On the non-realizability of braid groups by diffeomorphisms

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Basic setup

X - topological space.

$\mathrm{Diff}(X)$: group of C^k diffeomorphisms of X (rel. boundary)

($S \subset X$ subset: $\mathrm{Diff}(X,S)$ diffeos rel. S)

$\mathrm{Diff}_0(X,S)$: diffeos isotopic to id. rel. S

We study $\mathrm{Mod}(X,S) = \mathrm{Diff}(X,S)/\mathrm{Diff}_0(X,S)$

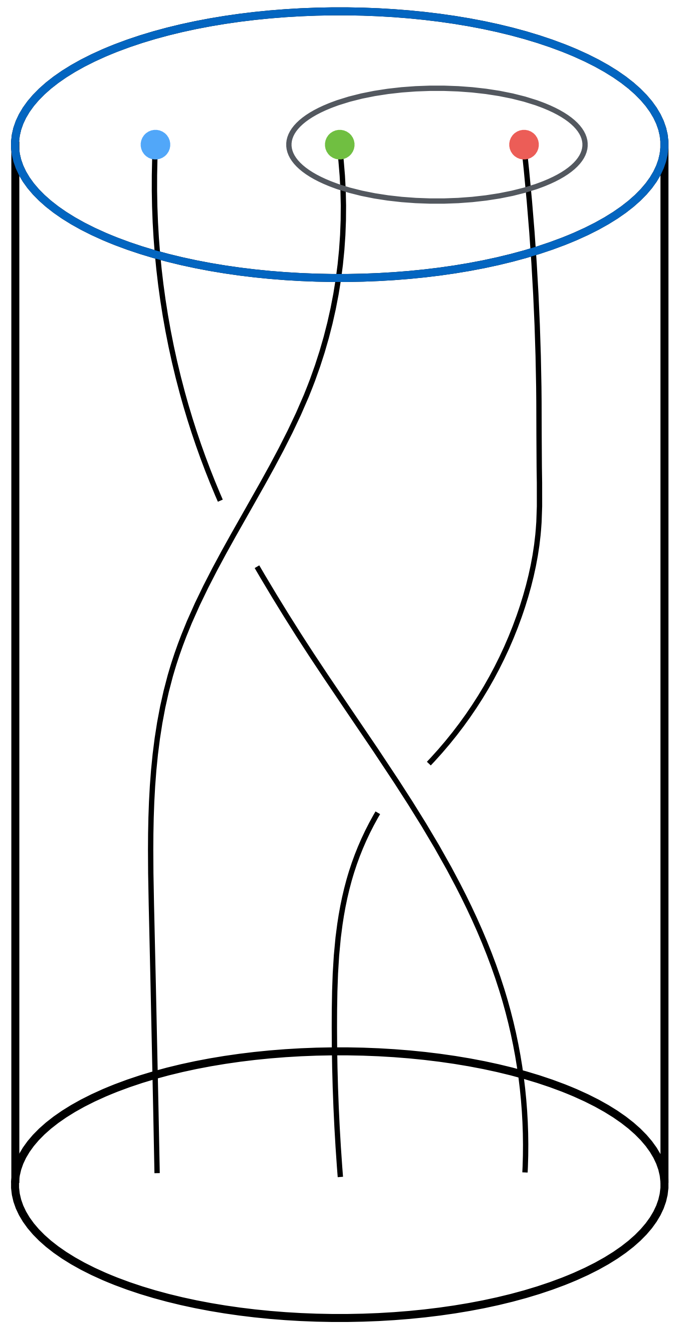
Key examples

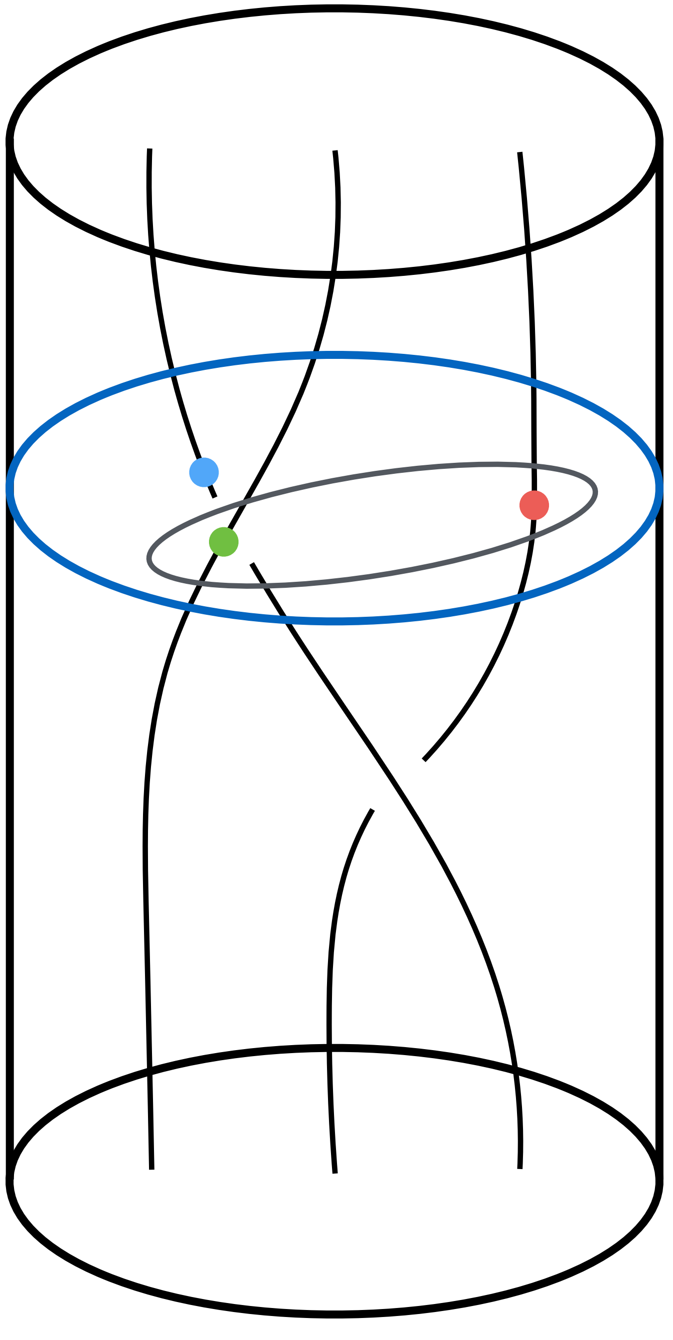
