HIGHER SPIN MAPPING CLASS GROUPS IN ALGEBRAIC AND FLAT GEOMETRY

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ABSTRACT. An *r*-spin structure is a choice of r^{th} root of the canonical bundle of a Riemann surface. Such structures arise in a variety of settings in geometry; in this lecture series, we will focus on their role in two places at the interface of algebraic geometry and topology: linear systems on algebraic surfaces (especially toric surfaces), and translation surfaces (also known as abelian differentials). In both these settings, there are "topological monodromy groups" valued in the mapping class group that encode important information about these families of Riemann surfaces and their degenerations, and the presence of *r*-spin structures is reflected in the underlying group theory. We will outline some recent developments in the theory of these "higher spin mapping class groups" that allow us to understand monodromy in the above problems, and ultimately to gain new insights into the behavior of these families. No specialized knowledge of topology or the mapping class group will be assumed. Portions of this work are joint with Aaron Calderon.

These notes are the source for a series of four lectures delivered at the winter school on Cremona groups, geometric topology, and algebraic geometry in Cuernavaca, Mexico in January 2020. They are intended as a companion to a series of recent papers [Sal16, Sal19, Cal19, CS19, CS20] by various subsets of the author and A. Calderon. We will refer to the relevant portions of the research papers throughout.

Assumed background. The author approaches this subject from the point of view of topology and geometric group theory and confesses that his understanding of the purely algebro-geometric aspects of the theory is not deep. Because of the nature of the audience, we will devote some time to the basic theory of the mapping class group, from which we draw heavily on the now-standard reference by Farb–Margalit [FM12]. The portions of the lectures dealing with abelian differentials necessarily only skim the surface of this deep body of material, and we refer interested readers to the survey of Wright [Wri15] for an honest introduction.

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1. MONODROMY PROBLEMS IN ALGEBRAIC AND FLAT GEOMETRY

The ultimate goal of these lectures is to explain how some developments in the theory of "higher spin mapping class groups" leads to an understanding of a variety of "monodromy problems". In this first lecture we will attempt to introduce the major characters in the story by way of motivating the subsequent excursion into the theory of the mapping class group.

Standing conventions. All algebraic geometry takes place over \mathbb{C} . We also have a clash of terminology to resolve: do we take dimension over \mathbb{C} (in which case we speak of algebraic *curves*, toric *surfaces*, etc.), or do we take real dimension (in which case the corresponding objects are surfaces and 4-manifolds)? We resolve to use terminology consistent with context, and will use the former convention in algebro-geometric settings and the latter when studying topology and the mapping class group.

1.1. **Problem 1: monodromy and vanishing cycles in toric surfaces.** A standard reference for toric varieties is the book of Fulton [Ful93], but in these lectures we will use very little of the theory and we will invoke what is needed as we go. A *toric variety* is a variety X with an open dense embedding of an *algebraic torus* $(\mathbb{C}^{\times})^n$, such that the action of $(\mathbb{C}^{\times})^n$ extends to an action on X. We will focus on n = 2 - *toric surfaces*.

Example 1.1. \mathbb{CP}^2 and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are smooth toric surfaces. $(\mathbb{C}^{\times})^2$ embeds in \mathbb{CP}^2 via

$$(z,w) \mapsto [1:z:w]$$

and into $\mathbb{CP}^1\times\mathbb{CP}^1$ via

$$(z,w)\mapsto [1:z]\times [1:w].$$

 $(\mathbb{C}^{\times})^2$ acts on \mathbb{CP}^2 via

$$(z,w) \cdot [a:b:c] = [a:zb:wc]$$

and on $\mathbb{CP}^1\times\mathbb{CP}^1$ by

$$(z,w) \cdot [a:b] \times [c:d] = [a:zb] \times [c:wb].$$

In Lecture 4 we will learn how to construct ample line bundles on toric surfaces using lattice geometry. For now you can imagine that the toric surface X is just \mathbb{CP}^2 and the line bundle \mathcal{L} is $\mathcal{O}(d)$ for d > 0. The space $\Gamma(\mathcal{L})$ of global sections gives rise to the associated *linear system* of curves in X: given a section $f \in \Gamma(\mathcal{L})$, you look at the divisor Div(f). For $(X, \mathcal{L}) = (\mathbb{CP}^2, \mathcal{O}(d))$, a section f of $\mathcal{O}(d)$ is a degree-dhomogeneous polynomial, e.g. $f = X^d + Y^d + Z^d$, and the divisor is just the solution set to this in \mathbb{CP}^2 .

The divisor Div(f) depends only on the projective class of f, and hence we projectivize. Set

$$\mathcal{U}_{\mathcal{L}} := \mathbb{P}(\Gamma(\mathcal{L})).$$

The first thing we must do if we want to study this family topologically is to stratify the space according to the topological type of the curve. For now we are interested only in the generic curve, which is smooth. The *discriminant* $\mathcal{D}_{\mathcal{L}}$ is a form on $\mathcal{U}_{\mathcal{L}}$ such that $V(\mathcal{D}_{\mathcal{L}})$ parameterizes non-smooth curves. Set

$$\mathcal{P}_{\mathcal{L}} := \mathcal{U}_{\mathcal{L}} \setminus (V(\mathcal{D}_{\mathcal{L}}));$$

this is the parameter space of *smooth* sections of \mathcal{L} . Accordingly, there is a *tautological family*

$$\mathcal{X}_{\mathcal{L}} := \{ (f, x) \in \mathcal{P}_{\mathcal{L}} \times X \mid x \in V(f) \}.$$

Topologically, the projection $\pi : \mathcal{X}_{\mathcal{L}} \to \mathcal{P}_{\mathcal{L}}$ endows $\mathcal{X}_{\mathcal{L}}$ with the structure of a *surface bundle*: a fiber bundle over $\mathcal{P}_{\mathcal{L}}$ with fiber a topological surface. In the case $(X, \mathcal{L}) = (\mathbb{CP}^2, \mathcal{L})$, the genus is $\binom{d-1}{2}$; for general \mathcal{L} we will see a beautiful formula in Lecture 4 for the genus in terms of lattice point counts in polygons (it is no accident that $\binom{d-1}{2}$) counts lattice points inside a right equilateral triangle of side length d - 3)!

