

Linear-central filtrations and representations of the braid group

Nick Salter

Portions joint with Kevin Kordek

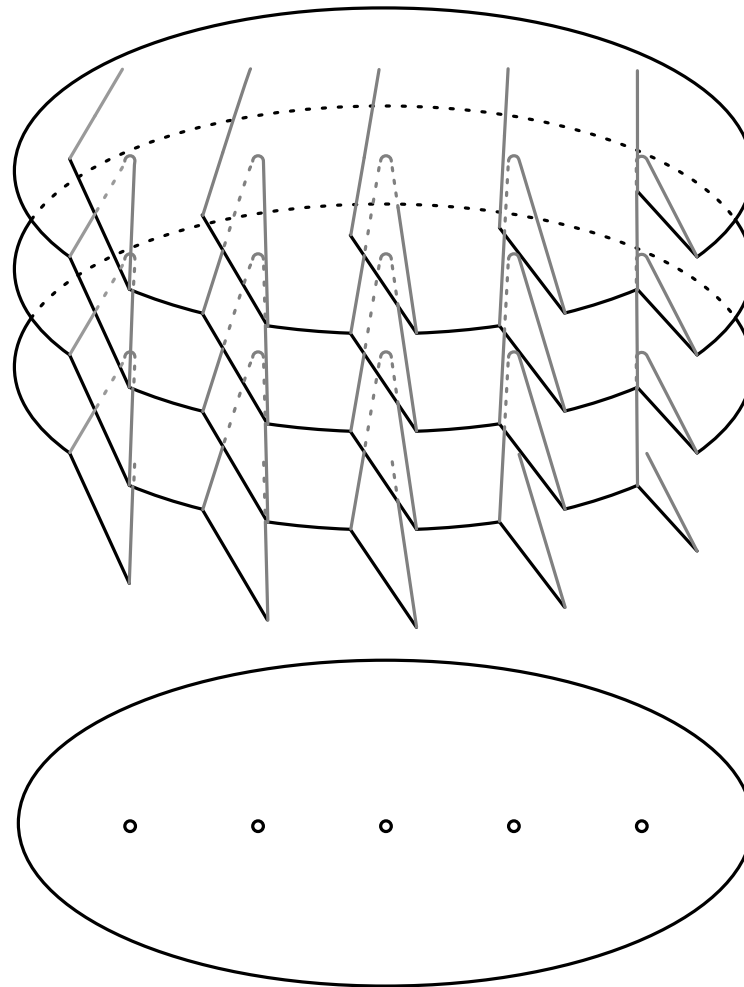
Columbia University

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The Burau representation

$$\tilde{D}_n$$

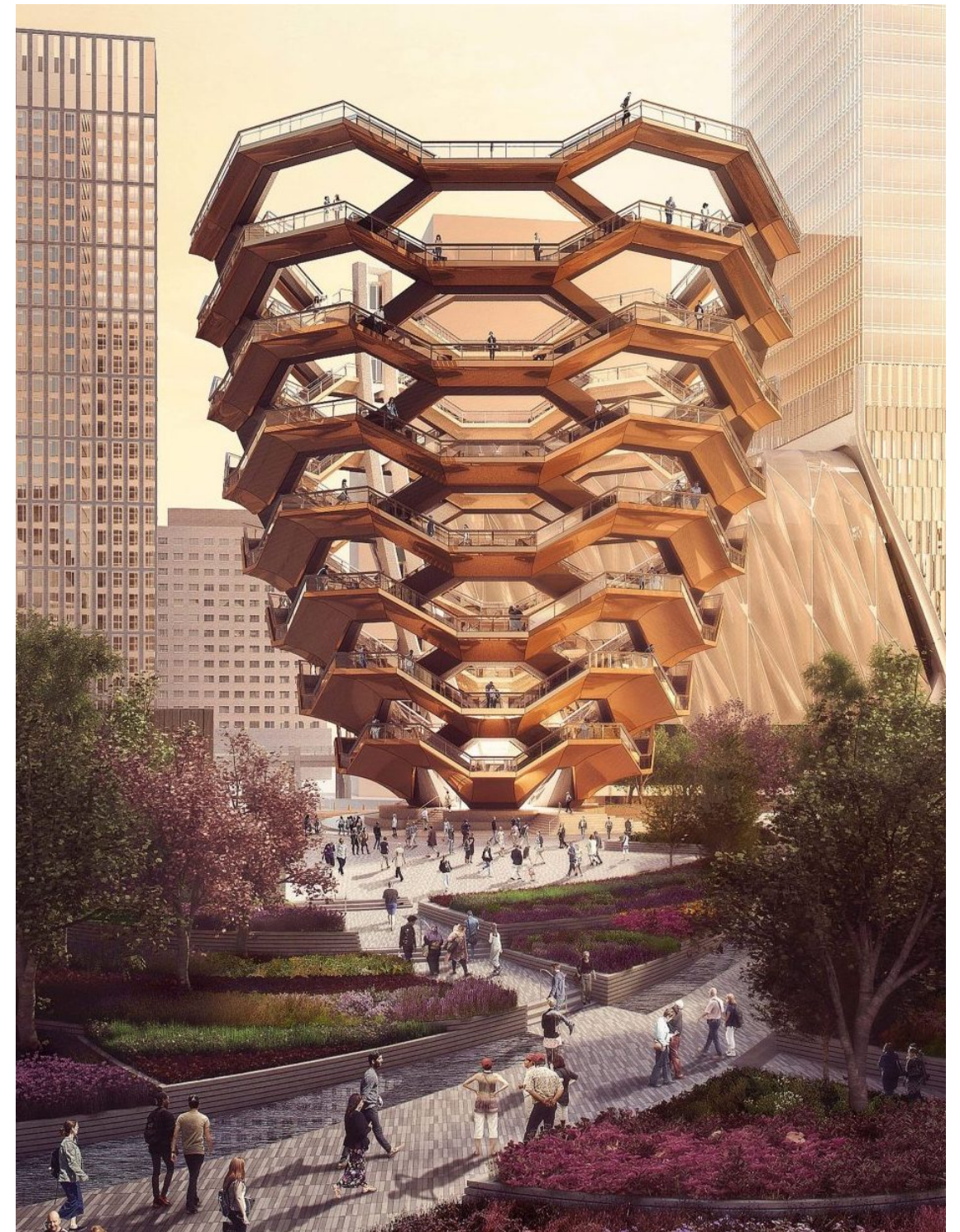
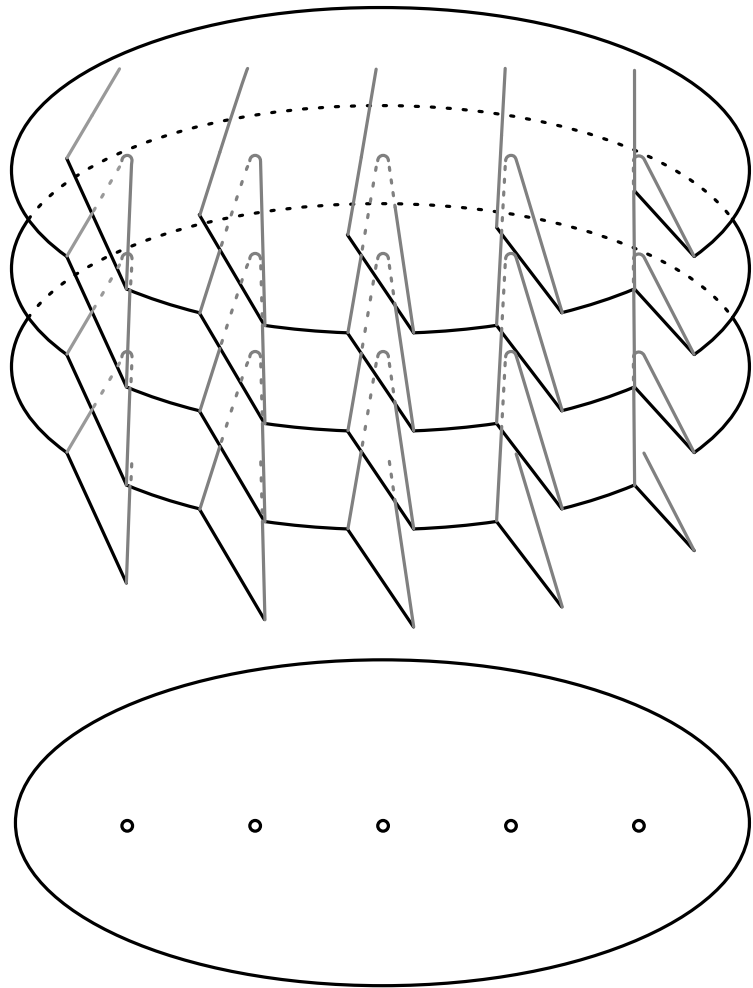
$$B_n \curvearrowright H_1(\tilde{D}_n)$$



(unreduced) Burau representation: $\beta : B_n \rightarrow \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} \oplus I_{n-i-1}$$

Broader impacts?



What does Burau know?

Burau connects to:

- Knot theory (Alexander polynomial) (Burau '36)
- Representation varieties (Long '89)
- Families of algebraic curves/Teichmüller theory (McMullen '09)
- Arithmetic groups (Venkataramana '14)
- Topology of moduli space (Hain conjecture) (Brendle-Margalit-Putman '14)

Burau is at the *crossroads of mathematics*

Pure braids as a congruence subgroup

Our study of Burau is inspired by the theory of *congruence subgroups*.

$$\begin{array}{ccc} \sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} \oplus I_{n-i-1} & & \\ \downarrow t=1 & & \\ I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1} & & \end{array}$$

$\beta \pmod{t-1}$ is the permutation representation of S_n

Slogan: PB_n is the “level- s ” subgroup (here $s = t-1$).

Burau is unitary

Much attention to *injectivity* of β (Moody, Long-Paton, Bigelow)

Old question (Birman, '74): What's the *image* of β ?

Known constraints:

- (1) Fixed vector
- (2) Unitarity (Squier '84)
- (3) $\text{Im}(\beta \bmod s) = S_n$ (not larger)

Define $\Gamma := \{A \in \text{GL}_n(\mathbb{Z}[t, t^{-1}]) \mid (1), (2), (3)\}$

A “strong approximation” to the image

Theorem (S.) For $n \geq 5$, $\text{Im}(\beta)$ is *dense* in Γ in the “s-adic topology”.

A, B close when $AB^{-1} = I + s^k M$ for some large k .

Equivalently, when initial terms of “Taylor series” at $t=1$ agree:

$$A = C_0 + s C_1 + \dots + s^{k-1} C_{k-1} + s^k A_k + \dots$$

$$B = C_0 + s C_1 + \dots + s^{k-1} C_{k-1} + s^k B_k + \dots$$

Homology of braid Torelli

“Braid Torelli”: $\mathcal{BI} = \ker(\beta \bmod t+1)$

Important in algebraic geometry (π_1 of branch locus of period map)

Basic structure mysterious. Finitely generated?

Theorem (Kordek - S.): For $n \geq 3$ odd, \mathcal{BI} admits an $\mathrm{Sp}(2g, \mathbb{Z})$ -equivariant surjection onto $\mathfrak{sp}(2g, \mathbb{Z})$

Via rep theory: this does not factor through any previously-known abelian quotient

Linear-central filtrations

Set $\tau = t - \zeta$ with $\zeta = \pm 1$ in theorems 1,2, respectively

Define $\Gamma[\tau^k] = \{A \in \Gamma \mid A \equiv I \pmod{\tau^k}\}$

Saw $B_n \cap \Gamma[\tau^k] = PB_n, \mathcal{BI}_n$

Easy lemma: $[\Gamma[\tau^k], \Gamma[\tau^\ell]] \leq \Gamma[\tau^{k+\ell}]$

Filtration $\dots \leq \Gamma[\tau^{k+1}] \leq \Gamma[\tau^k] \leq \dots$ is *linear-central* (Bass-Lubotzky '94)

Acquire \mathbb{Z} -Lie algebra $\mathfrak{g} = \bigoplus_{k \geq 1} \Gamma[\tau^k] / \Gamma[\tau^{k+1}]$

This carries action of $\Gamma / \Gamma[\tau] = S_n, \mathrm{Sp}(2g, \mathbb{Z})$

Application: homology of pure braid group

To illustrate our techniques, here's a new take on a classic story:

$$H^1(PB_n; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{2}}$$

Arnol'd's winding number functionals: coordinate-dependent

Representation stability: $H^1(PB_n; \mathbb{Q}) \cong V(0) \oplus V(1) \oplus V(2)$

This hides a much more natural description!

$$H^1(PB_n; \mathbb{Z}) \cong \mathrm{Sym}^2(V)$$

(V : standard rep over \mathbb{Z})

Application: homology of pure braid group

Method: $PB_n \twoheadrightarrow \Gamma[s]/\Gamma[s^2] \cong \text{Sym}^2(V)$

$$\beta(A_{ij}) = \begin{pmatrix} t^2 - t + 1 & 1 - t \\ t - t^2 & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + s \begin{pmatrix} t & -1 \\ -t & 1 \end{pmatrix}$$

(ii, ij, ji, jj entries only)

Set $X_{ij} = E_{ii} + E_{jj} - E_{ij} - E_{ji}$: then $\beta(A_{ij}) = X_{ij}$

And $\text{Sym}^2(V) = \langle X_{ij} \rangle$!

Why do we see symmetry here? Unitarity

Where next?

Work in progress: $\text{Im}(\beta) = \Gamma$

(Method totally different: geometric group theory)

Prototype for a whole family of theorems: braid group representations should behave like (arithmetic?) lattices.

- Roots of unity beyond Venkataramana?
- Jones rep?

Application: “Spectrum” of a knot invariant:
which polynomials are e.g. Jones polynomials?