Example: $X = \Sigma_g$

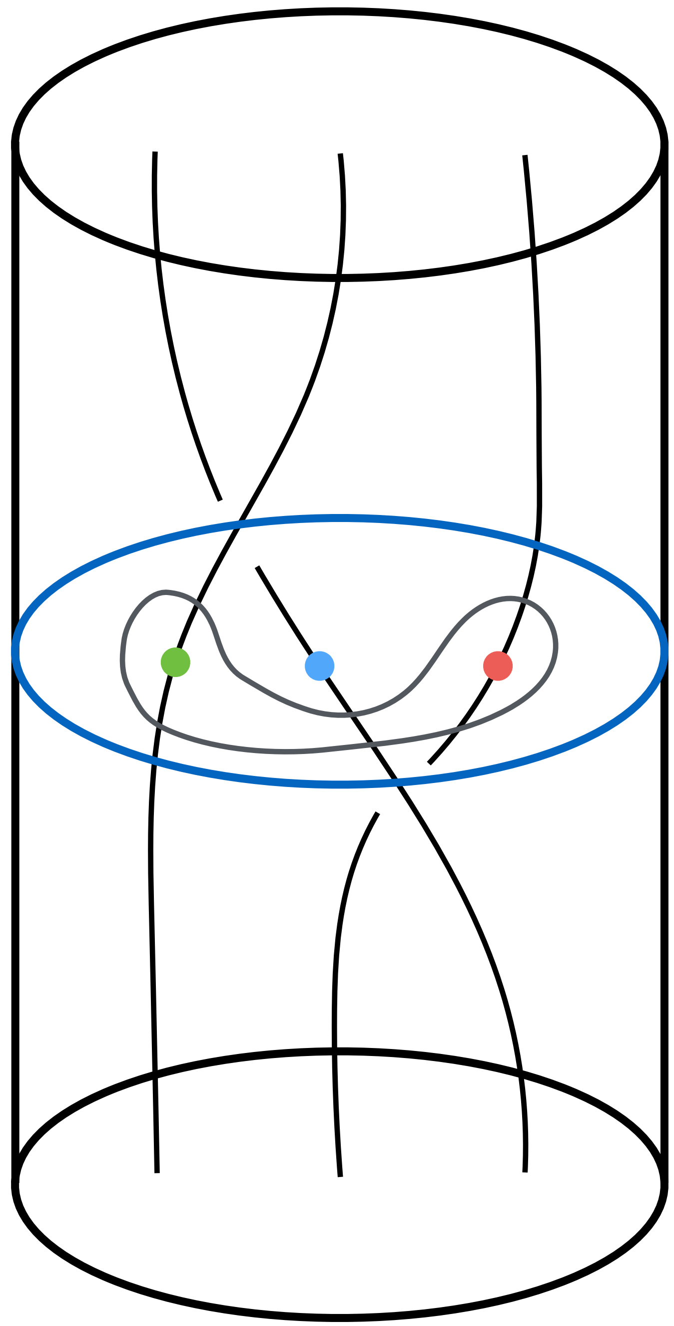
$\mathrm{Mod}(\Sigma_g)$: mapping class group of surface

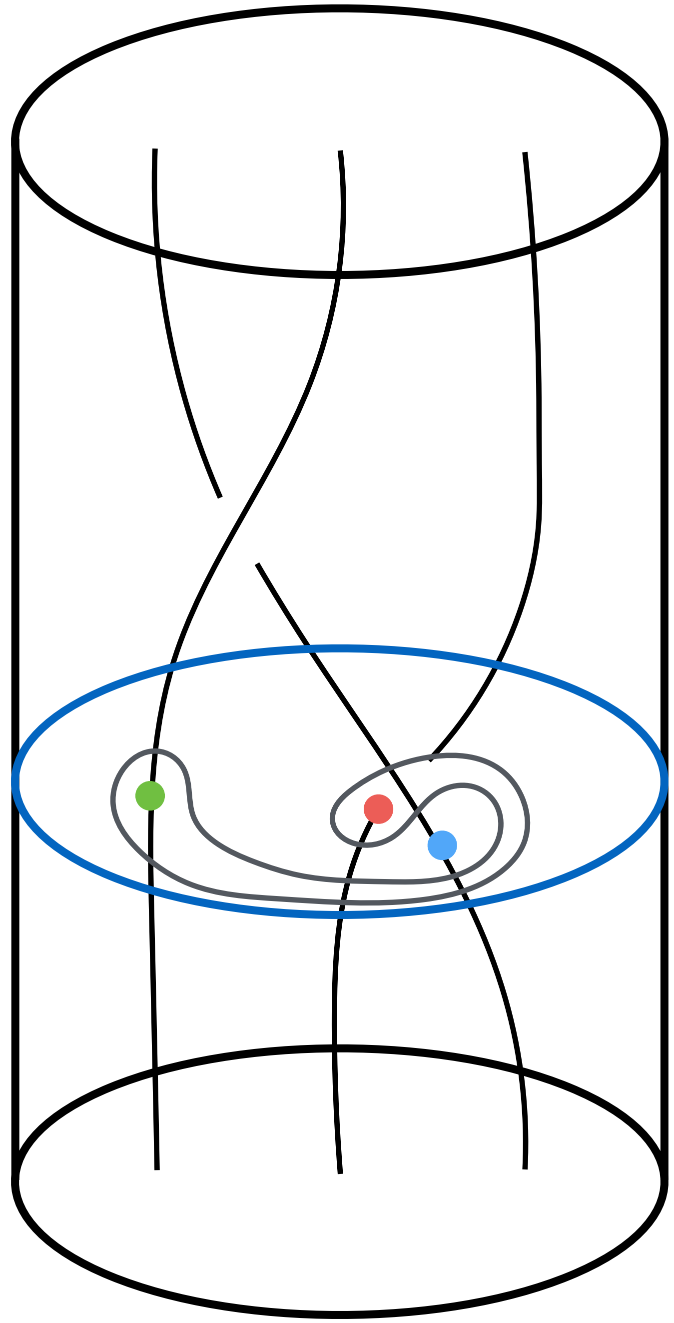
Main concern today: $X = D^2$, $S = \{n \text{ points}\}$

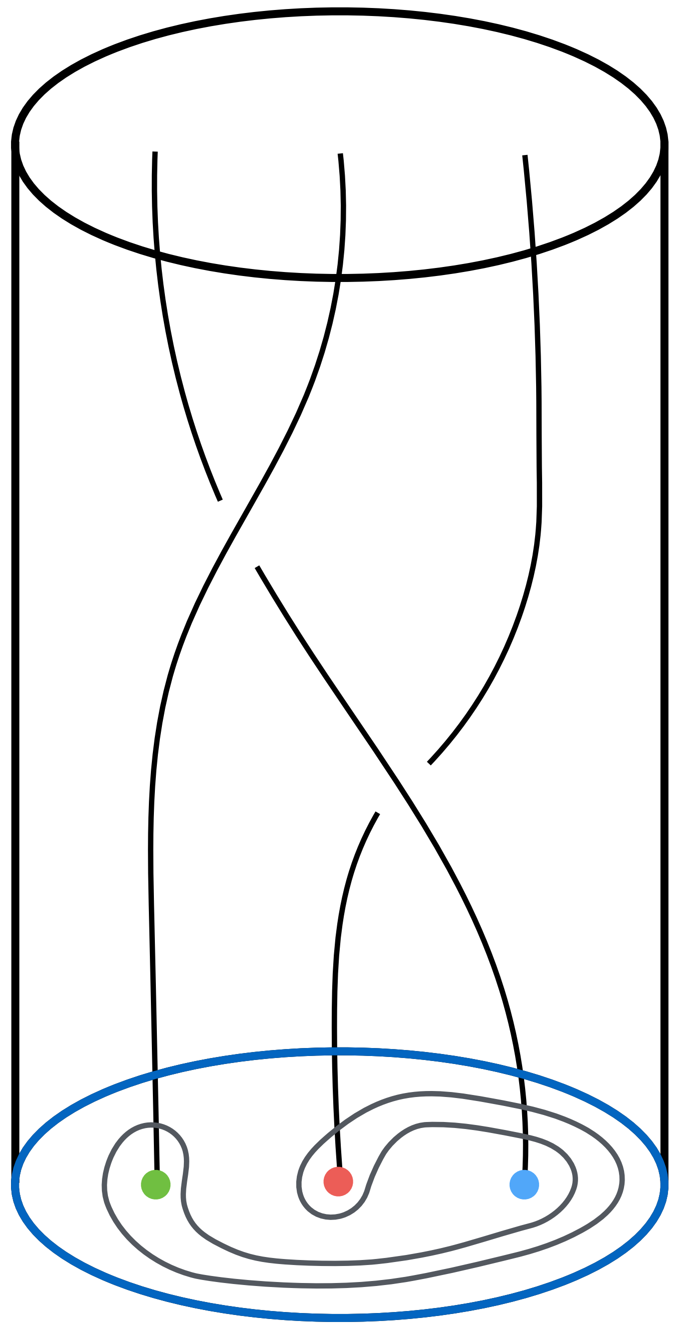
$\mathrm{Mod}(X,S) = B_n$ (braid group on n strands)











The basic question

Does the map $\pi : \mathrm{Diff}(X, S) \to \mathrm{Mod}(X, S)$

admit a *section*, i.e. a map

$$\sigma : \mathrm{Mod}(X, S) \to \mathrm{Diff}(X, S)$$

for which $\pi \circ \sigma = \mathrm{id}$?

We say that $\mathrm{Mod}(X,S)$ is *(not) realizable by diffeomorphisms* if the answer is yes (resp. no).

Morita's non-lifting theorem

The case of $\mathrm{Mod}(\Sigma_g)$ is well-understood.

Theorem (S. Morita, 1987):

$Mod(\Sigma_g)$ is not realizable by C^2 diffeomorphisms, for all $g \geq 18$

The idea

Use cohomology.

Existence of section implies a diagram

$$H^*(\mathrm{Mod}(\Sigma_g)) \xrightarrow{\pi^*} H^*(\mathrm{Diff}(\Sigma_g)) \xrightarrow{\sigma^*} H^*(\mathrm{Mod}(\Sigma_g))$$

for which $\sigma^* \circ \pi^* = 1$

In particular, π^* is an injection.

The contradiction

Morita constructs an element $e_3 \in H^6(\mathrm{Mod}(\Sigma_g))$

He applies the *Bott vanishing theorem*
from foliation theory to show that $\pi^*(e_3) = 0$

Today's question

Is B_n realizable by diffeomorphisms?

Following Morita, would like to understand

$$\ker \pi^* : H^*(B_n) \to H^*(\mathrm{Diff}(D^2, S))$$

Problem!

Theorem (Nariman, 2015):
$\pi^* : H^*(B_n) \to H^*(\mathrm{Diff}(D^2, S))$ is an injection.

Alternative approaches

Many proofs of Morita's theorem by now.

Bestvina-Church-Souto:
Milnor-Wood inequality.

Franks-Handel:
Use dynamics to produce fixed points.
Use these fixed points to construct
homomorphisms that can't exist.

Main Theorem (I)

This last approach inspires our method.

Theorem (S., Tshishiku, 2015):

For $n \geq 5$ the braid group is not realizable by diffeomorphisms. More generally, the "surface braid group" $B_n(\Sigma_g^b)$ is not realizable by diffeomorphisms for $n \geq 5$ if $b \geq 1$, and for $n \geq 6$ otherwise.

Main Theorem (II)

By exploiting the hyperelliptic mapping class group, we obtain a new and quite simple proof of Morita's theorem in the best possible range:

Theorem (S., Tshishiku, 2015):

For all $g \geq 2$ *, the mapping class group* $\mathrm{Mod}(\Sigma_g)$ *is not realizable by* C^1 *diffeomorphisms.*

Outline of proof (I)

Diffeos realizing elements of B_n must fix points in S.

Exploit this to manufacture homomorphisms
$f : B_n \to A$ with A abelian.

Use dynamics/geometry to show these are
highly non-degenerate.

Exhibit subgroups G of B with $H^1(G, \mathbb{Z}) = 0$

Steps 2 and 3: constructing homomorphisms

Derivative map: $D_x : B_n \to GL_2^+(\mathbb{R})$

Analyze centralizers: image must be abelian.

If trivial, use *Thurston stability:*

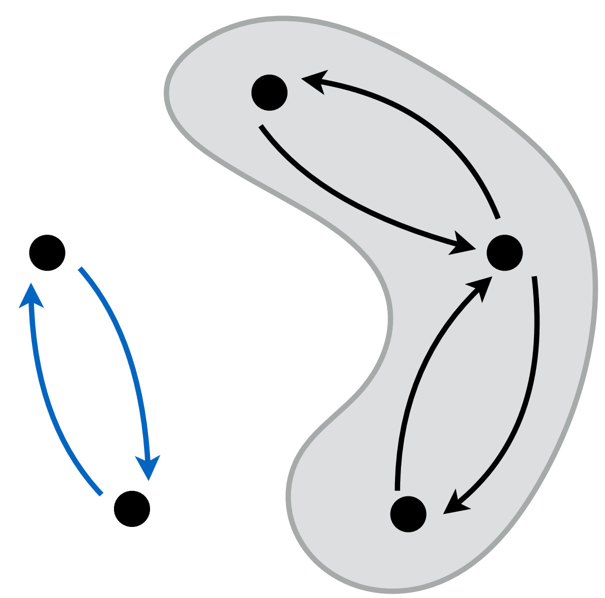
Theorem (Thurston, 1974):

Let G be f.g. acting via C^1 diffeomorphisms on $\mathbb{R}^n$ with a global fixed point x. If $D_x : G \to GL_n(\mathbb{R})$ is trivial, there is a map $f : G \to \mathbb{R}$ with nontrivial image.

Step 4: "bad" braid subgroups

Fact: For $n \geq 5$, the commutator subgroup $[B_n, B_n]$ is finitely generated, and $H^1([B_n, B_n], \mathbb{Z}) = 1$

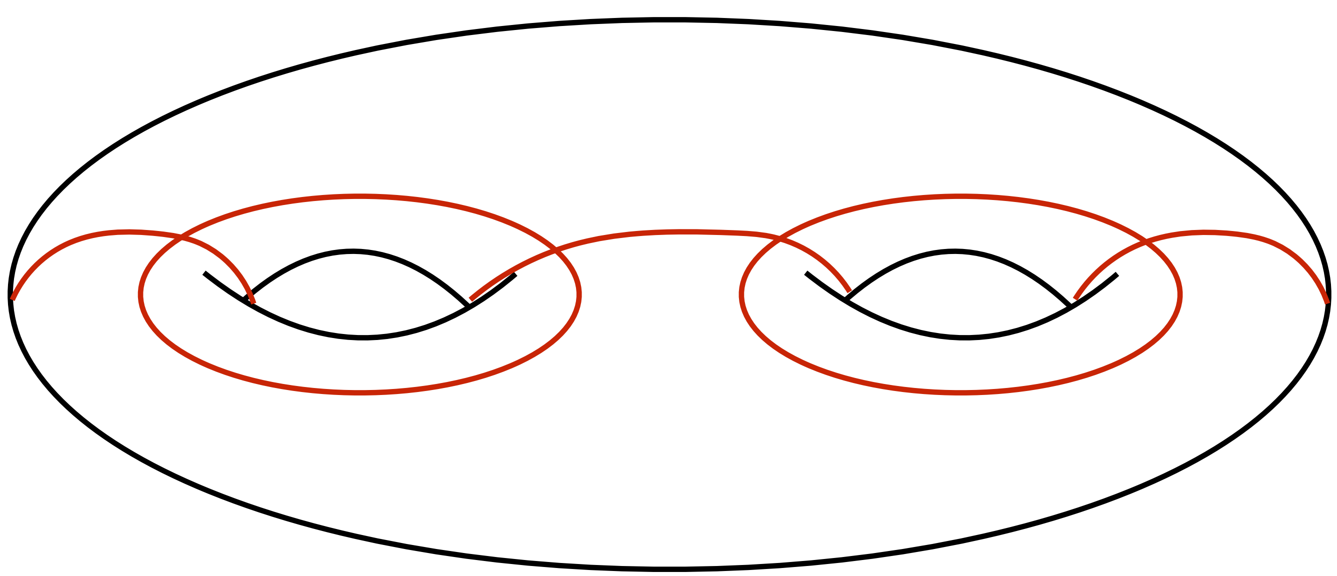
Proof: Just do it! Vaguely reminiscent of other arguments in the theory of diffeomorphism groups.



From braids to surfaces

To prove Theorem 2, exploit the map

$$f : B_{2g+2} \to \mathrm{Mod}(\Sigma_g)$$

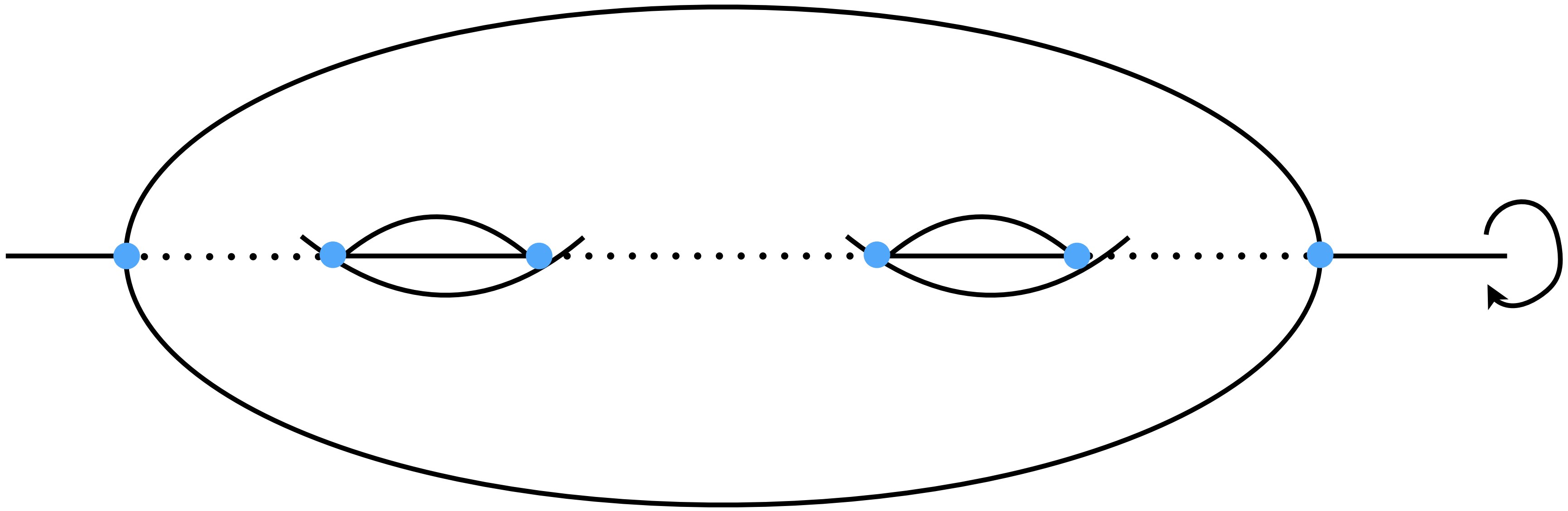


Outline of proof (II)

To use previous methods: need fixed points

im(f) commutes with hyperelliptic involution ι

$\sigma(\iota)$ has $2g+2$ fixed points (Lefschetz)



Outline (cont'd)

Basic principle of group actions:
im(f) preserves these points.

Obtain map $\phi : B_{2g+2} \to S_{2g+2}$

We show this map has to be the standard one.

Thus we can find fixed points, proceed as before.

Questions

- Study other subgroups of mapping class groups. When is a general $G \leq \mathrm{Mod}(\Sigma_g)$ realizable by diffeomorphisms?
- In particular, are there surface subgroups of $\mathrm{Mod}(\Sigma_g)$ that are not realizable? 3-manifold subgroups?
- Are there dynamical approaches to other lifting problems, e.g. for finding a section of
$$\mathrm{Mod}(\Sigma_{g,*}) \to \mathrm{Mod}(\Sigma_g)$$

Thanks!