Donaldson's question. I came to study plane curves motivated solely by the hope that this surface bundle would have some interesting topology that could be captured by the mapping class group. At exactly the same time, Remi Crétois and Lionel Lang were studying the same family (indeed the exact same problem!) in the larger context of smooth toric surfaces. One of their main motivations was to better understand

the topological theory of nodal degenerations of curves in toric surfaces. The basic question here was posed by Donaldson [Don00]; here is a succinct formulation:¹

Question 1.2 (Donaldson). *Fix a linear system on a smooth toric surface* (X, \mathcal{L}) . *Which simple closed curves can be vanishing cycles for nodal degenerations in \mathcal{L}?*

There is much to make sense of above. First, recall the notion of a *nodal degeneration*. This is a family C_{ε} of curves locally modeled on $XY = \varepsilon$. The curves C_{ε} for $\varepsilon \neq 0$ are smooth, and the central fiber $C_0 = XY$ is said to have a *node*. Topologically, each C_{ε} is homeomorphic to a cylinder and is shaped like a hyperboloid. As $\varepsilon \to 0$, the "waist" of the hyperboloid shrinks until C_0 is homeomorphic to a cone. This waist curve is called the *vanishing cycle* for the nodal degeneration.



FIGURE 1. A vanishing cycle.

To make honest sense of Donaldson's question, we need to know what is precisely meant by "which curves", since different vanishing cycles live over different points of \mathcal{L} : how do we compare simple closed curves in different fibers? We solve this by choosing a basepoint $B \in \mathcal{P}_{\mathcal{L}}$, a smooth curve. A nodal degeneration is a path $\alpha : [0,1] \rightarrow \mathcal{U}_{\mathcal{L}}$ such that $\alpha(0) = B, \alpha(1)$ is a curve with a single node, and $\alpha(0,1) \subset$ $\mathcal{P}_{\mathcal{L}}$ lies in the locus of smooth curves. Then the vanishing cycle c_{α} for the nodal degeneration α is a well-defined isotopy class of curve in B.

In Lecture 4 we will give a complete answer to Donaldson's question. The answer is formulated in terms of a gadget called a *r-spin structure*. We will postpone a precise statement of the answer until we have defined these. For now, let me advertise one small but intriguing consequence.

The important point to make now is the strategy of the argument, which also serves as an outline for the remainder of the lectures.

¹In his full question, Donaldson does not restrict his attention to the case of linear systems on toric *surfaces*.



FIGURE 2. For $d \ge 6$, three vanishing cycles for nodal degenerations of a degree-*d* plane curve cannot be arranged as shown.

- In Lecture 2 we will introduce the *mapping class group* Mod(Σ_g) of a topological surface Σ_g. This is a "topological automorphism group" for Σ_g in the same way that the symplectic group Sp(2g, Z) is a "homological automorphism group" for H₁(Σ_g; Z).
- (2) Also in Lecture 2, we will define a *topological monodromy representation* $\rho_{\mathcal{L}}$: $\pi_1(\mathcal{P}_{\mathcal{L}}) \to \operatorname{Mod}(\Sigma_g)$ (more generally, we will construct such a monodromy representation for *any* topological surface bundle).
- (3) Our goal will be to understand the image of $\rho_{\mathcal{L}}$. In Lecture 4 we will see how to read off the answer to Donaldson's question from $\text{Im}(\rho_{\mathcal{L}})$. As $\text{Mod}(\Sigma_g)$ is reasonably well-understood (in particular, there is a simple set of generators), it would be wonderful if we could easily show $\rho_{\mathcal{L}}$ was surjective. Unfortunately, this is rarely the case (but see the work of Crétois–Lang [CL18]).
- (4) In Lecture 3, we discuss *r*-spin structures, and see that the monodromy representations of (2) *stabilize* a certain *r*-spin structure φ_L. This means that ρ_L is valued in a "*r*-spin stabilizer subgroup" Mod(Σ_g)[φ_L]. Thus to understand Im(ρ_L), we need to understand the subgroup Mod(Σ_g)[φ_L]. If we want to show that ρ_L surjects onto Mod(Σ_g)[φ_L], we first need to find a usable set of generators. This is the heart of the entire argument; we give an outline of how this is accomplished.
- (5) Having identified a useful generating set for Mod(Σ_g)[φ_L], we still need to exhibit them in Im(ρ_L). This was accomplished by Crétois–Lang in [CL18], using methods of tropical geometry. We will say something about this in Lecture 4.

1.2. **Problem 2: abelian differentials.** I am going to switch gears now and talk about a seemingly-unrelated problem. The point, of course, is that these problems have

much more in common than it seems at first! All of the work in these lectures on abelian differentials is joint with Aaron Calderon.

We fix a genus $g \ge 3$ and consider the (*projectivized*) Hodge bundle $\mathbb{P}\Omega_g$. This is (at least morally) the vector bundle over the moduli space \mathcal{M}_g whose fiber over $C \in \mathcal{M}_g$ is the space $\mathbb{P}\Omega^0(C)$ of (projective) holomorphic differentials (a holomorphic differential is also called an *abelian differential*). A 1-form $\omega \in \Omega^0(C)$ has a divisor $\mathrm{Div}(\omega)$, i.e. the locus where ω vanishes. This induces a partition κ of 2g - 2, according to the multiplicities of the zeroes. The space we are interested in is a fixed *stratum* \mathcal{H}_{κ} : this is the set of differentials $\omega \in \Omega^0(C)$, varying over all $C \in \mathcal{M}_g$, with $\mathrm{Div}(\omega)$ inducing the partition κ .

There is again a tautological family $\pi : \mathcal{X}_{\kappa} \to \mathcal{H}_{\kappa}$, and the basic problem we seek to answer is to compute the associated monodromy representation.² The main reason we are interested in this has to do with *flat geometry*. Here is a fact that I think is equal parts wonderful and surprising:

Fact 1.3. Up to suitable equivalence relations, abelian differentials are in one-to-one correspondence with Euclidean polygons with edges identified by translation ("translation surfaces").

That is, the following pictures are actually describing holomorphic one-forms!



FIGURE 3. Two abelian differentials on genus 2 surfaces, realized as translation surfaces. The differential on the left has two zeroes of order 1, drawn in red and blue, and the differential on the left has a single zero of order 2.

²As we will see in Lecture 4, \mathcal{H}_{κ} can in fact be disconnected, and the monodromy representations on different components can be quite different!

The study of abelian differentials is a wonderful blend of algebraic geometry, flat geometry, dynamics, and more. We cannot hope to give an introduction to the geometric/dynamical aspects of the theory, and we refer the interested reader to Wright's survey article [Wri15]. The main thing we will exploit about this is the fact that it is very easy to "write down" families of Abelian differentials, by instead drawing families of translation surfaces.

We think that the study of the monodromy of strata is worthy of study in its own right, but in Lecture 4 we will say more about some specific corollaries that fall out of our calculation.

1.3. **The common thread:** *r***-spin structures.** To conclude this lecture, we give a first look at the structure that ties together Problems 1 and 2: *r*-spin structures. Here is a first definition; three more will appear in Lecture 3 (c.f. Theorem 3.4).

