Holomorphic maps between configuration spaces

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Configuration spaces…

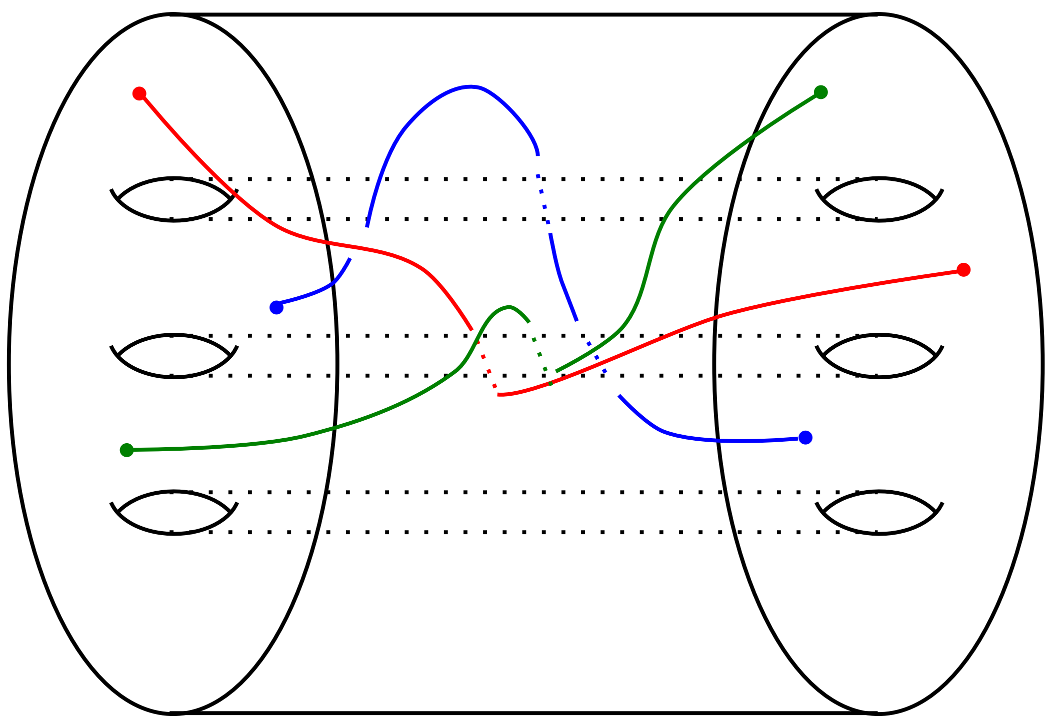
X: topological space

$\mathrm{Conf}_n(X)$: space of unordered n-tuples of distinct points

$\mathrm{PConf}_n(X)$: space of ordered n-tuples of distinct points

When X is a smooth manifold, so are $\mathrm{Conf}_n(X)$ and $\mathrm{PConf}_n(X)$

Focus on case of surfaces.
Then these are $K(\pi,1)$ spaces
for *surface braid groups.*



…and the maps between them

One interesting family of questions:

Fix X, Y, m, n. Classify all maps
$f : \mathrm{(P)Conf}_m(X) \to \mathrm{(P)Conf}_n(Y)$

"Superrigidity": expect maps to be induced from natural operations on the space level.

