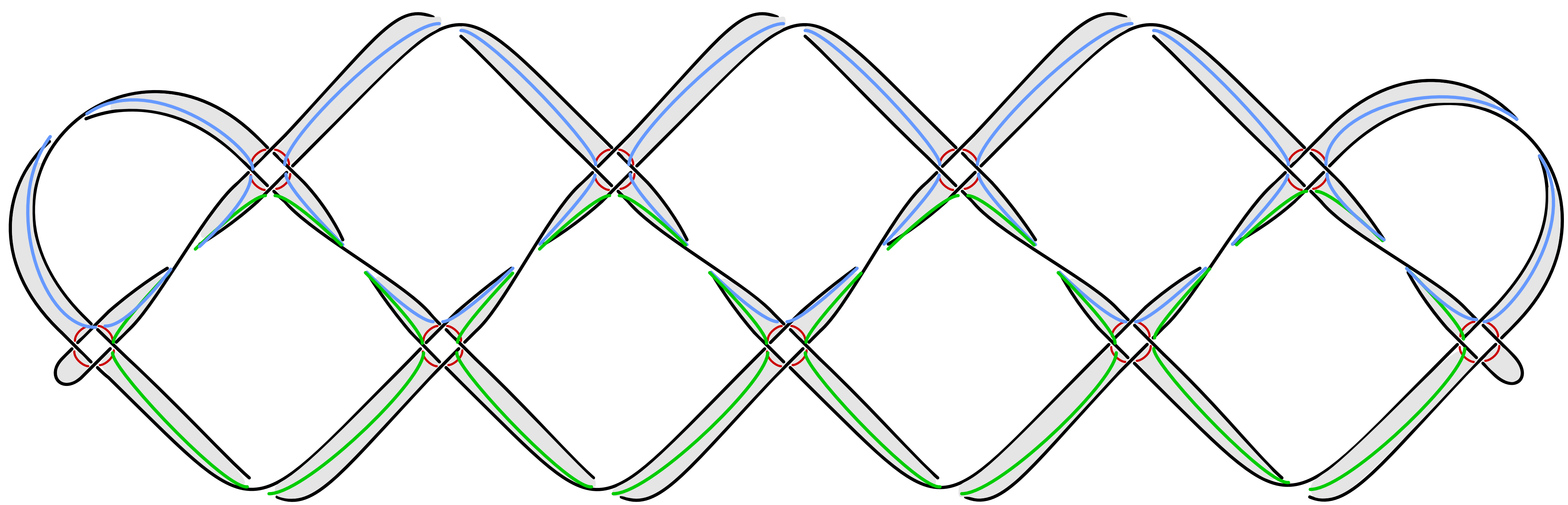


Example

$$f(X, Y) = X^3 + Y^{10}$$

Step 4: Identify vanishing cycles

Two kinds: associated to crossings, and to regions

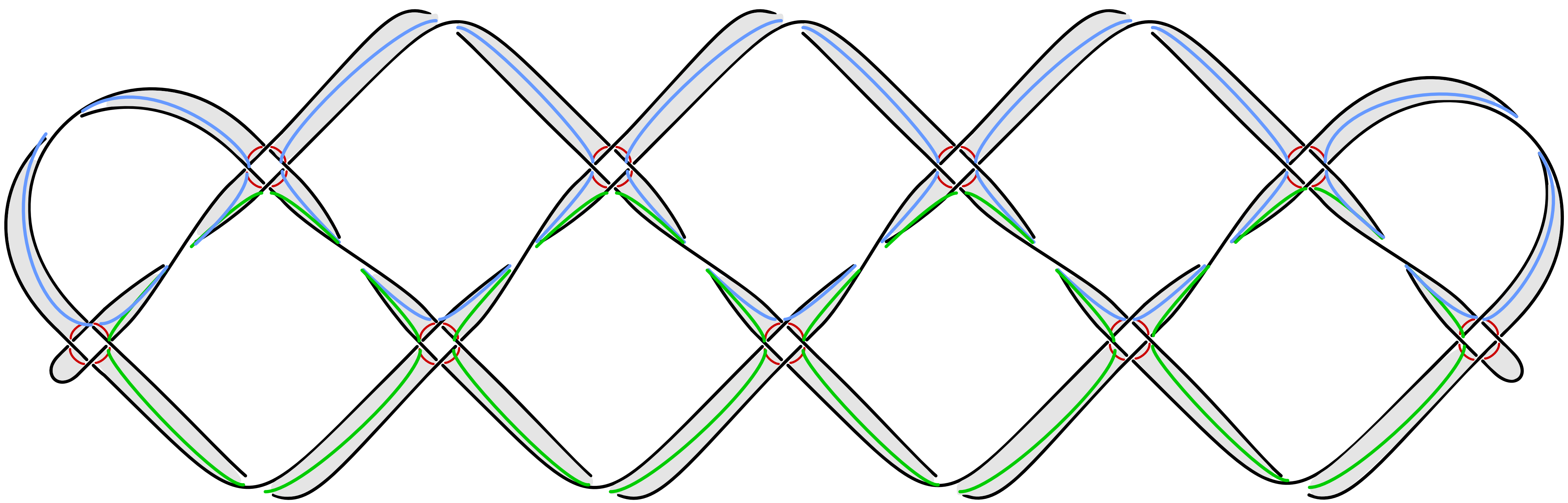


Example

$$f(X, Y) = X^3 + Y^{10}$$

Step 5: Quote Calderon-S:

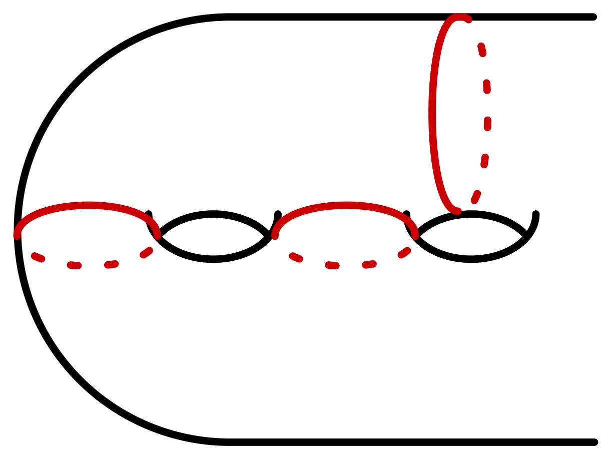
The 18 Dehn twists depicted below generate a framed mapping class group on the surface Σ_9^1



More on vanishing cycles

As a further corollary, we can answer the following question:

Which curves can be realized as vanishing cycles?



?

Corollary (PC-S): A nonseparating simple closed curve is a vanishing cycle if and only if it has zero winding number w.r.t. the canonical framing.

E.g. can't be satisfied for all three curves above (Poincaré-Hopf).