Definition 1.4. Let *X* be a smooth algebraic curve with canonical bundle³ K_X . An *r*-spin structure is a line bundle *L* such that $L^{\otimes r} = K_X$, i.e. an r^{th} root of K_X .

r-spin structures naturally appear in Problems 1 and 2. To explain how this happens for toric surfaces, we consider the *adjunction formula*, which computes the canonical bundle of a smooth section C of a line bundle \mathcal{L} on X:

$$K_C = (\mathcal{L} \otimes K_X) \mid_C .$$

For example, in the case $(X, \mathcal{L}) = (\mathbb{CP}^2, \mathcal{O}(d))$, we have $K_{\mathbb{CP}^2} = \mathcal{O}(-3)$, and so the canonical bundle of a smooth plane curve *C* of degree *d* is induced by restricting $\mathcal{O}(d-3) \in \operatorname{Pic}(\mathbb{CP}^2)$ to *C*. Thus $\mathcal{O}(1) \mid_C$ is an *r*-spin structure for r = d - 3.⁴ The key point here is that the globally-defined line bundle $\mathcal{O}(1)$ restricts to an *r*-spin structure on every smooth plane curve: *smooth plane curves of degree d carry canonical* (d-3)-*spin structures*.

Something very similar is true for abelian differentials in a fixed stratum \mathcal{H}_{κ} . Recall that $\kappa = \{\kappa_1, \ldots, \kappa_n\}$ is the partition of 2g - 2 induced by the zeroes of $\omega \in \mathcal{H}_{\kappa}$. We set

$$r = \gcd\{\kappa_1, \ldots, \kappa_n\}.$$

³Recall that the *canonical bundle* of a smooth variety X is the top exterior power of the cotangent bundle; in the case of X a curve this is just the cotangent bundle itself.

⁴As another suggestive remark, this d - 3 is the *same* d - 3 appearing in the lattice point count formula alluded to above.

By definition, a point $(C, \omega) \in \mathcal{H}_{\kappa}$ has divisor $\text{Div}(\omega) = \sum \kappa_i p_i$ for distinct points $p_i \in C$. Thus the divisor

$$\frac{1}{r}\operatorname{Div}(\omega) = \sum \frac{\kappa_i}{r} p_i$$

determines a canonical r^{th} root of the canonical bundle, i.e. a canonical r-spin structure.

2. INTRODUCTION TO MAPPING CLASS GROUPS, SURFACE BUNDLES, AND MONODROMY

This goal of this lecture is to give a construction of the *monodromy representation* of a surface bundle. We recall the setup. Let $p : E \to B$ be a Σ_g -bundle. We will define the *mapping class group of* Σ_g , written $Mod(\Sigma_g)$, or Mod_g for short, and a homomorphism

$$\rho_p: \pi_1(B) \to \operatorname{Mod}_g.$$

This is a topological refinement of the sort of monodromy one typically encounters in algebraic geometry, where one looks at how $\pi_1(B)$ acts on the homology of the fibers.

A standing convention in the remainder of the lectures: we always take the genus *g* to be at least 2.

2.1. The mapping class group. We first discuss the target group Mod_g . For a booklength introduction to the subject we recommend *A primer on mapping class groups* by Farb and Margalit [FM12].

In Lecture 1 we motivated the mapping class group as the group of "topological automorphisms" of Σ_g , in a similar spirit to how $\operatorname{Sp}(2g, \mathbb{Z})$ is the group of "homological automorphisms" of Σ_g . The most naïve definition of "topological automorphism group" of Σ_g would then be the group $\operatorname{Diff}^+(\Sigma_g)$ of all (orientation-preserving) diffeomorphisms of Σ_g . This is unsuitable for our purposes, however. First, this group is *massive* and it's a hopeless task to say anything meaningful about its algebraic structure. Secondly, this group records far too much information. To construct a monodromy representation to $\operatorname{Diff}^+(\Sigma_g)$ will require a choice of auxiliary "flat connection" which doesn't even exist for all surface bundles, and is non-unique even when it does.

The solution we take is to forget the fine structure of Diff and just work with diffeomorphisms up to *isotopy*. Recall that two diffeomorphisms f_0 , f_1 are *isotopic* if there exists a homotopy f_t between f_0 and f_1 such that f_t is a diffeomorphism for all

 $t \in [0, 1]$. Equivalently, an isotopy is a *path* in Diff⁺(Σ_g). This motivates our definition of Mod_g:

$$\operatorname{Mod}_g := \pi_0(\operatorname{Diff}^+(\Sigma_g)) \cong \operatorname{Diff}^+(\Sigma_g)/\operatorname{Diff}_0(\Sigma_g).$$

Here, $\text{Diff}_0(\Sigma_g)$ is the path-component of the identity, i.e. the subgroup of diffeomorphisms isotopic to the identity.

How much do we throw away when we ignore isotopy? The answer is, *nothing meaningful*. In particular we still have well-defined actions on the standard topological invariants of Σ_g , namely homology and homotopy. Below, recall that Out(G) is the group of *outer automorphisms* of *G*, i.e. Aut(G)/Inn(G), where Inn(G) is the group of "inner" automorphisms (*G* acting on itself by conjugation).

Lemma 2.1. The mapping class group has a well-defined action on both $\pi_1(\Sigma_g)$ and $H_1(\Sigma_g; \mathbb{Z})$. That is, there are homomorphisms

$$\Pi: \mathrm{Mod}_g \to \mathrm{Out}(\pi_1(\Sigma_g))$$

and

$$\Psi: \operatorname{Mod}_q \to \operatorname{Sp}(2g, \mathbb{Z}).$$

Remark 2.2. The theorem of Dehn-Nielsen-Baer asserts that Π is "almost an isomorphism". If we extend the definition of Mod_g to allow our mapping classes to reverse orientation (a group typically written $\operatorname{Mod}_g^{\pm}$), then $\Pi : \operatorname{Mod}_g^{\pm} \to \operatorname{Out}(\pi_1(\Sigma_g))$ is an isomorphism. Thus $\Pi(\operatorname{Mod}_g)$ is the index-2 subgroup $\operatorname{Out}^+(\pi_1(\Sigma_g))$ defined as those outer automorphisms that preserve the "orientation" on $\pi_1(\Sigma_g)$.

Dehn twists. It is a basic fact that $Sp(2g, \mathbb{Z})$ is generated by *transvections*, automorphisms of the form

$$T_v(x) = x + \langle x, v \rangle v,$$

where $\langle \cdot, \cdot \rangle$ denotes the symplectic (intersection) pairing on $H_1(\Sigma_g; \mathbb{Z})$. In fact, you can write down a simple explicit collection of 2g + 1 transvections that generate, in much the same way that you can show that $SL_n(\mathbb{Z})$ is generated by an explicit collection of elementary matrices. There is an analogous statement for the mapping class group, where we give a topological incarnation to a transvection as a so-called *Dehn twist*.

We start by defining a Dehn twist on a cylinder: hold one end fixed, and apply a full rotation to the other boundary component. See Figure 4. This is the local model for a general Dehn twist. Let $a \subset \Sigma_g$ be a simple closed curve, and enlarge a into a



FIGURE 4. Left: the local model of a Dehn twist about the blue curve. Right: Dehn twisting on a surface. The red curve has been smoothed out by an isotopy.

cylinder C_a with core curve a. The *Dehn twist* T_a is the element of Mod_g represented by performing a full twist on the cylinder C_a , leaving $\Sigma_g \setminus C_a$ fixed. You can check that the *mapping class* T_a is well-defined independently of the various choices made. The theorem below is a fundamental result on mapping class groups. The finite generation statement is due to Dehn and Lickorish, and the generating set is due to Humphries.

Theorem 2.3 (Dehn–Lickorish, Humphries). For $g \ge 2$, Mod_g is generated by the collection of 2g + 1 Dehn twists shown in Figure 5



FIGURE 5. The Humphries generators for Mod_g , illustrated for g = 4.

Thus Dehn twists play a similar role to transvections in the theory of $\text{Sp}(2g, \mathbb{Z})$. In fact, they are very closely related:

Fact 2.4. Let $a \subset \Sigma_g$ be a simple closed curve. Then

$$\Psi(T_a) = T_{[a]}$$

where the element $T_{[a]} \in \text{Sp}(2g, \mathbb{Z})$ is the transvection about the homology class $[a] \in H_1(\Sigma_q; \mathbb{Z})$.

Dehn twists are the fundamental building blocks of mapping classes. To give you a taste of the subject, and to set the stage for Lecture 3, we present below some of the basic relations satisfied by collections of Dehn twists.

Proposition 2.5 (Dehn twists and Artin groups).

- (1) If simple closed curves a and b are disjoint, then the corresponding Dehn twists commute: $T_aT_b = T_bT_a$.
- (2) If simple closed curves a and b intersect transversely exactly once, then the corresponding Dehn twists satisfy Artin's braid relation: $T_aT_bT_a = T_bT_aT_b$.

If these relations look familiar, that's because this is the same set of relations that define the *braid group* B_n . This is a powerful fact (and is far from being an accident): mapping class groups and braid groups are closely related. Group relations in the braid group (e.g. the existence of a center) can be ported into the mapping class group by embedding the generators of B_n as Dehn twists in the right configuration. More generally, an *Artin group* is any group where all relations are of the above form. Other Artin groups (those of "finite type") have similar interesting relations in them which will play a very important (if somewhat occluded) role in the guts of the main theorem discussed in Lecture 3.

Change-of-coordinates. In linear algebra, we are used to the idea of changing bases and working coordinate-free. There is a similar sort of principle at work in the mapping class group which is worth briefly discussing here.

One of the basic facts in linear algebra is that any linearly-independent set can be extended to a basis. This implies that GL_n acts *transitively* on the set of "configurations" of vectors of a given cardinality, where by "configuration" here we simply mean the property of being linearly-independent. For surfaces, simple closed curves play the role of vectors, but here "configuration" can mean any number of things, typically referring to some prescribed pattern of intersection. Where linear algebra uses the basis-extension lemma to take one configuration to another, changeof-coordinates in the mapping class group uses the classification of surfaces. We give

a simple illustration of this. Recall a simple closed curve $c \subset \Sigma_g$ is *nonseparating* if $\Sigma_g \setminus c$ is connected; this is well-defined on the level of isotopy.

Lemma 2.6. Mod_g acts transitively on the set of isotopy classes of nonseparating simple closed curves.

Proof. Let *c* and *d* be nonseparating. The surfaces $\Sigma_g \setminus c$ and $\Sigma_g \setminus d$ are both connected, both have Euler characteristic $\chi(\Sigma_g)$ and two boundary components (and are oriented). By the classification of surfaces, there exists a diffeomorphism $f : \Sigma_g \setminus c \rightarrow \Sigma_g \setminus d$. This takes boundary components to boundary components and so can be extended to a diffeomorphism of Σ_g to itself that takes *c* to *d* as required. \Box

More sophisticated incarnations of this principle are at work in essentially every substantial theorem about the mapping class group; it is used ubiquitously to invoke the existence of desired configurations of curves, e.g. on arbitrary subsurfaces, or to guarantee the ability to extend configurations of curves to larger configurations, etc. It is what allows us to work coordinate-free, i.e. without having to draw explicit pictures of very complicated curves, or having to find and work with some elaborate scheme for parameterizing simple closed curves. See [FM12, Section 1.3] for a more in-depth discussion.

2.2. **Surface bundles and monodromy.** Recall that a *surface bundle* of genus *g* is a fiber bundle

$$p: E \to B$$

with fibers diffeomorphic to Σ_g . In Lecture 1 we met the surface bundles we will eventually study: the family of smooth sections of a linear system on a smooth toric surface, and the family of abelian differentials in a given stratum. How does one actually study such things? Again we look to a "linear" analogue for inspiration.

The most straightforward analogue would be the theory of vector bundles. Here, a central role is played by the characteristic classes of the bundle. While there is a theory of characteristic classes for surface bundles (c.f. [Mor01, Chapter 4]), these are fairly coarse invariants, and in any case, the theory is badly underdeveloped.⁵

⁵We understand the so-called "stable classes" very well thanks to the theorem of Madsen–Weiss. But an Euler characteristic computation of Harer–Zagier reveals that only a vanishingly small portion of the ring of characteristic classes of surface bundles are stable, and almost nothing is known about these unstable classes.

A better analogy is to theory of local systems (locally-constant sheaves). Here, the *monodromy* of a flat vector bundle is a fundamental (indeed, *complete*) invariant.

(Optional) Monodromy of local systems. Let \mathcal{F} be a local system over B with stalks an abelian group A, and let $b \in B$ be a basepoint. From this we seek to construct a homomorphism

$$\rho: \pi_1(B, b) \to \operatorname{Aut}(A)$$

Since \mathcal{F} is a local system, there exists a covering \mathcal{U} of B such that $\mathcal{F} \mid_U$ is the constant sheaf A for $U \in \mathcal{U}$. Hence if $x, y \in U$ are points, there is a canonical identification of stalks $\mathcal{F}_x \cong \mathcal{F}_y$.

Let $\gamma : [0,1] \to B$ be a loop based at B. We can partition [0,1] into segments $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that for each $1 \leq i \leq n$, there is a containment $\gamma(t_{i-1}, t_i) \subset U$ for some $U \in \mathcal{U}$. Thus there are canonical isomorphisms $f_i : \mathcal{F}_{\gamma(t_{i-1})} \to \mathcal{F}_{\gamma(t_i)}$, and composing, we get an automorphism

$$\rho(\gamma) := f_n \circ \cdots \circ f_1 : \mathcal{F}_b \to \mathcal{F}_b.$$

Moreover, it is straightforward to show that $\rho(\gamma)$ truly depends only on the homotopy class of γ .

A rigorous construction of the monodromy map for surface bundles would proceed along these lines: we would show that all of the choices made along the way only change the diffeomorphism we produce by an isotopy.

Monodromy for surface bundles. For expediency's sake we will give a somewhat ad-hoc definition of monodromy for surface bundles and will elide some subtleties.⁶ For a more comprehensive discussion, see [FM12, Section 5.6]. We start with the simple case where $B = S^1$.

Example 2.7 (Surface bundles over S^1 : the mapping torus construction). Let $f \in$ Diff⁺(Σ_q) be chosen. Associated to this is the *mapping torus* M_f , a closed 3-manifold:

$$M_f = \Sigma_g \times [0, 1]/(x, 1) \sim (f(x), 0).$$

The projection onto the second coordinate defines a fibration $p: M_f \to S^1$. We make the following (not difficult) assertions:

⁶For the careful reader: we are ignoring the fact that the monodromy representation is well-defined only up to global conjugacy (that is, there is really only a Mod_g -conjugacy class of homomorphisms $\rho : \pi_1 B \to Mod_g$). To convert the discussion below into factual assertions, replace equality of mapping classes with equality up to conjugacy throughout.

- (1) The diffeomorphism type of M_f as a 3-manifold depends only on the class [f] ∈ Mod_g. Moreover, mapping tori M_{f1} and M_{f2} are isomorphic as Σ_g-bundles (i.e. via a diffeomorphism covering id : S¹ → S¹) if and only if there is an equality f₁ = f₂ holding in Mod_g.
- (2) Conversely, if $p: M \to S^1$ is a Σ_g -bundle over S^1 , then M has the structure of a mapping torus: there is a bundle isomorphism $M \cong M_f$ for some uniquely-specified $f \in \operatorname{Mod}_q$.⁷

We define the *monodromy* of $p: M_f \to S^1$ to be the associated element $f \in Mod_q$.

We can use this fact to define the monodromy representation for an arbitrary surface bundle. Let $p : E \to B$ be a Σ_g -bundle. Choose a basepoint $b \in B$. Let $\gamma : S^1 \to B$ be a loop based at b. We can pull E/B back along γ to obtain a surface bundle M_{γ} over S^1 . By the above, there is an element $f_{\gamma} \in Mod_g$. We define the *monodromy representation* of $p : E \to B$ to be the homomorphism

$$\rho: \pi_1(B, b) \to \operatorname{Mod}_g; \qquad \gamma \mapsto f_\gamma.$$

It is not hard to see that $\rho(\gamma)$ depends only on the homotopy class of γ and hence is well-defined as a function, and to convince yourself that ρ is a homomorphism.

Examples. The monodromy of a surface bundle around a loop measures how the fiber gets "twisted" as it moves along the loop. To close this lecture we give some examples of how this works in practice.

Example 2.8 (Vanishing cycles and the Picard–Lefschetz formula). We return to the nodal degeneration family C_{ε} discussed in Lecture 1. We can view this as a surface bundle over $\Delta^* = \{\varepsilon \in \mathbb{C} \mid 0 < |\varepsilon| \leq 1\}$. What is the monodromy of the loop determined by $t \mapsto C_{e^{2\pi i t}}$? The *Picard-Lefschetz formula* says that the homological monodromy is given by the transvection T_v , where $v \in H_1(C_1; \mathbb{Z})$ is the core curve of the cylinder, i.e. the *vanishing cycle*. In light of Fact 2.4 it should not be surprising to learn that the Mod_q-monodromy is the Dehn twist T_a about the vanishing cycle.

⁷We advise the reader that the classification of mapping tori up to mere *diffeomorphism* (i.e. not preserving the surface bundle structure) is a much more subtle and rich problem. Thurston classified the set of ways that a given 3-manifold M can be given the structure of a surface bundle [Thu86]. This includes the following fascinating trichotomy: a given M either has zero structures as a mapping torus, or else $M \cong M_f$ for a unique (up to conjugacy) $f \in Mod_g$, or else $M \cong M_{f_i}$ for an infinite collection of mapping classes $f_i \in Mod_{g_i}$, where the genera of possible fibers g_i is necessarily unbounded. This was one of the first facts about surface bundles that really grabbed my attention!

To see how to compute the monodromy in this example, we follow the presentation given by [AGZV12, pp. 2-5]. See Figure 6. We work with the equivalent model $C_{\varepsilon} = V(X^2 + Y^2 - \varepsilon)$ of a nodal degeneration. The projection onto the *X*-coordinate represents C_{ε} as a double cover of \mathbb{C} , branched at $X = \pm \sqrt{\varepsilon}$. We can view the cylinder C_{ε} as two copies of \mathbb{C} glued together along a slit running between ε , $-\varepsilon$. As we orbit the singularity once along the path $\varepsilon(t) = e^{2\pi i t}$, the slit makes only a half-rotation. What was originally a curve running directly from top to bottom of the cylinder now has been twisted once: this is a Dehn twist!



FIGURE 6. The monodromy of a vanishing cycle is a Dehn twist.

This is a very important example that lies at the heart of our study of vanishing cycles in toric surfaces. It tells us that we can keep track of vanishing cycles by keeping track of Dehn twists in our monodromy.

Example 2.9 (Shearing of translation surfaces). Consider the translation surface X_0 depicted below at left. We consider two translation surfaces to be equivalent if one can be cut up and rearranged (using only Euclidean translations) to give the other. Thus the two translation surfaces X_0 and X_1 at left and right correspond to the same point $(C, \omega) \in \mathcal{H}_{\kappa}$ (here $\kappa = \{1, 1\}$). We can interpolate between X_0 and X_1 by the family X_t , and the assignment $t \mapsto X_t$ determines a loop in \mathcal{H}_{κ} . What is the monodromy of this loop? Convince yourself that the monodromy is the Dehn twist

along the core curve of the cylinder indicated below. Such a deformation is called a *shear*. These seem to play a fundamental role in the structure of $\pi_1(\mathcal{H}_{\kappa})$ (they do not in general *generate* $\pi_1(\mathcal{H}_{\kappa})$, but they may generate a finite-index subgroup; it's one of many many things waiting to be understood about $\pi_1(\mathcal{H}_{\kappa})$!).



FIGURE 7. The monodromy of a shear is a Dehn twist.

3. HIGHER SPIN STRUCTURES AND THEIR MAPPING CLASS GROUPS

In Lecture 1 we met two (families of) surface bundles: the family of smooth sections of a line bundle on a toric surface, and the family of abelian differentials in a fixed stratum. We also saw a first glimpse of the structure connecting these two: the existence of canonical r^{th} roots of the canonical bundles of the fibers, i.e. r-spin structures. The purpose of this lecture is to explain how the presence of this structure is reflected in the associated monodromy representations. In the first part of the lecture, we will study r-spin structures themselves and recast them from a more

topological point of view. Then in the second part, we use this new perspective to study the stabilizer subgroups $Mod_g[\phi]$.

3.1. *r*-spin structures. To an algebraic geometer, the definition of an *r*-spin structure given above (Definition 1.4) is perfectly satisfactory: it is just an r^{th} root of the canonical bundle. To a topologist, however, this leaves a lot to be desired. It is not even clear if there is any topological meaning that can be assigned to an *r*-spin structure, nor whether (as there ought to be) there is an action of the mapping class group on the set of *r*-spin structures. In this section, we will see how *r*-spin structures correspond to purely topological gadgets called $\mathbb{Z}/r\mathbb{Z}$ winding number functions. These were originally defined and investigated⁸ by Humphries–Johnson [HJ89].

Definition 3.1 ($\mathbb{Z}/r\mathbb{Z}$ winding number functions). For a topological surface Σ , let $S(\Sigma)$ denote the set of isotopy classes of oriented simple closed curves on Σ . A $\mathbb{Z}/r\mathbb{Z}$ *winding number function* is a function

$$\phi: \mathcal{S}(\Sigma) \to \mathbb{Z}/r\mathbb{Z}$$

that satisfies the following axioms:

- (1) (Reversibility) If \bar{c} is the same curve *c* endowed with the opposite orientation, then $\phi(\bar{c}) = -\phi(c)$.
- (2) (Twist-linearity) If $a, b \in \mathcal{S}(\Sigma)$, then

$$\phi(T_b(a)) = \phi(a) + \langle a, b \rangle \phi(b).$$

(3) (Homological coherence) Let $S \subset \Sigma$ be a subsurface with boundary components c_1, \ldots, c_k , each oriented so that *S* lies to the left. Then

$$\sum_{i=1}^{k} \phi(c_i) = \chi(S).$$

Remark 3.2 (r divides 2g - 2). We know that if L is an r-spin structure, then r must divide 2g - 2 since degree of a line bundle is multiplicative. As a first hint as to the connection between r-spin structures and $\mathbb{Z}/r\mathbb{Z}$ winding number functions, we show here that homological coherence implies that $r \mid 2g - 2$ for winding number functions as well. Let $c \subset \Sigma_g$ be a curve separating Σ_g into subsurfaces $S \cong \Sigma_{1,1}$ and $S' \cong \Sigma_{g-1,1}$. When we orient c with S to the left, homological coherence tells us that

⁸In a somewhat more general context.

 $\phi(c) = -1$. Then reversibility and homological coherence tells us that $\phi(c) = 2g - 3$ as well, since *c* also bounds *S'* to the right. Thus -1 = 2g - 3 in $\mathbb{Z}/r\mathbb{Z}$, i.e. $r \mid 2g - 2$.

As a sub-remark, this example illustrates an important fact about winding number functions: they are not cohomology classes (at least on Σ_g itself)! The curve *c* is null-homologous, but carries a nonzero ϕ -value. In Theorem 3.4 below we will understand that ϕ *is* a cohomology class, just on a different space.

Remark 3.3. The terminology suggests that we should think of the value $\phi(c)$ as some sort of "winding number" of the oriented curve c. One way to construct winding number functions that makes this precise is as follows. Let ξ be a vector field on Σ that has isolated zeroes p_1, \ldots, p_n . If c is a C^1 -immersed curve on $\Sigma_g \setminus \{p_1, \ldots, p_k\}$, then we can compute the winding number of the forward-pointing tangent vector c'(t) with respect to ξ , giving us an integral winding number of c. You can check (alternatively, appeal to the Poincaré-Hopf theorem) that the winding number of cthen changes by $\operatorname{ord}_{\xi}(p_i)$ when c is isotoped across the zero p_i . Thus, setting

$$r = \operatorname{gcd}\operatorname{ord}_{\xi}(p_1), \ldots, \operatorname{ord}_{\xi}(p_k),$$

we see that this procedure gives a well-defined $\mathbb{Z}/r\mathbb{Z}$ -winding number function on Σ_g .

Observe that this structure appears on an abelian differential. A differential (C, ω) determines a vector field (at least up to a fiberwise action of \mathbb{R}^+) on C by taking tangent vectors v with $\omega(v) \in \mathbb{R}_{\geq 0}$, and r as we have defined it here agrees with the definition of r we supplied in Lecture 1. Below we see that this is not a coincidence.

Theorem 3.4. Let *C* be a Riemann surface of genus *g*. Let *UTC* denote the unit tangent bundle of *C*; this is an S^1 bundle over *C* with fiber represented by a loop ζ (counterclockwise rotation in the tangent space at a single point). Then for $r \mid 2g - 2$, the following sets are in natural bijection:

- (1) The set of r-spin structures on C (r^{th} roots L of K_C),
- (2) The set of $\mathbb{Z}/r\mathbb{Z}$ fiberwise connected covers of UTC up to a choice μ of primitive r^{th} root of unity,
- (3) The coset of $H^1(UTC; \mathbb{Z}/r\mathbb{Z})$ defined by the condition $\phi(\zeta) = \mu$ (identifying $\mathbb{Z}/r\mathbb{Z}$ with the group of r^{th} roots of unity),
- (4) The set of $\mathbb{Z}/r\mathbb{Z}$ winding number functions on *C*.

Proof. We will provide a sketch of the main ideas.

 $[(1) \iff (2)]$ Let L be an r-spin structure. We adopt the following notation: for a line bundle V over C, let E(V) denote the total space of the bundle, and let $E(V)^{\circ}$ be obtained from E(V) by deleting the zero section. As $L^{\otimes r} = K_C$, there is a $\mathbb{Z}/r\mathbb{Z}$ cover $E(L) \to E(K_C)$ which is unbranched when restricted to $E(L)^{\circ} \to E(K_C)^{\circ}$. Fiberwise, this is represented by the covering $z \mapsto z^r$. Thus an r-spin structure determines a $\mathbb{Z}/r\mathbb{Z}$ unbranched cover of $E(K_C)^{\circ}$. The unit tangent bundle UTC is a fiberwise deformation-retract of $E(K_C)^{\circ}$, and so an r-spin structure gives rise to an element of (2). The covers so arising all lift ζ to an arc in \mathbb{C} connecting 1 to $e^{2\pi i/r}$, thus fixing a choice of primitive root of unity. It is not hard to see that different r-spin structures give rise to distinct covers (if L, L' are distinct, there must be some loop $\gamma \in C$ which has different holonomy for L and for L' and hence the associated covers assign different lifts to γ), and there are r^{2g} of each, establishing the bijection.

[(2) \iff (3)] This is basic covering space theory. For an abelian group *A* and a space *X*, the set of *A*-covers of *X* is in bijection with $H^1(X; A)$. If $\gamma \in X$ is a loop and $\phi \in H^1(X; A)$ is a class, the assignment $\gamma \mapsto \phi(\gamma) \in A$ determines which sheet of the cover γ lifts to, so that the choice of μ in (2) corresponds to the coset condition $\phi(\zeta) = \mu$ in (3).

 $[(3) \iff (4)]$ The direction $(3) \implies (4)$ proceeds via the *Johnson lift*. Let $c \in S(C)$ be an oriented simple closed curve. A loop in *UTC* is the same data as a *framed curve* on *C*; i.e. a curve *c* on *C* equipped with a choice of nonvanishing vector everywhere along *c*. The *Johnson lift* of $c \in S(C)$ is the loop $\hat{c} \subset UTC$ obtained by framing *c* via the forward-pointing tangent vector. If $\phi \in H^1(UTC; \mathbb{Z}/r\mathbb{Z})$ is an element of (3), we define the function

$$\phi: \mathcal{S}(C) \to \mathbb{Z}/r\mathbb{Z}; \qquad c \mapsto \phi(\widehat{c}),$$

i.e. by evaluating the cohomology class ϕ on the 1-cycle \hat{c} . The twist-linearity and homological coherence properties now follow from the fact that ϕ is a cohomology class on UTC.

The reverse direction (4) \implies (3) is due to Humphries–Johnson. They proceed by showing that twist-linearity and homological coherence imply that ϕ (a $\mathbb{Z}/r\mathbb{Z}$ winding number function) is determined by its values on a set of 2g + 1 curves determining a basis (under the Johnson lift) for $H_1(UTC;\mathbb{Z})$. **Remark 3.5.** As alluded to in the proof, it is *a priori* known that sets (1), (2), (3) each have cardinality r^{2g} , and hence when $r \mid 2g - 2$, there are exactly $r^{2g} \mathbb{Z}/r\mathbb{Z}$ -winding number functions on Σ_g as well.

3.2. The mapping class group action; stabilizers. Following Theorem 3.4, we will view an *r*-spin structure as a $\mathbb{Z}/r\mathbb{Z}$ winding number function. This allows us to understand how the mapping class group acts on the set of *r*-spin structures. Mod_{*g*} acts on the set of *r*-spin structures (via its action on winding number functions) from the left via

$$(f \cdot \phi)(c) = \phi(f^{-1}(c)).$$

For a fixed *r*-spin structure ϕ , we define the *spin structure stabilizer group* as

$$\operatorname{Mod}_{g}[\phi] = \{ f \in \operatorname{Mod}_{g} \mid f \cdot \phi = \phi \}.$$

Following the discussions in Lectures 1 and 2, we can now say that for both problems we are studying, the monodromy representation is valued in a particular $Mod_g[\phi]$. We will eventually show that in both these cases, the monodromy *surjects* onto $Mod_g[\phi]$. In order to do this, we need to give an explicit description of a generating set. That is our objective for the remainder of the lecture, and indeed forms the technical heart of the entire discussion.

Admissibility. To formulate the statement, we introduce a key idea. As discussed in Lecture 2, Dehn twists are the basic building blocks of mapping classes. We would like to understand the collection of Dehn twists that stabilize ϕ . To understand this, we consider the twist-linearity formula. This tells us that if $\phi(a) = 0$, then necessarily $\phi(T_a(b)) = \phi(b)$ for all $b \in S(\Sigma_g)$, i.e. $T_a \in \text{Mod}_g[\phi]$. And conversely, if $\phi(b) \neq 0$ and b is nonseparating, then you can find a curve c with $\langle b, c \rangle = 1$, and then $\phi(T_b(c)) \neq \phi(c)$. In other words, c witnesses the fact that T_b does not preserve ϕ .

(As a remark, twist-linearity also implies that if *a* is a separating curve, then $T_a \in \text{Mod}_g[\phi]$ since in that case $\langle a, \cdot \rangle$ is identically zero. Separating twists will play an important role later on, but we will not consider them to be "admissible" in the sense to be defined below).

Definition 3.6 (Admissible curve, admissible twist, admissible subgroup). A nonseparating curve $a \subset \Sigma_g$ is said to be *admissible* (with respect to an implied ϕ) if $\phi(a) = 0$ (necessarily for either choice of orientation). The corresponding Dehn twist $T_a \in Mod_g[\phi]$ is called an *admissible twist*, and the group

$$\mathcal{T}_{\phi} = \langle T_a \mid a \text{ admissible} \rangle \leq \operatorname{Mod}_g[\phi].$$

is called the *admissible subgroup*.

Since there are finitely many *r*-spin structures, the subgroup $\operatorname{Mod}_g[\phi]$ is of finite index in Mod_g . Many finite-index subgroups do not contain any Dehn twists at all (of course every $\Gamma \leq \operatorname{Mod}_g$ contains *powers* of every Dehn twist), and even though Mod_g itself is generated by Dehn twists, this absolutely does not directly imply that $\operatorname{Mod}_g[\phi]$ is generated by the Dehn twists it contains (the admissible twists). Remarkably, this is true for spin structure stabilizer groups.

Theorem 3.7 (Generation by admissible twists). Let ϕ be an *r*-spin structure on Σ_g for $g \geq 5$. Then there is an explicit finite collection a_1, \ldots, a_k of admissible curves such that

$$\operatorname{Mod}_{g}[\phi] = \langle T_{a_{1}}, \ldots, T_{a_{k}} \rangle.$$

Moreover, if r = 2g - 2 we can take k = 2g and otherwise we can take k = 2g + 1.

Remark 3.8. There are more precise formulations of Theorem 3.7 that supply you with exact configurations of twists that generate $Mod_g[\phi]$; see [Sal19, Theorem 9.5] and [CS19, Theorem B]. I am choosing to be imprecise here in the statement of Theorem 3.7 for expediency's sake, since giving precise formulations would be tedious and ultimately not worth the expenditure in time. In subsequent work I'll invoke the precise formulations; you'll just have to trust me when I assert that the given collections of twists generate.

Example 3.9. To illustrate the theorem, Figure 8 shows some examples of the sorts of generating sets produced by Theorem 3.7.

Remark 3.10 (Arf invariants and orbits of *r*-spin structures). The difference in the structure of the configurations of curves in the second and third examples above is explained as follows. If ϕ and ψ are *r*-spin structures in the same Mod_g -orbit, then their stabilizers are conjugate subgroups in Mod_g and hence the generating sets "look the same" (c.f. our discussion of the change-of-coordinates principle in Lecture 2). The orbit structure of Mod_g on *r*-spin structures is classified by something called the "Arf invariant". In the interest of simplicity, I will attempt to avoid a discussion of the Arf invariant, but see [Sal19, Sections 3.4, 4.2] and/or [CS19, Section 2.3].



FIGURE 8. Three examples of the sorts of generating sets produced by Theorem 3.7. The left generates the stabilizer of a 3-spin structure on a surface of genus 10. The top right generates an 8-spin structure stabilizer, and the bottom right generates a 4-spin stabilizer subgroup. In each case, the spin structure is the one implicitly but uniquely specified by the condition that each curve shown is admissible.

For now, it suffices to know that if r is odd, there is exactly one orbit and if r is even there are exactly two. The different configurations in the examples above show generating sets for different Arf invariants.

3.3. **Proving Theorem 3.7.** A full proof of Theorem 3.7 is quite involved: the original version in [Sal19] occupies the bulk of a 61 page paper, and the improvements needed for [CS19] took another 30 or so (though the second paper largely improves arguments from the first instead of adding new ones). Nevertheless it is not difficult to explain the argument in outline. It breaks up into two halves. In the first half, we start with an explicit finite collection of admissible twists, and we show that this finite collection of twists generates the full admissible subgroup \mathcal{T}_{ϕ} . Then in the second half we show that the admissible subgroup is the full stabilizer: $\mathcal{T}_{\phi} = \text{Mod}_{g}[\phi]$.

Step 1: Generating admissible twists. The basic principle at work here is a common tool in geometric group theory: we make use of an auxiliary simplicial complex (in fact, just a graph). In this case, our goal is to show that the finite set $\{T_{a_1}, \ldots, T_{a_k}\}$ generates *all* admissible twists. We achieve this by building the *admissible curve graph* $C_{\phi}(\Sigma_g)$. The vertices in this graph correspond to admissible curves, and we join a, a' by an edge if they are disjoint. Observe that $Mod_g[\phi]$ acts on this graph by simplicial automorphisms, but that Mod_g does not.

Lemma 3.11. The admissible curve graph $C_{\phi}(\Sigma_g)$ is connected for all $g \geq 3$.

Proof. (Sketch) We use the "hitchhiking principle": we find a path between arbitrary vertices a, b by first finding a path between them in a related graph that is known to be connected, and then joining each pair of adjacent vertices in this auxiliary graph by a path in $C_{\phi}(\Sigma_g)$. In this case, the auxiliary graph is the graph of genus 1 subsurfaces, known to be connected for $g \ge 3$ by the "Putman trick" (c.f. [Put08] or [Sal19, Theorem 7.1]). We enclose a, b inside genus 1 subsurfaces S_a and S_b and connect these by a sequence of disjoint subsurfaces. Using the theory of winding number functions, we can find admissible curves on every genus-1 subsurface, and thus build our path of admissible curves by following the path of disjoint genus-1 subsurfaces.

Morally, the proof of step 1 proceeds as follows: we develop methods to show that if we start with a certain configuration of admissible curves in some localized region $R \subset C_{\phi}(\Sigma_g)$, we can "expand out" one step, writing the admissible twists T_a for a any curve in the 1-neighborhood of R as a product of the admissible twists in R. Since $C_{\phi}(\Sigma_g)$ is connected, as we expand this way starting from our starting configuration, we are guaranteed to eventually hit all admissible twists.⁹

Step 2: From admissible twists to the full stabilizer. Step 1 found a finite generating set for the admissible subgroup \mathcal{T}_{ϕ} . We now need to show that $\mathcal{T}_{\phi} = \text{Mod}_g[\phi]$. The way we do this is by examining both groups from the point of view of a filtration on

⁹For readers interested in reconciling this remark with the paper [Sal19], here are some guideposts. The "methods" alluded to for expanding outwards in the complex of admissible curves is a reference to the theory of "spin subsurface push subgroups" discussed in [Sal19, Section 8]. For the technical reasons explained in [Sal19, proof of Proposition 8.2], we actually structure our argument around the complex of genus 2 subsurfaces, which is connected only for $g \ge 5$. This is the source of the stipulation $g \ge 5$ in Theorem 3.7.

the mapping class group. This is called the *Johnson filtration*; here is a brief summary of the relevant portion of the theory.

Definition 3.12 (Johnson filtration). We will only need to know about the first two terms. The first term is the *Torelli group* \mathcal{I}_g , defined as the kernel of the symplectic representation $\Psi : \operatorname{Mod}_g \to \operatorname{Sp}(2g, \mathbb{Z})$. The second term is the *Johnson kernel* \mathcal{K}_g . In [Joh80], Johnson defined a homomorphism

$$\tau: \mathcal{I}_q \to V_q,$$

where V_g is a certain finite-rank \mathbb{Z} -module, and defined \mathcal{K}_g as the kernel of τ . In a subsequent paper [Joh85], he showed that \mathcal{K}_g is generated by separating twists:

$$\mathcal{K}_g = \langle T_c \mid c \text{ separating} \rangle$$

Remark 3.13. Recall that we observed from the twist-linearity formula that if *c* is separating, the twist T_c preserves ϕ . Thus we have a containment $\mathcal{K}_g \leq \operatorname{Mod}_g[\phi]$.

To show that $\mathcal{T}_{\phi} = \text{Mod}_g[\phi]$, it suffices to show that they "look the same" from the perspective of the Johnson filtration. This has three pieces. For the interested reader, we note where to find the corresponding arguments in the papers [Sal19, CS19].

- (1) Identify the image $\Psi(\operatorname{Mod}_g[\phi])$, and show that $\Psi(\mathcal{T}_{\phi}) = \Psi(\operatorname{Mod}_g[\phi])$ [Sal19, Lemmas 5.4 and 6.4].
- (2) Identify the image $\tau(\operatorname{Mod}_{g}[\phi])$, and show that $\tau(\mathcal{T}_{\phi}) = \tau(\operatorname{Mod}_{g}[\phi])$ [CS19, Lemma 6.5].
- (3) Show that $\mathcal{K}_g \leq \mathcal{T}_{\phi}$ [CS19, Lemma 6.4].

We close the lecture with a brief comment on each of these steps. In each case, the constructive step (constructing suitable elements in \mathcal{T}_{ϕ}) has two main ingredients: a more sophisticated version of the change-of-coordinates principle that takes ϕ into account (this allows us to "work coordinate-free"), and a facility with relations in the mapping class group (e.g. that allow us to express separating twists as products of nonseparating twists).

(1) If *r* is odd then we can show directly that Ψ(*T*_φ) = Sp(2*g*, ℤ) by constructing admissible curves in arbitrary homology classes and running the admissible twists through Ψ. If *r* is even then Mod_g[φ] preserves a mod-2 spin structure, which is well-defined on the level of homology and hence Ψ(Mod_g[φ]) stabilizes this; we can again directly show that Ψ(*T*_φ) is as large as possible.

- (2) The image τ(Mod_g[φ]) turns out to be closely related to a classical portion of the theory of the Torelli group - the *Chillingworth homomorphism*. