Stratified braid groups

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Stratified braid groups

$\mathrm{Conf}_n(\mathbb{C})$: space of unordered n-tuples $\{z_1, \ldots, z_n\} \subset \mathbb{C}$ of distinct points

Or, space of monic squarefree polynomials:

$$\{z_1, \ldots, z_n\} \leftrightarrow p(z) = (z - z_1)\ldots(z - z_n)$$

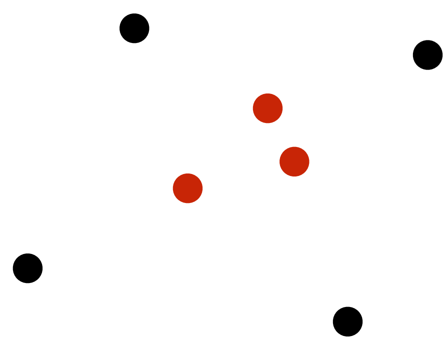
Object of interest today: a *stratification* on $\mathrm{Conf}_n(\mathbb{C})$.

Definition Fix a partition $\kappa = \{k_1, \ldots, k_p\}$ of n-1.

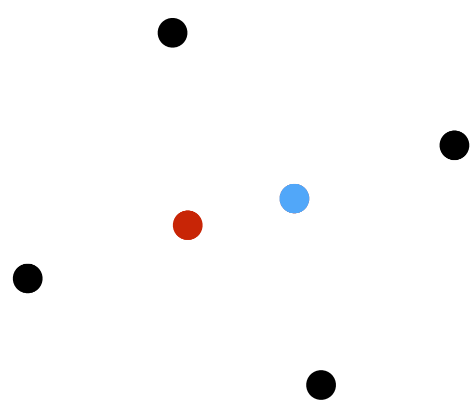
Stratum $\mathrm{Conf}_n(\mathbb{C})[\kappa]$: polynomials $p(z) \in \mathrm{Conf}_n(\mathbb{C})$, roots of $p'(z)$ have multiplicity κ.

Note: root of p' = critical point of p

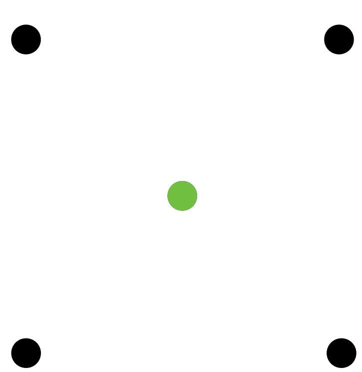
Stratified braid groups



$\mathrm{Conf}_4(\mathbb{C})[1^3]$



$\mathrm{Conf}_4(\mathbb{C})[2,1]$



$\mathrm{Conf}_4(\mathbb{C})[3]$

Why?

- A naturally-appearing structure on the space of polynomials

- Can be studied from many different points of view:

    - Singularity theory: discriminant complement
    - Algebraic geometry: Hurwitz spaces, meromorphic differentials
    - Geometry: translation surfaces
    - Topology: fundamental groups, $K(\pi,1)$ spaces

- Hope that this is simple enough to actually make progress!

Goal of the talk: explain a little bit of these points of view and how they interact

Main questions

(1) What are the fundamental groups

$$\mathscr{B}_n[\kappa] := \pi_1(\mathrm{Conf}_n(\mathbb{C})[\kappa])\ ?$$

One description: subquotients of B_n given in terms of Hurwitz spaces

(2) Inclusion $\mathrm{Conf}_n(\mathbb{C})[\kappa] \to \mathrm{Conf}_n(\mathbb{C})$ induces hom.

$$\rho : \mathscr{B}_n[\kappa] \to B_n$$

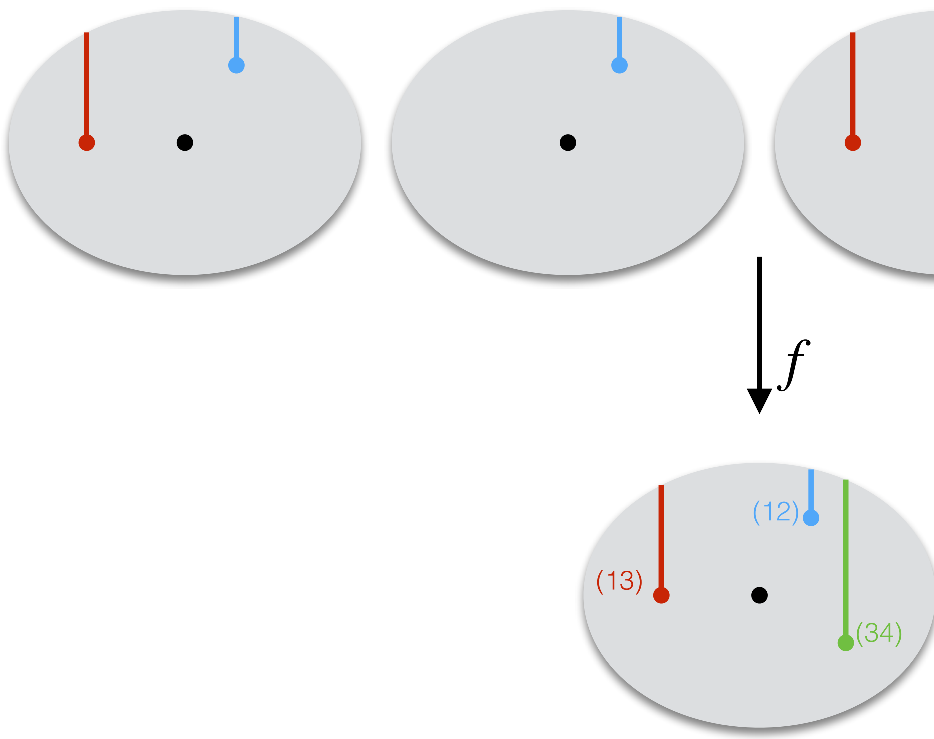
What is the image?

Will present a complete answer.

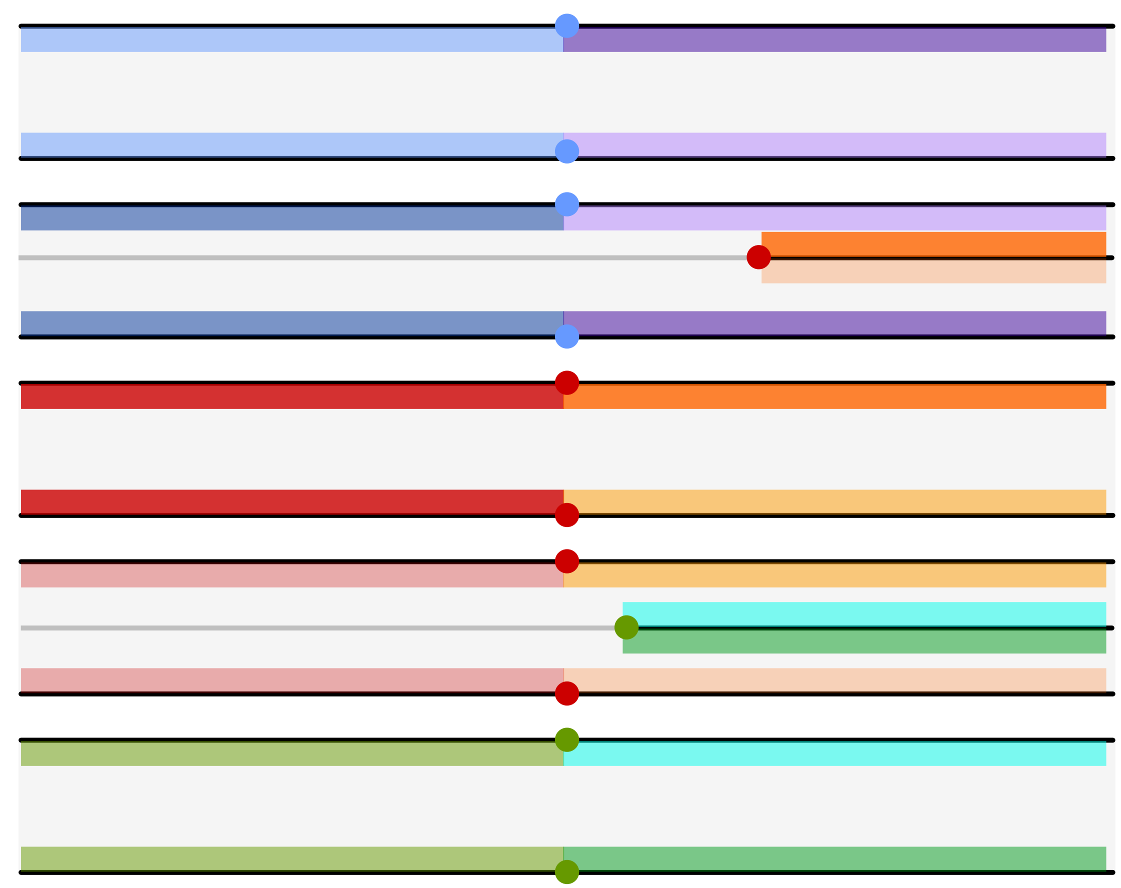
(3) Is each $\mathrm{Conf}_n(\mathbb{C})[\kappa]$ a $K(\pi,1)$?

🤷‍♂️

Several points of view



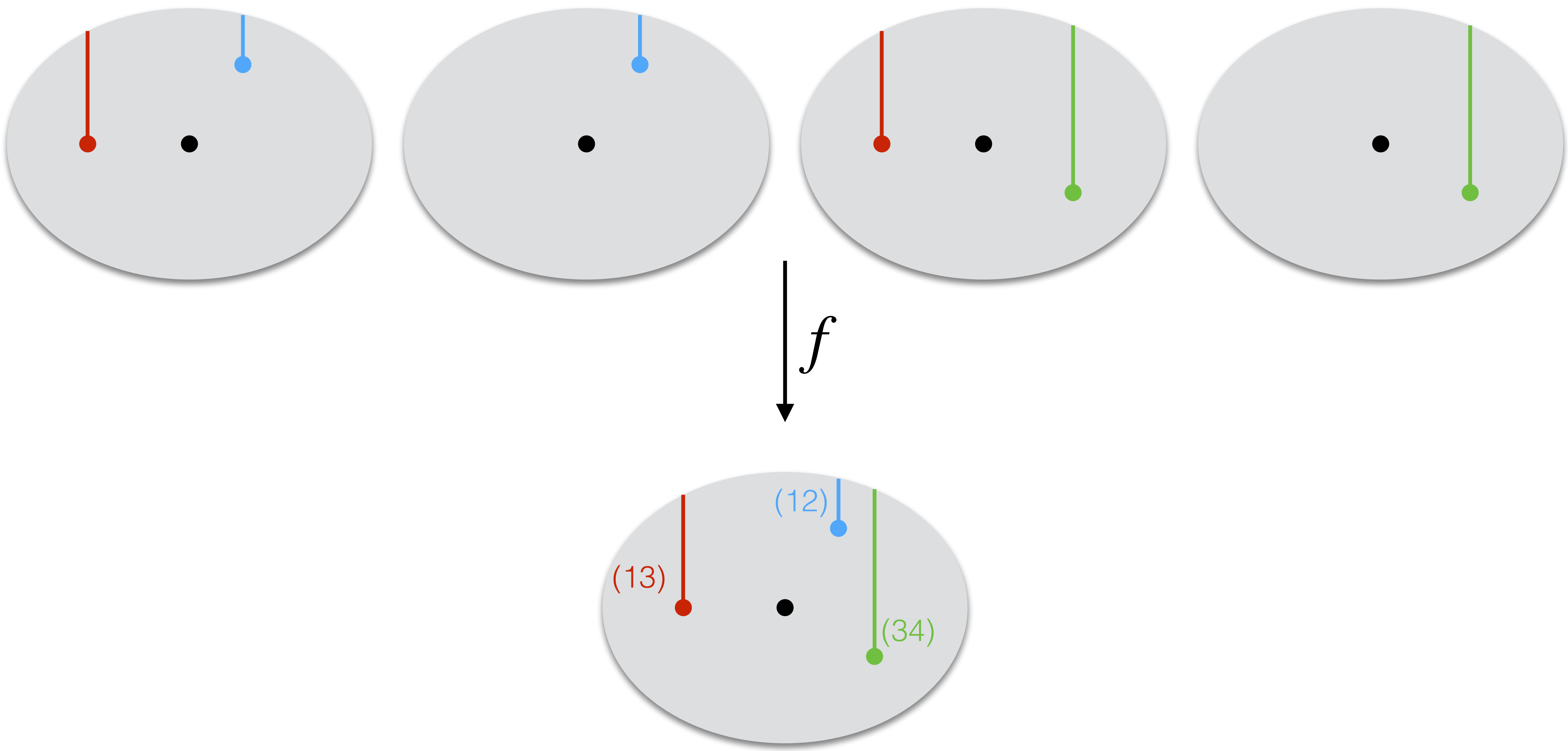
Remainder of the talk: explain how both of these give pictures of $\mathrm{Conf}_n(\mathbb{C})[\kappa]$, and comment on how they explain fundamental group, monodromy, and more.



POV 1: Hurwitz spaces

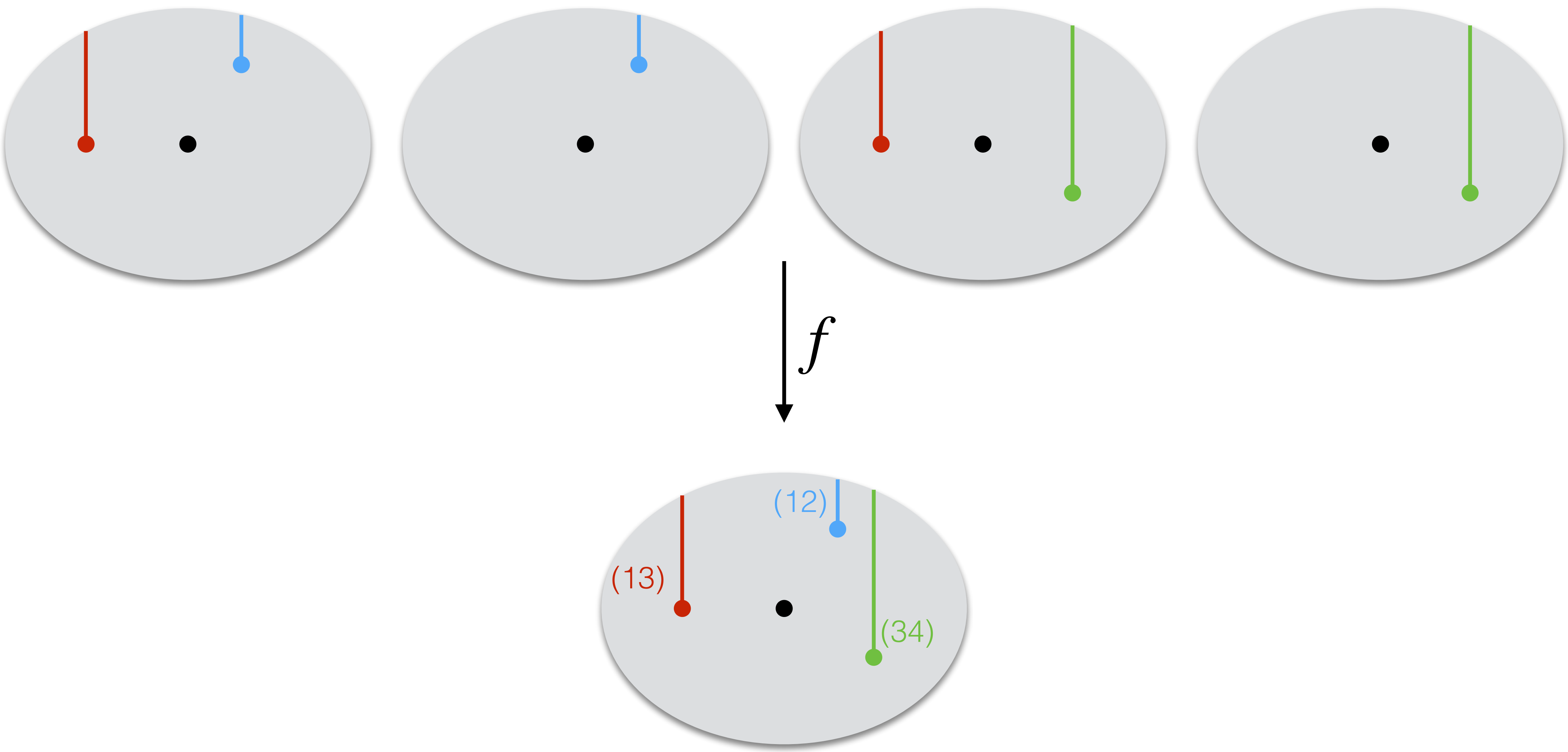
As before: $\kappa = \{k_1, \ldots, k_p\}$ partition of n-1.

$\mathbf{Hur}(\kappa)$: space of n-sheeted branched covers $f : \mathbb{C} \to \mathbb{C}$ with p *cyclic* branched points of orders $k_1 + 1, \ldots, k_p + 1$.



POV 1: Hurwitz spaces

$\mathrm{Hur}(\kappa)^{\circ} \subset \mathrm{Hur}(\kappa)$: $0 \in \mathbb{C}$ is a regular value.



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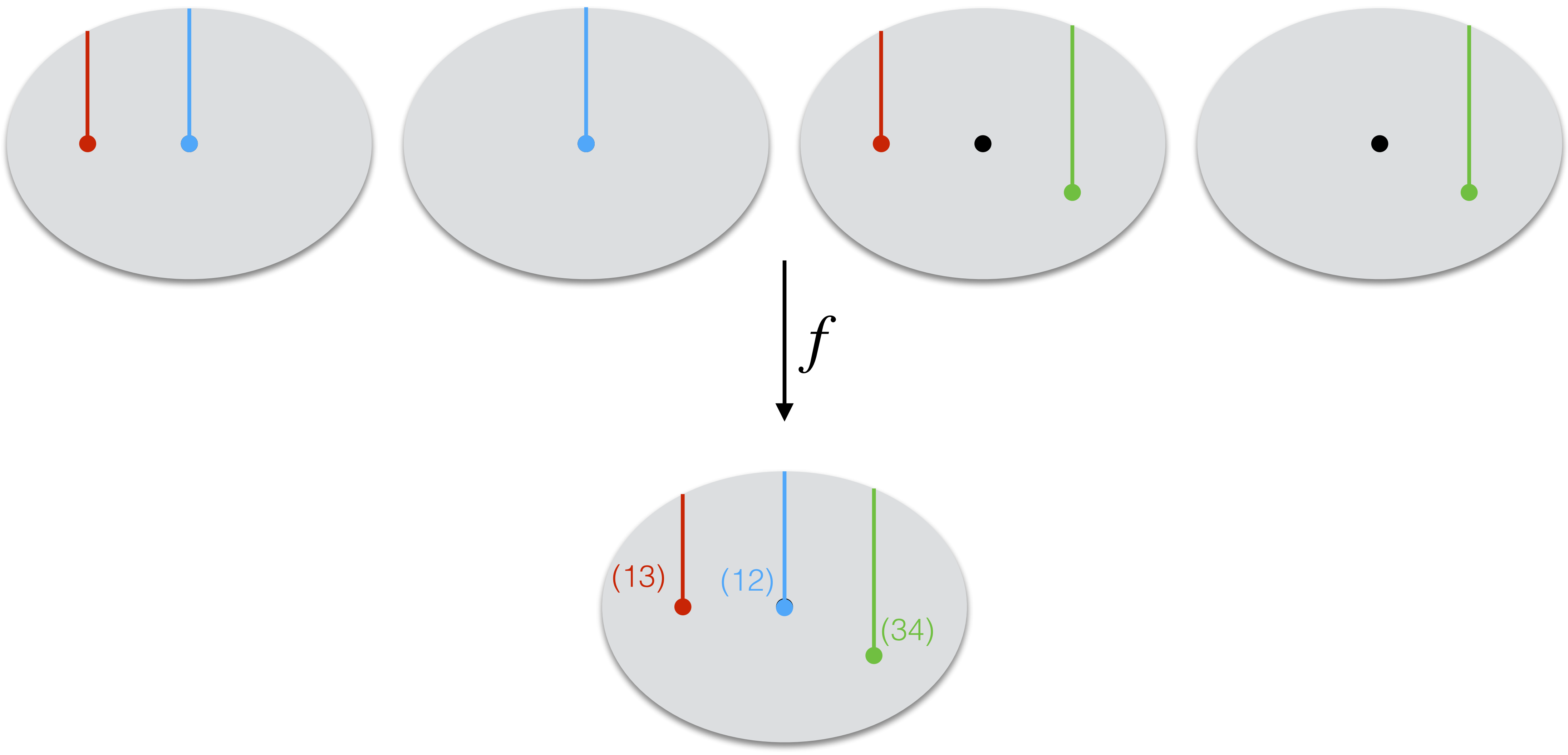

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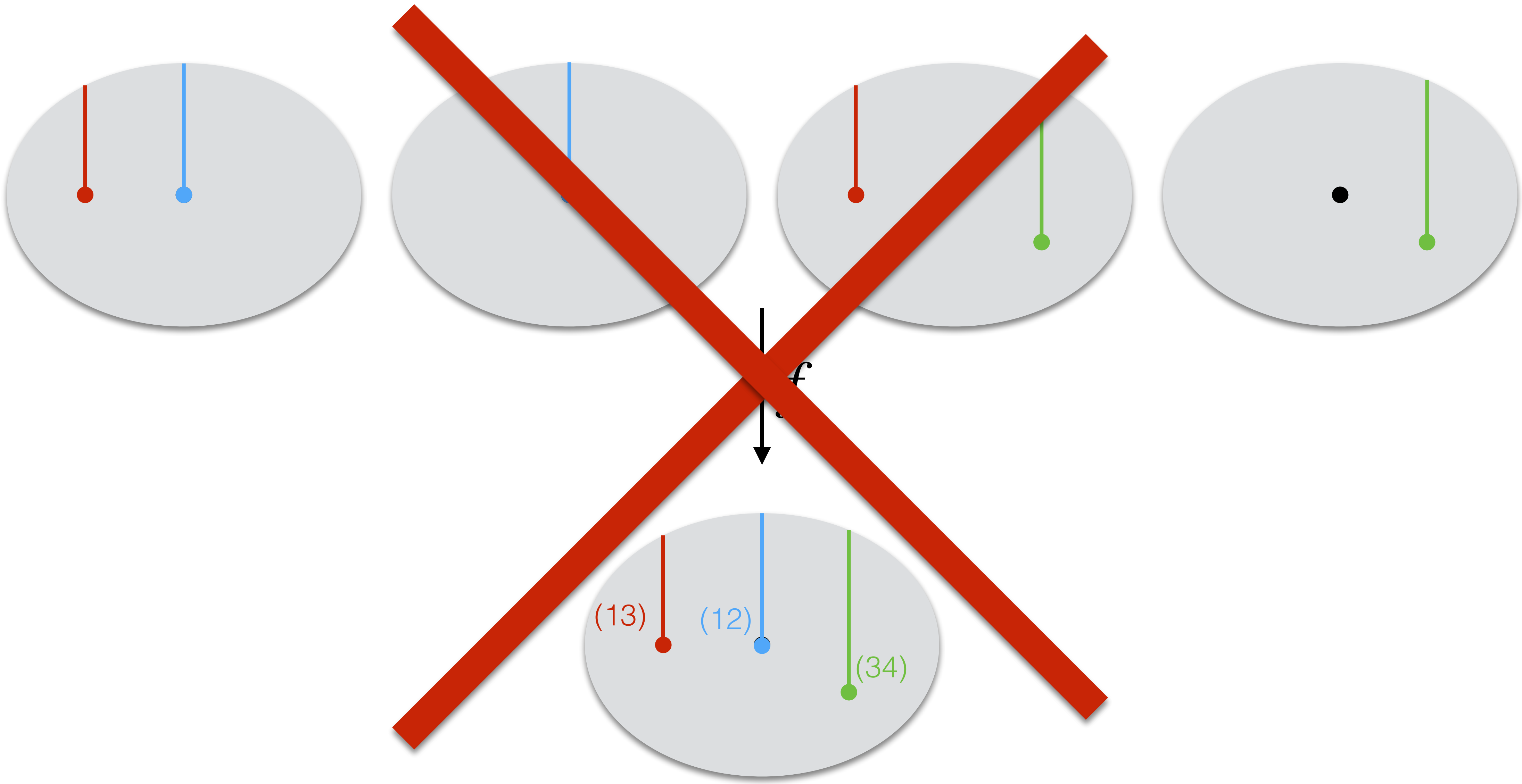
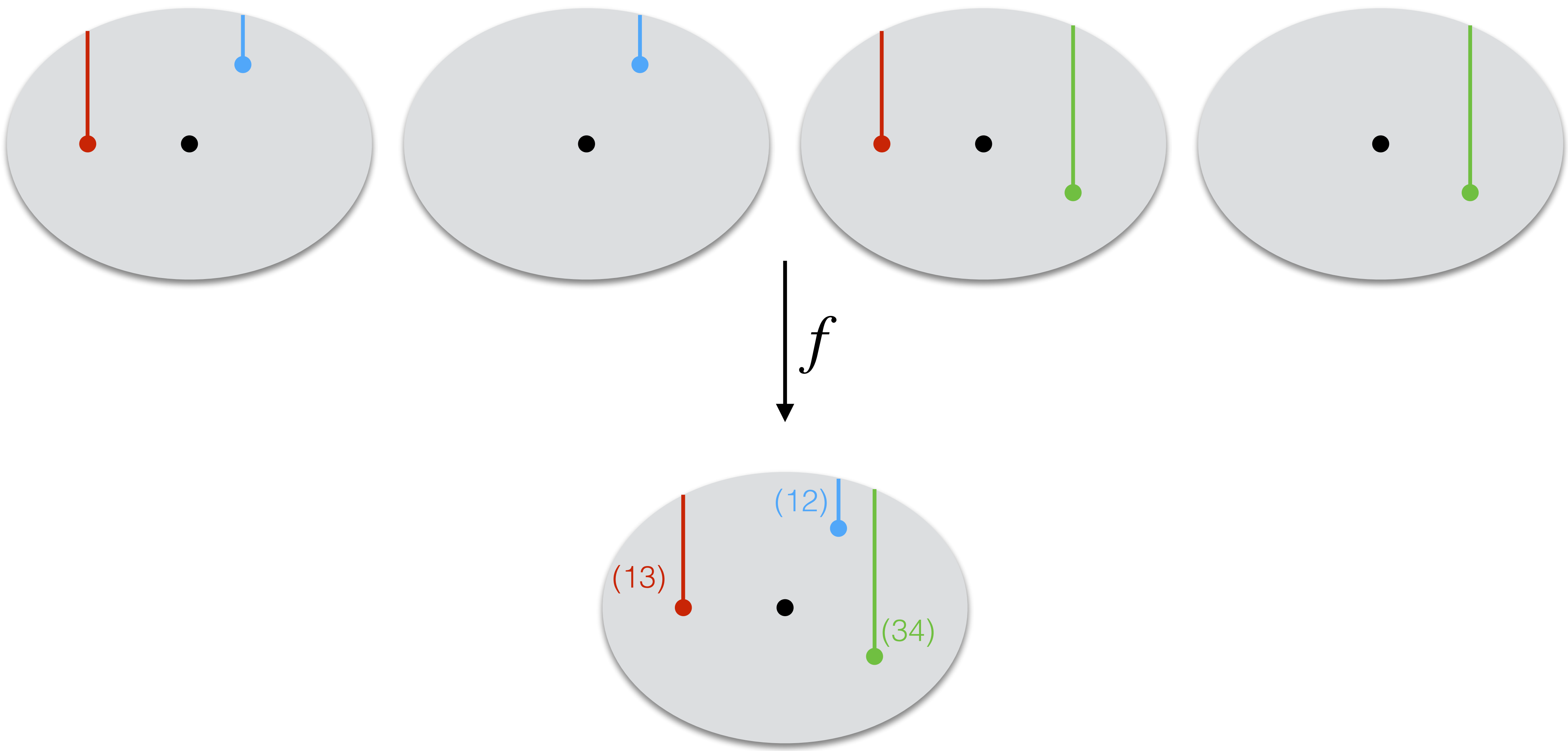


POV 1: Hurwitz spaces

Can join critical *values* as long as the critical *points* remain distinct.

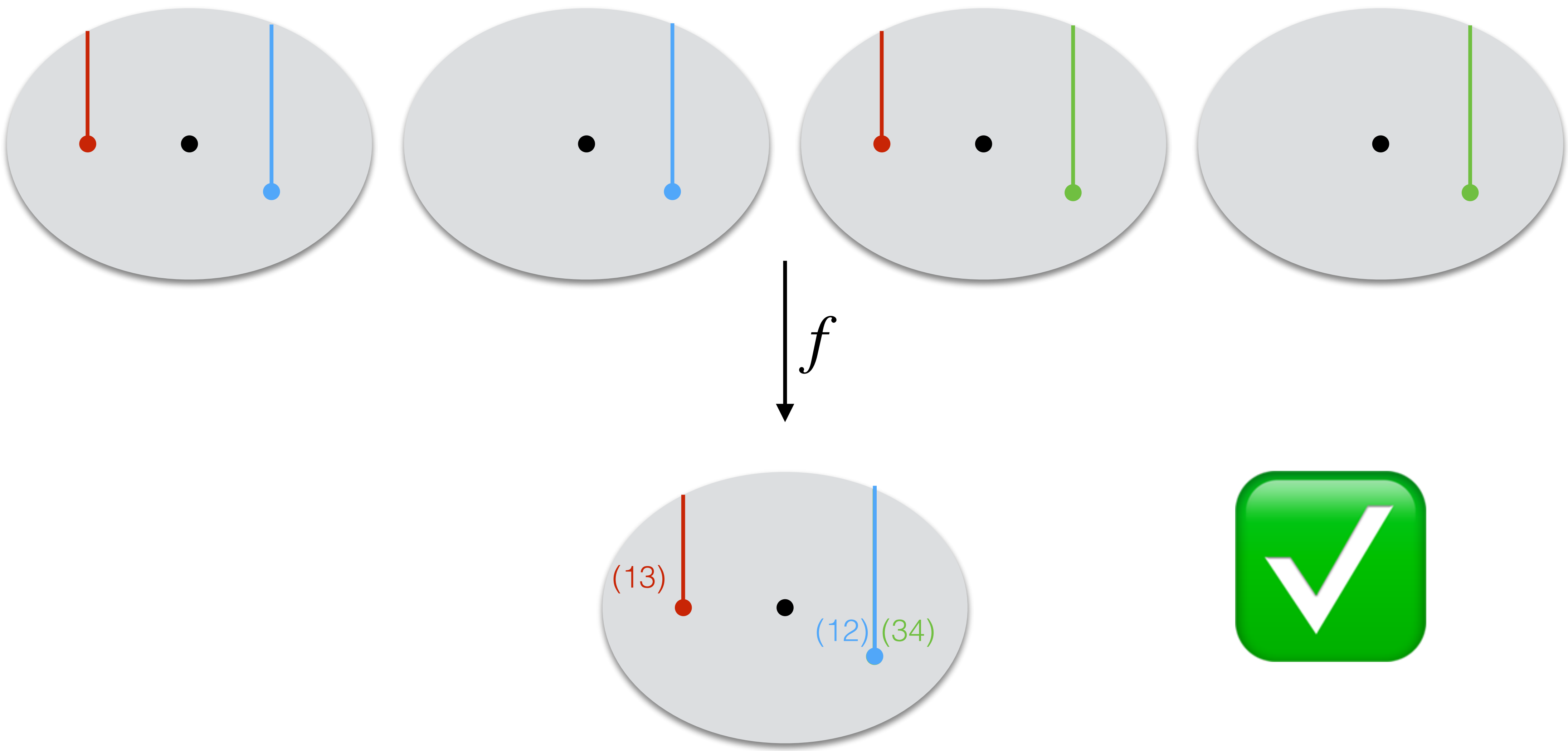


POV 1: Hurwitz spaces

Can join critical *values* as long as the critical *points* remain distinct.

These are smaller Hurwitz spaces.

In terms of permutations, can join $\sigma \in S_p$ to τ iff the supports are disjoint.



POV 1: Hurwitz spaces

Theorem There is a decomposition

$$\mathrm{Conf}_n(\mathbb{C})[\kappa] = \bigcup \mathrm{Hur}(\kappa')^\circ$$

where κ' ranges over all "admissible degenerations" of κ.

Corollary There is a presentation

$$\mathscr{B}_n[\kappa] \cong \pi_1(\mathrm{Hur}(\kappa)^\circ)/\langle\langle \mu_{\kappa',i} \rangle\rangle$$

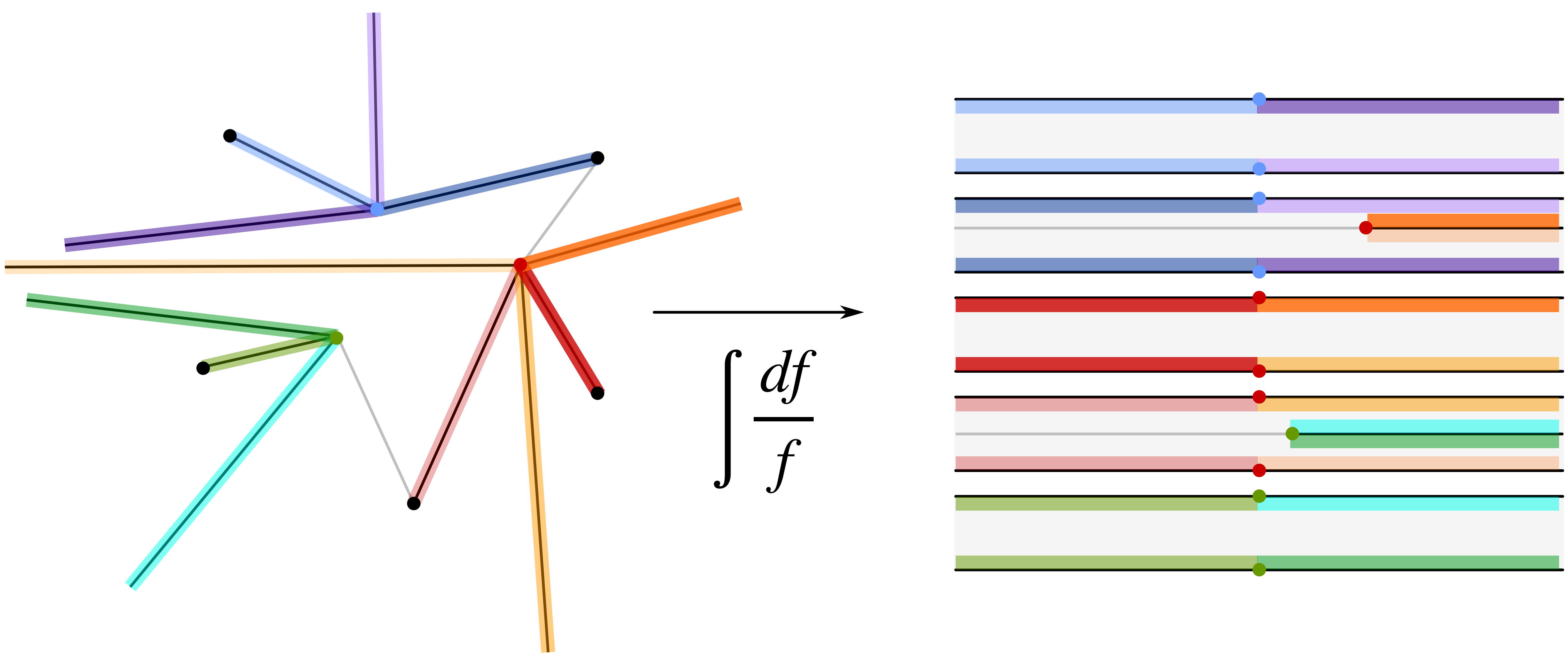
where $\{\mu_{\kappa',i}\}_i$ comprise a set of meridians around components of $\mathrm{Hur}(\kappa')^\circ$ with a *single* degeneration

Note: in general, $\mathrm{Hur}(\kappa')^\circ$ has many components.

POV 2: translation surfaces

There is another way to study this space.

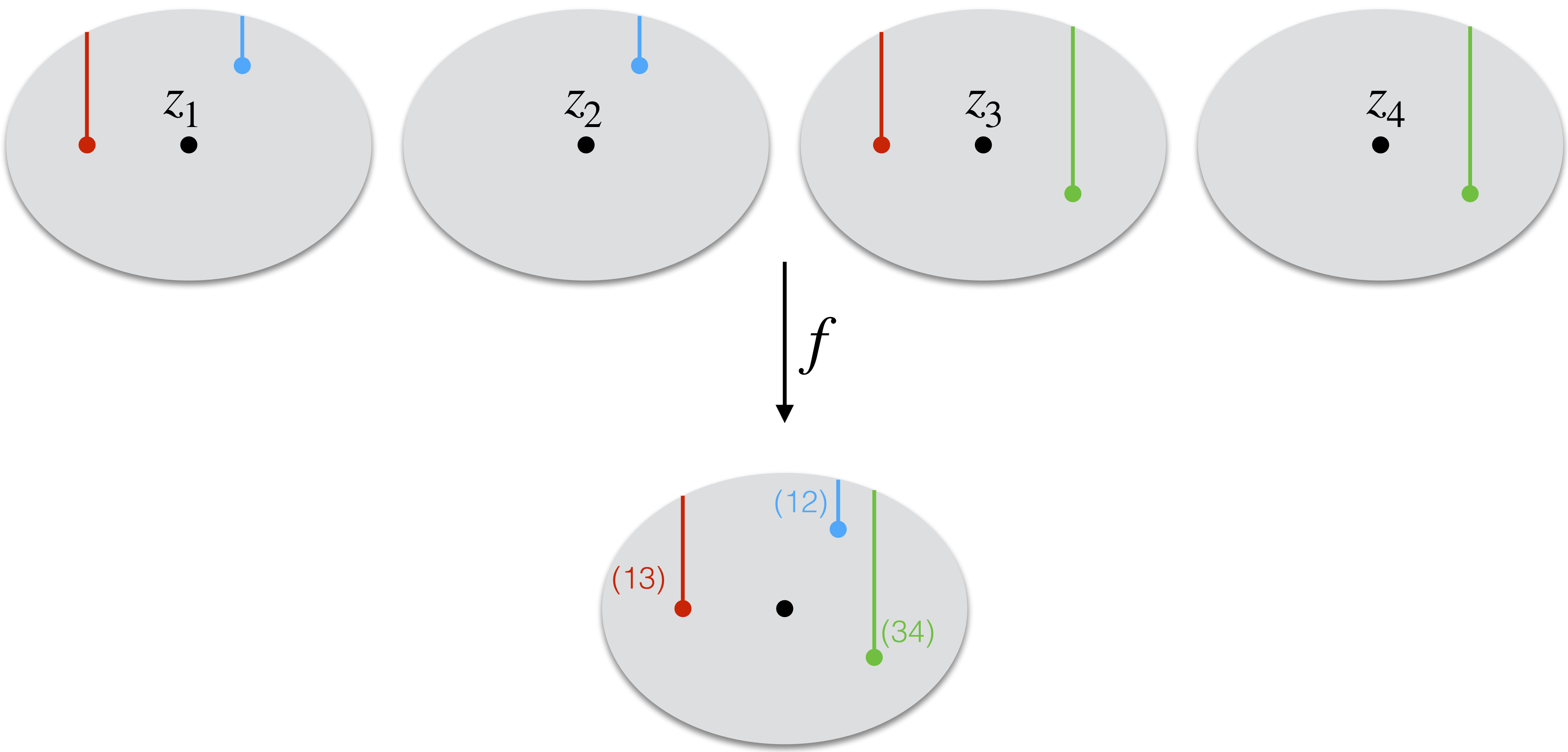
Correspondence: meromorphic differentials ↔ *translation surfaces*



Monodromy

Both pictures inform the monodromy question.

From Hurwitz POV, seeing generators is easy.



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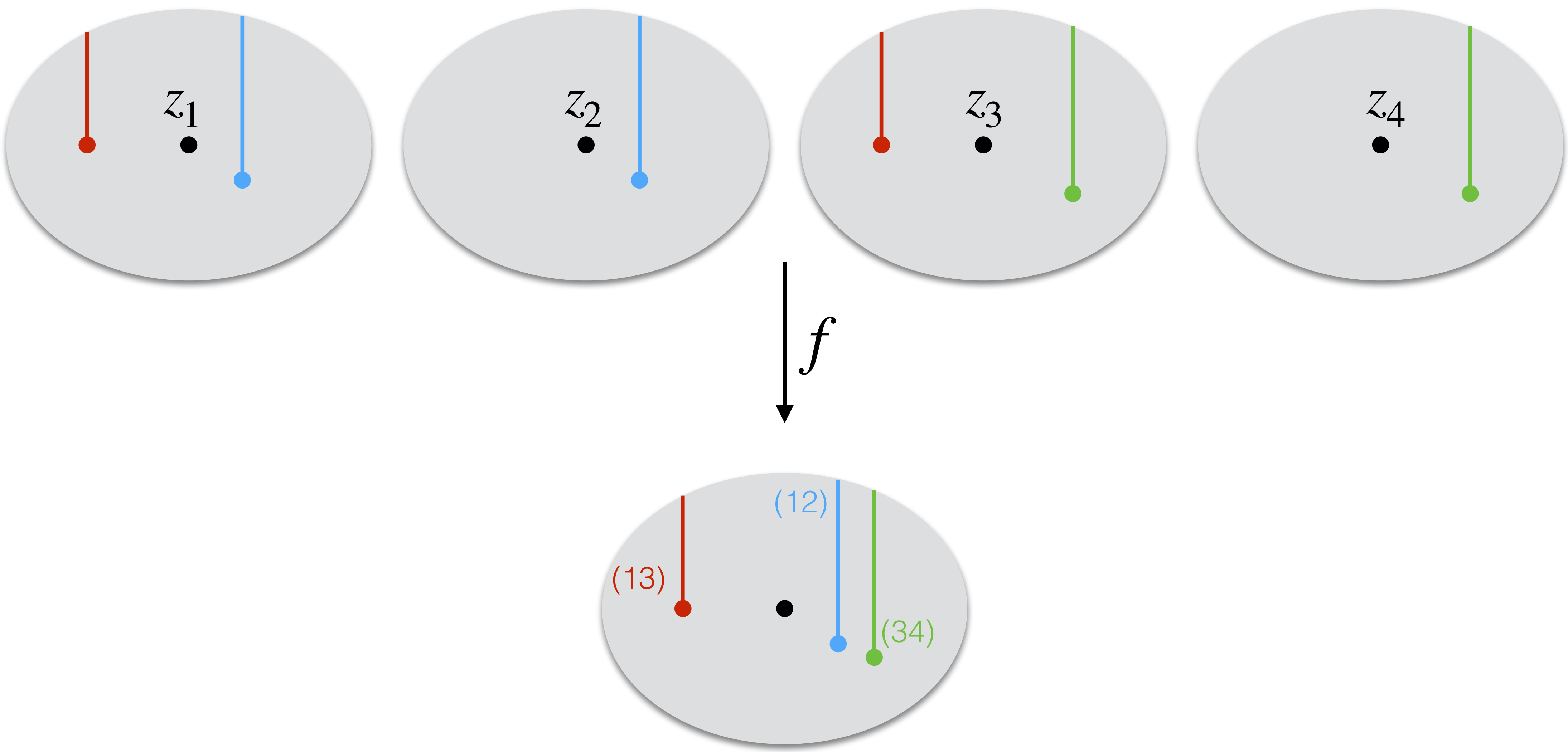

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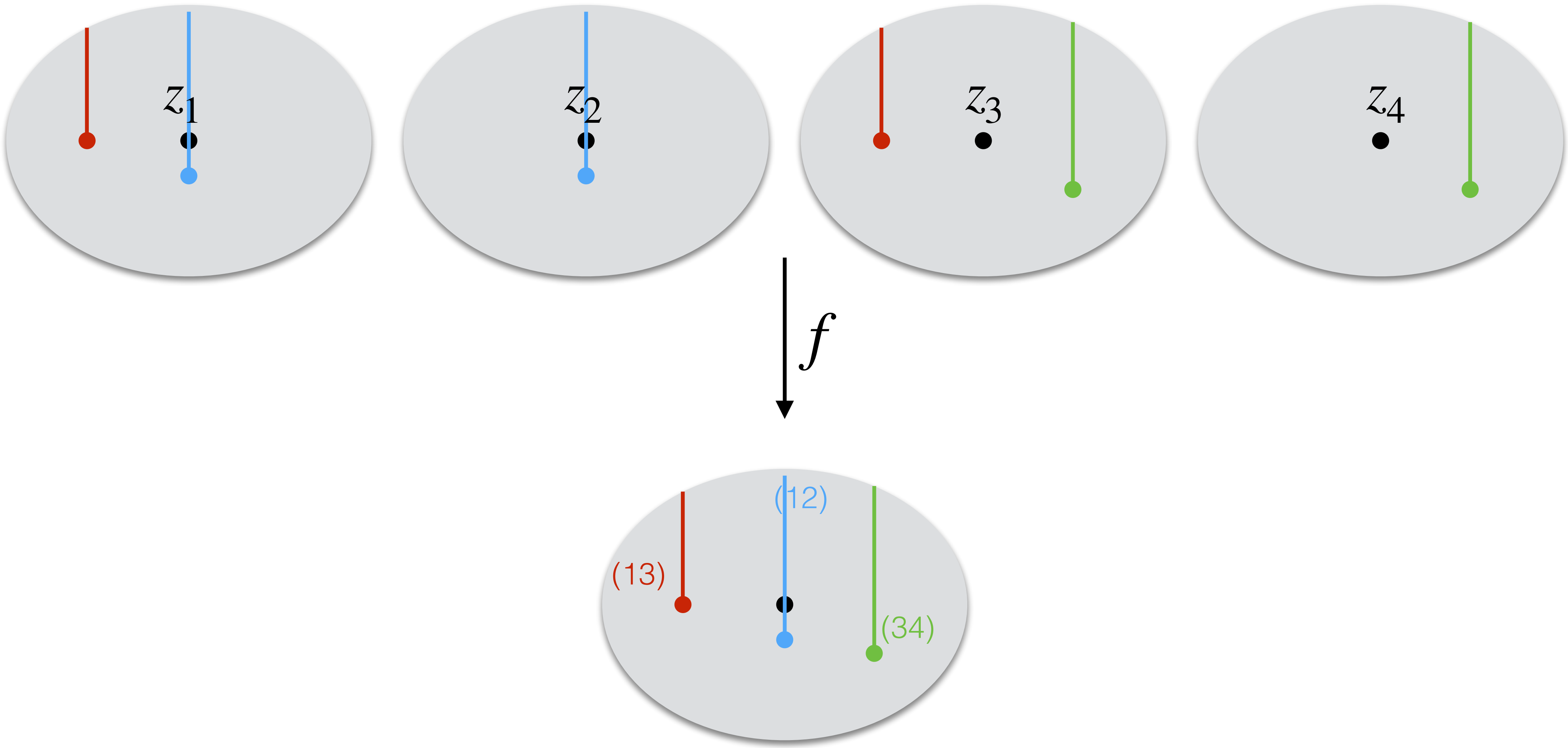

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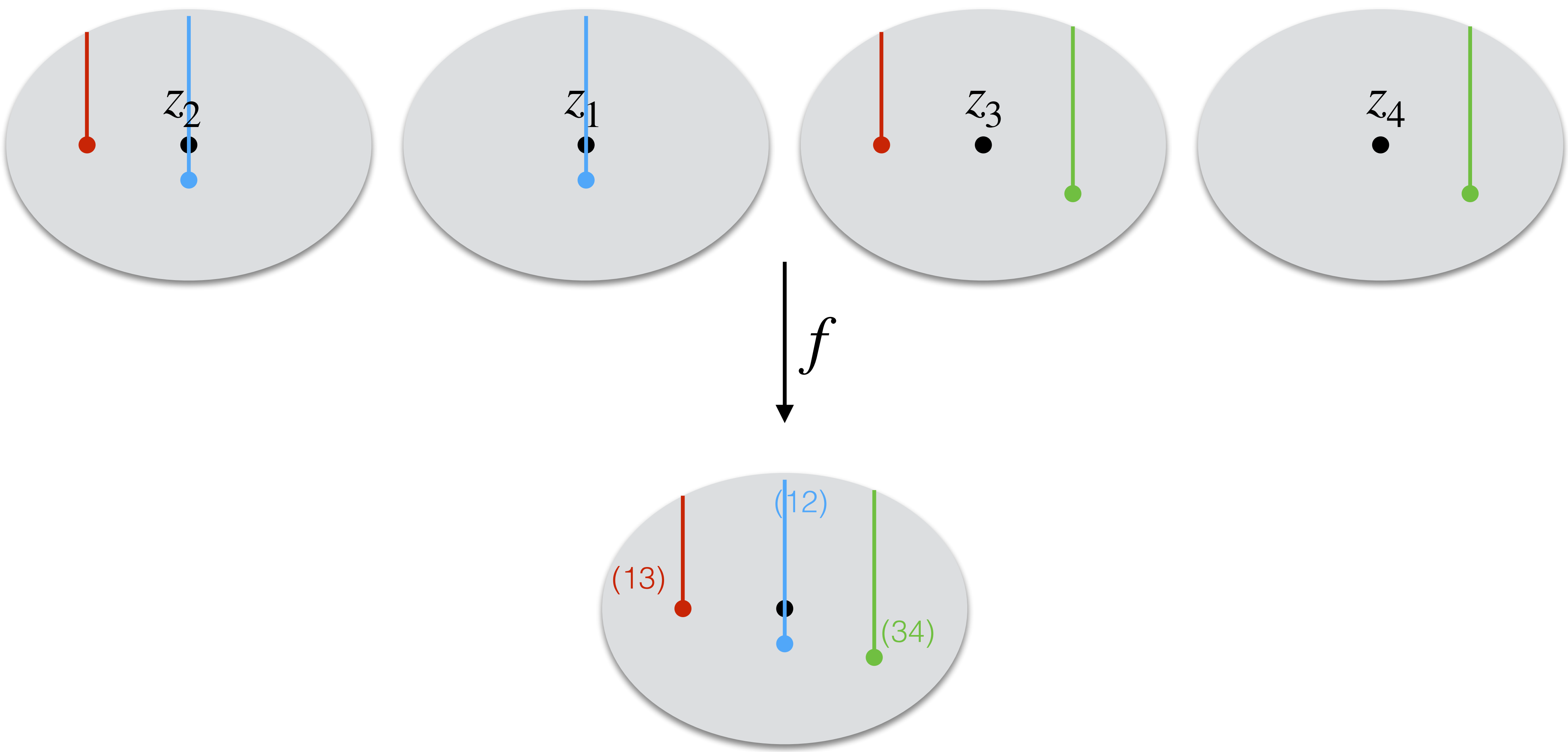

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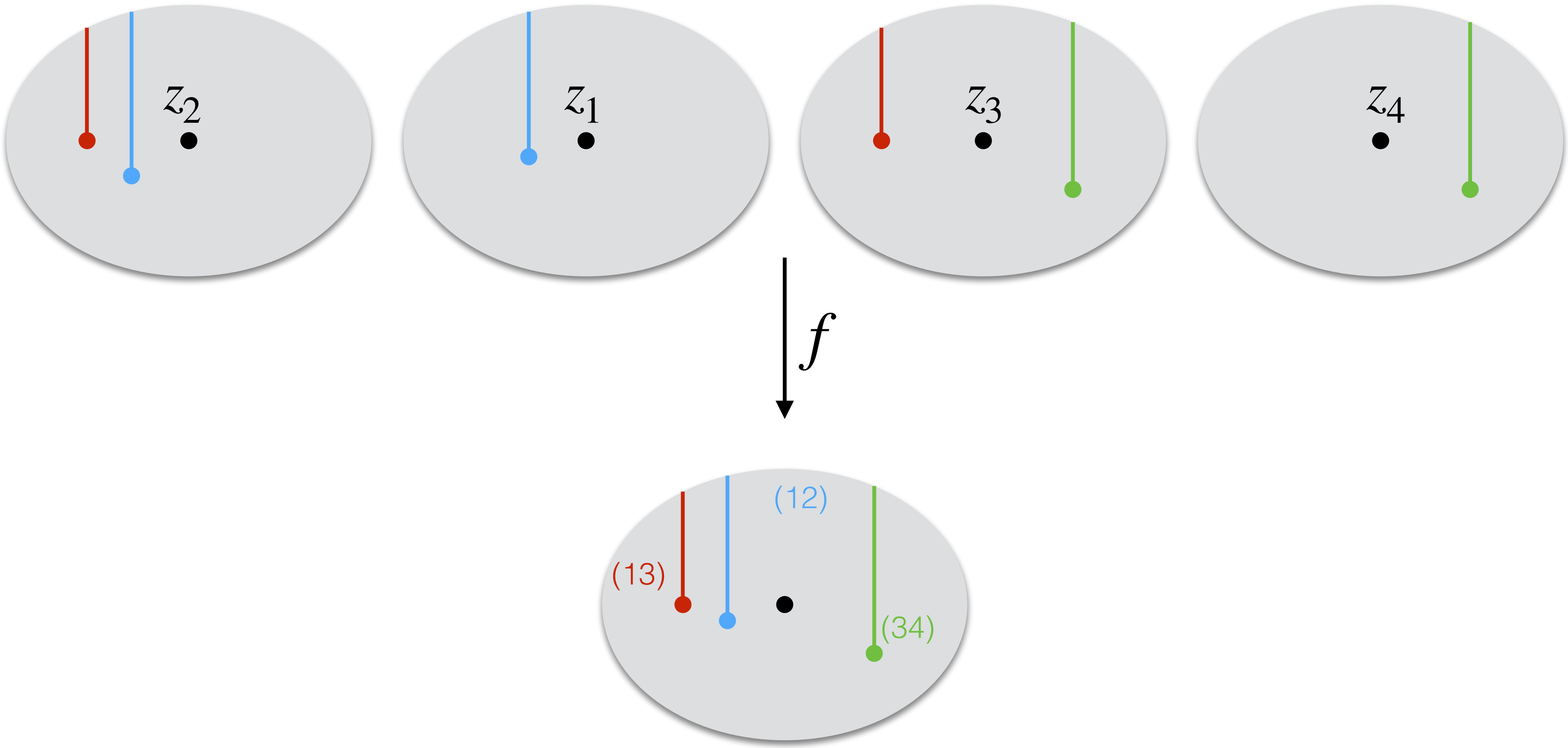
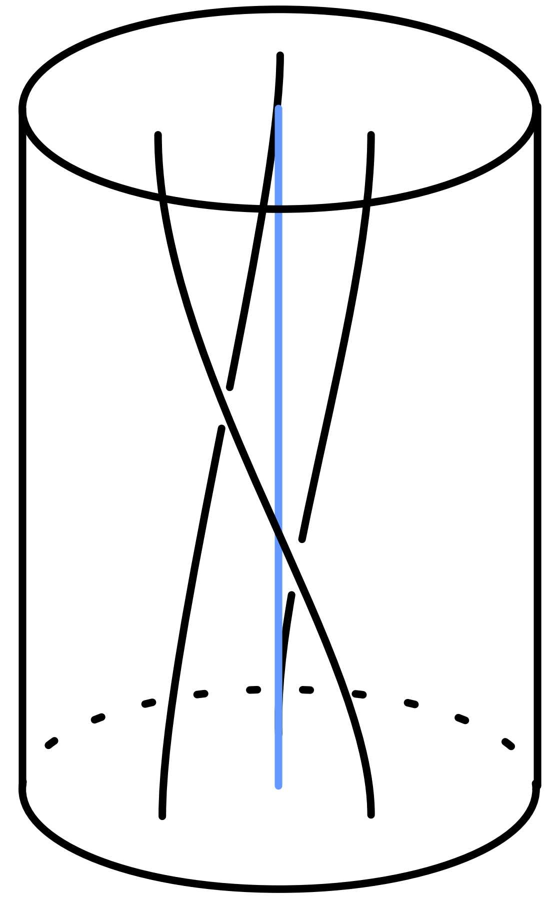


Monodromy

Clear from Hurwitz space picture: local monodromy at a critical point of order k is a rotation σ_k of $k+1$ points.



Simple braid calculation:

Given elements σ_a, σ_b on $a+1$ (resp. $b+1$) points, can do Euclidean algorithm to get $\sigma_{\gcd a,b}$.

Guess: monodromy $B_n[\kappa] \leqslant B_n$ is group Γ_n^r generated by elements σ_r, where $r = \gcd(\kappa)$.

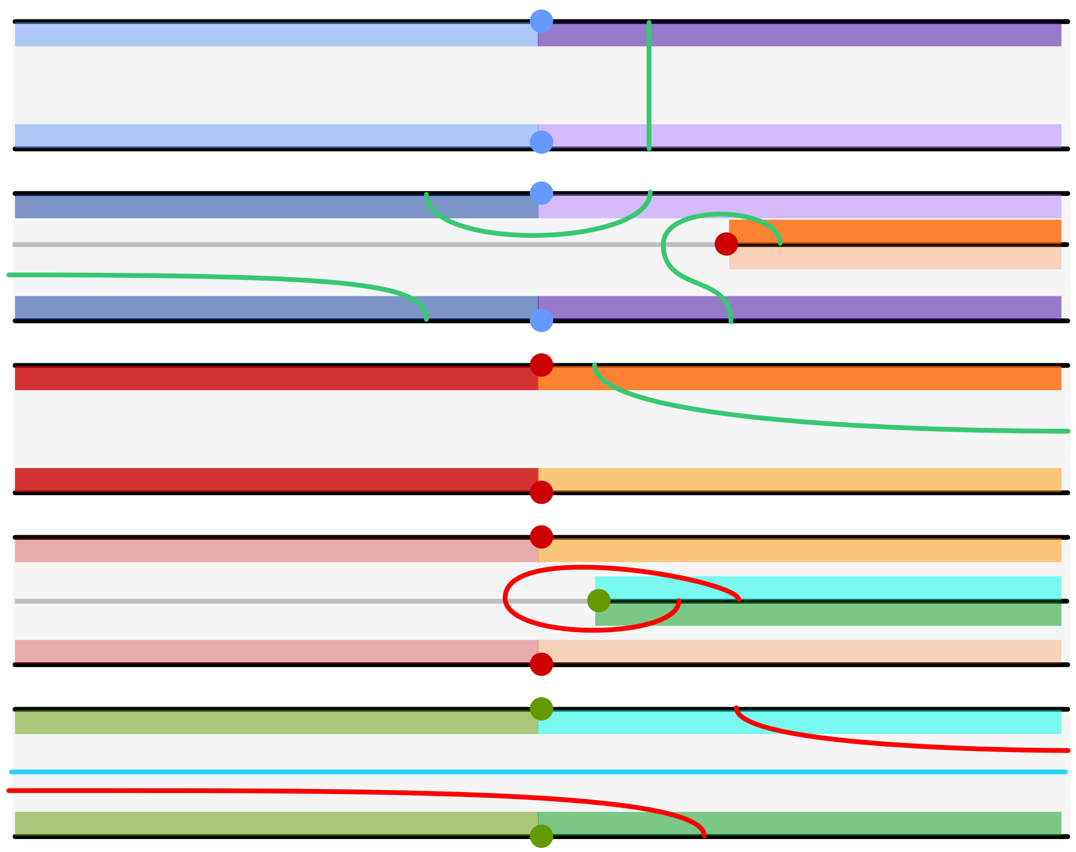
But:

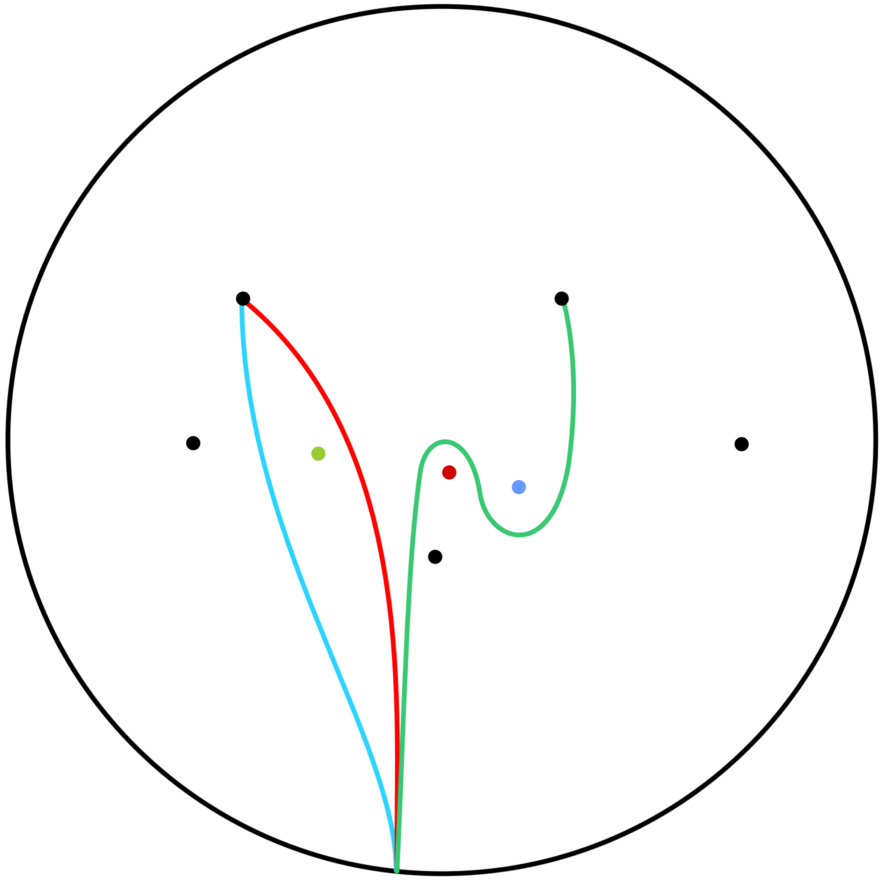
(1) How do you generate Γ_n^r?

(2) What is Γ_n^r? Is it finite-index in B_n? Does it have another description?

Winding numbers

We answer these by looking at the translation surface picture.

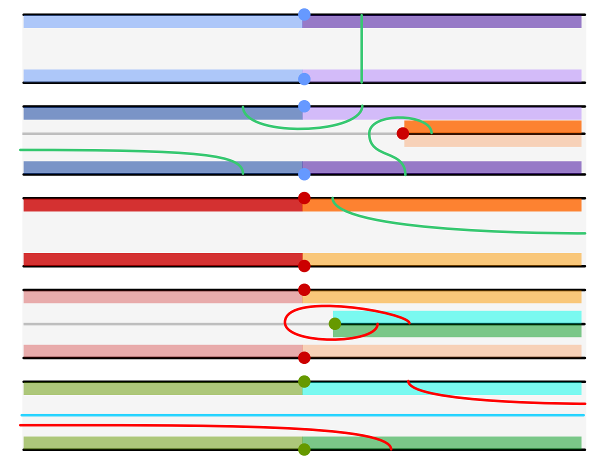


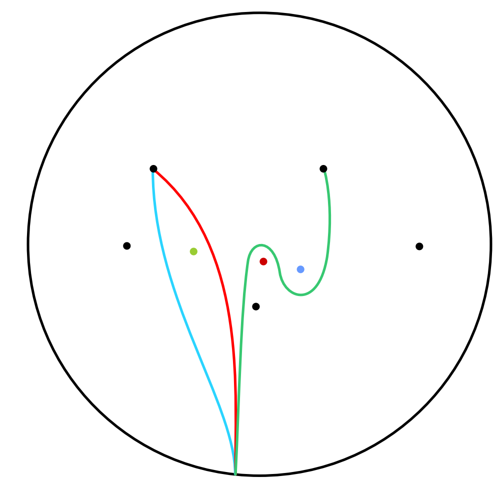


Winding numbers

We answer these by looking at the translation surface picture.

We show that there is a well-defined notion of *winding number* of arcs, visible from the translation surface structure.





Crucial observation: Any braid obtained by deforming our *flat* surfaces *must preserve the winding numbers of all arcs.*

Move an arc across a critical point of order k_i: WN changes by k_i.

Without tracking critical points, can only measure WN mod $r := \gcd(\kappa)$.

Get "change of WN" (crossed) homomorphism $\phi_r : B_n \to (\mathbb{Z}/r\mathbb{Z})^n$.

$$B_n[\kappa] \leqslant \ker(\phi_r)$$

Monodromy

Theorem

For n sufficiently large w.r.t. $r = \gcd(\kappa)$, the following are equivalent:

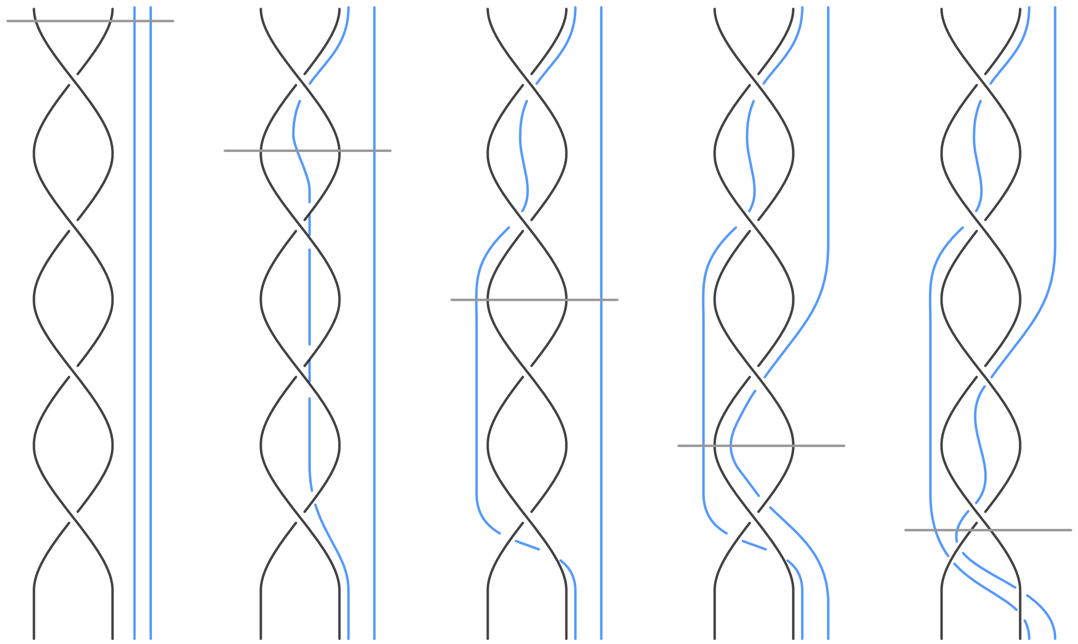
- The monodromy $B_n[\kappa] \leqslant B_n$,
- The group Γ_n^r generated by rotations σ_r of $r+1$ points,
- The kernel of $\phi_r : B_n \to (\mathbb{Z}/r\mathbb{Z})^n$.

In particular, the index $[B_n : B_n[\kappa]]$ is finite.

Note also that $B_n[\kappa] = B_n$ if $r = 1$; here we need $n \geq 5$.

One-sentence proof:

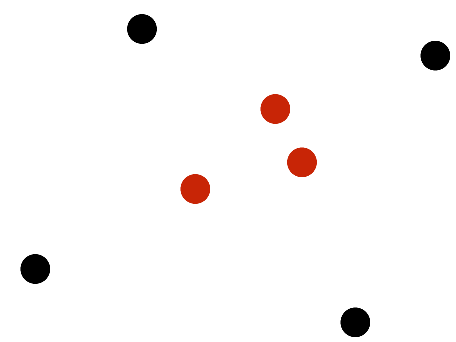
Develop a *factorization algorithm* to express braids in $\ker(\phi_r)$ as products of σ_r



Gauss-Lucas for equicritical families

As a final note, want to discuss a connection with a very classical story.

Theorem (Gauss-Lucas) The critical points of a polynomial p lie inside the convex hull of the roots.



Question: Fix a partition κ of $n-1$. What motions (braids) of $n+p$ points can you see in $\mathrm{Conf}_n(\mathbb{C})[\kappa]$? In other words, what is the monodromy $\rho : \mathrm{Conf}_n(\mathbb{C})[\kappa] \to B_{n+p}$?

Gauss-Lucas requires the "critical braid" to lie in the convex hull of the "root braid".

Our monodromy theorem says *this is not sufficient* when $r > 1$, but in fact this is general.

Gauss-Lucas for equicritical families

Our monodromy theorem says *this is not sufficient* when $r > 1$, but in fact this is general.

When tracking critical points, winding numbers lift from $\mathbb{Z}/r\mathbb{Z}$ to $\mathbb{Z}$.

And for instance, this braid is "convex" but violates the winding number condition.