…and the maps between them

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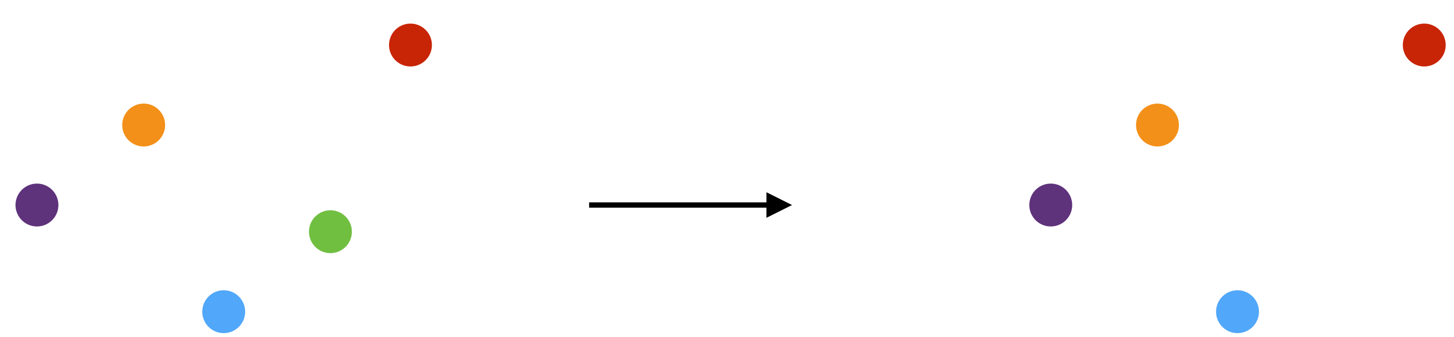
$f : (\mathrm{P})\mathrm{Conf}_m(X) \to (\mathrm{P})\mathrm{Conf}_n(Y)$

Example: X any space

$$\pi_k : \mathrm{PConf}_n(X) \to \mathrm{PConf}_{n-1}(X)$$

Forget the k^{th} point.



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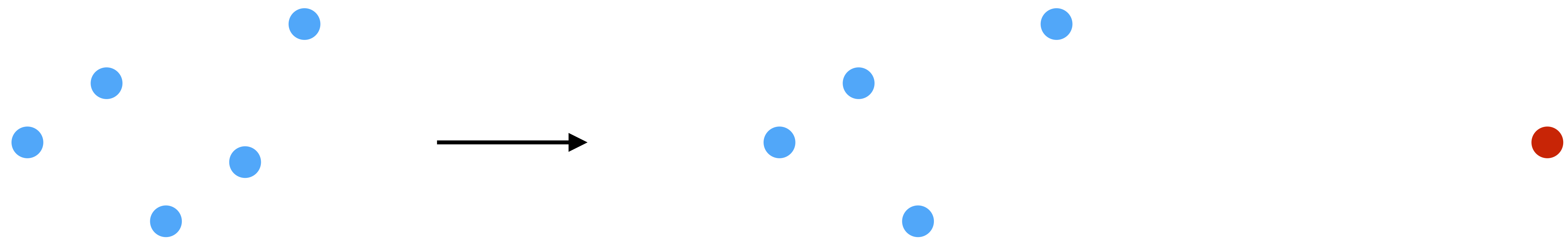
Fix X, Y, m, n. Classify all maps
$f : \text{(P)Conf}_m(X) \to \text{(P)Conf}_n(Y)$

Example: $X = \mathbb{C}$

$$\alpha : \text{Conf}_n(\mathbb{C}) \to \text{Conf}_{n+1}(\mathbb{C})$$

Add a new point “near infinity”



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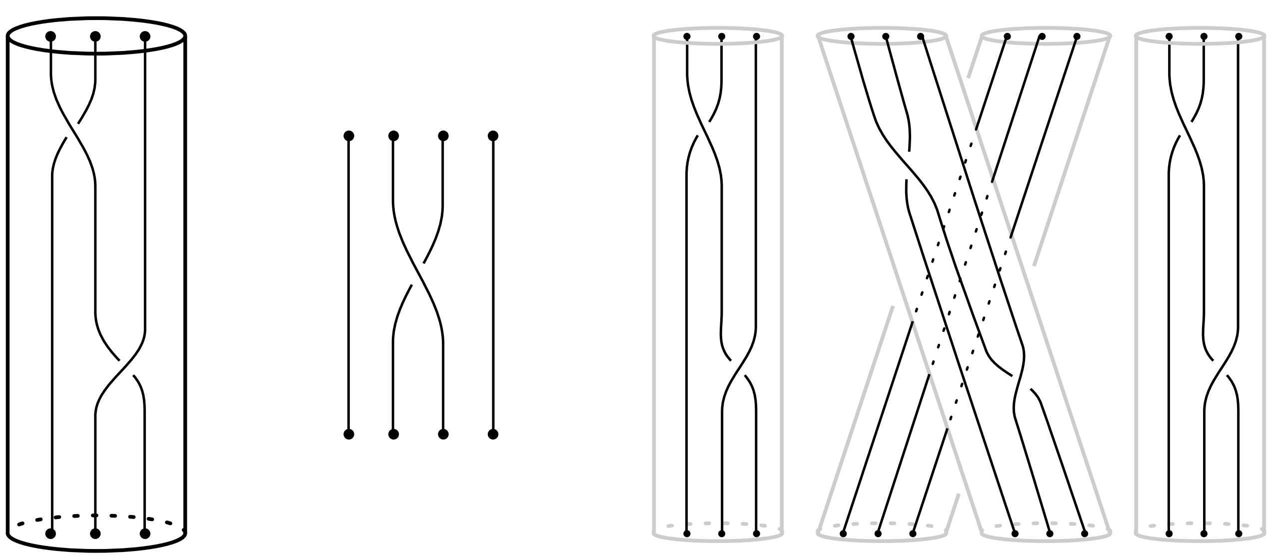
$$\beta : \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{kn}(\mathbb{C})$$

"Cabling". Split each strand into its own k-stranded braid.

A very flexible procedure that yields a huge variety of maps



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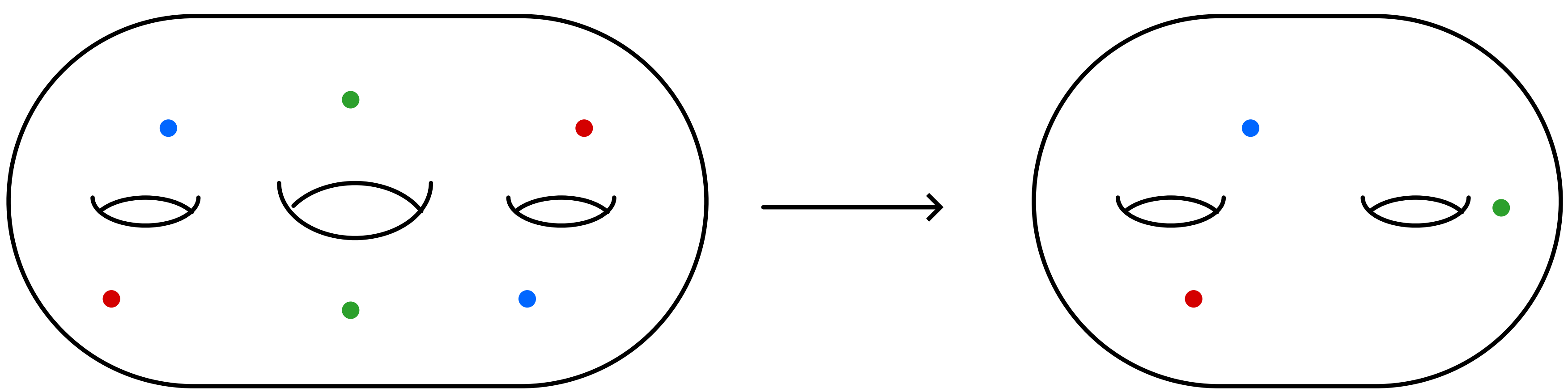
Example:

X smooth algebraic curve of genus g≥2
$p : Y \to X$ unbranched cover of degree d

$$\gamma : \mathrm{Conf}_n(X) \to \mathrm{Conf}_{dn}(Y)$$



"Lifting": send $\zeta = \{z_1, \ldots, z_n\}$ to $p^{-1}(\zeta)$

Configuration spaces as varieties

Now endow X with the structure of a Riemann surface.

Then $(\mathrm{P})\mathrm{Conf}_n(X)$ is a complex manifold (even a smooth variety)

Refinement of main question:

When is $f : (\mathrm{P})\mathrm{Conf}_m(X) \to (\mathrm{P})\mathrm{Conf}_n(Y)$
homotopic to a holomorphic map?

Both “forget” and “lift along cover” are holomorphic

Fun challenge: is “add near infinity” holomorphic?

The holomorphic landscape

The table below summarizes our work on this problem.

PConf	$g(Y) = 0$	1	≥ 2
$g(X) = 0$	Full classification	All constant	All constant
1	??	Full classification	All constant
≥ 2	??	Full classification	Reduction to “effective de Franchis” problem

The holomorphic landscape (II)

And here is our work in the impure setting.

Conf	$g(Y) = 0$	1	≥ 2
$g(X) = 0$	Partial classification (see also Lin)	All constant	??
1	??	Abundant: lifting	??
≥ 2	??	??	Abundant: lifting

Twisting

Our classification results involve "twisting"

Definition:

Let $f : (\mathrm{P})\mathrm{Conf}_m(X) \to (\mathrm{P})\mathrm{Conf}_n(Y)$ and
$A : (\mathrm{P})\mathrm{Conf}_m(X) \to \mathrm{Aut}(Y)$
be holomorphic.

The *twist* $f^A : (\mathrm{P})\mathrm{Conf}_m(X) \to (\mathrm{P})\mathrm{Conf}_n(Y)$ is given by the formula $f^A(\zeta) = A(\zeta)(f(\zeta))$

Example:

$X = Y = \mathbb{C}$.

Take $A(z_1, \ldots, z_n) = -\dfrac{z_1 + \ldots + z_n}{n} \in \mathbb{C} \leqslant \mathrm{Aut}(\mathbb{C})$.

Then id^A normalizes center of mass to 0.

Note: twisting can change homotopy class!
The twist of a constant map can be interesting.

More precise statements

Theorem: For m ≥ 5 and n ≤ 2m, up to twisting, every holomorphic $f : \mathrm{Conf}_m(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$ is either constant, the identity, or a "root map".

- Lin previously established case $m = n$.
- Our techniques extend to show that *no* cabling map can be holomorphic.

Theorem: Let X, Y be compact Riemann surfaces of genus 1. Then every nonconstant holomorphic $h : \mathrm{PConf}_m(X) \to \mathrm{PConf}_n(Y)$ is induced by an isomorphism $X \cong Y$ and is a twist of a forgetful map.

More precise statements

Theorem: Let X, Y be compact Riemann surfaces of genus ≥ 2 (not necessarily the same) and let $h : \mathrm{PConf}_m(X) \to \mathrm{PConf}_n(Y)$ be holomorphic.

Then up to twisting, either $X \cong Y$ and h is a forgetful map, or else h factors via a forgetful $p : \mathrm{PConf}_m(X) \to X$ and a holomorphic $f : X \to \mathrm{PConf}_n(Y)$.

"Effective de Franchis problem": given X, Y, what is the maximum number of distinct holomorphic $f_i : X \to Y$ (no requirement that graphs be disjoint). Quite open! General bounds exponential in $g(X), g(Y)$.

Theorem: Let X, Y be compact Riemann surfaces of genera $g(X), g(Y) \geq 2$, and let $f : X \to \mathrm{PConf}_n(Y)$ be holomorphic. Then $n \leq 4g(X)g(Y)$.

Proof techniques

General theme: promotion of group-theoretic rigidity to space level.

Start with a list of possible maps on π_1 (e.g. by Chen-Kordek-Margalit).

Then analyze which can arise holomorphically.

Use Teichmüller theory (Imayoshi-Shiga),
as well as classical complex analysis (Picard,
uniformization, max. modulus)

Dimension reduction: fix $\zeta = \{z_1, \ldots, z_{n-1}\}$, take $z_n \in X \backslash \zeta$.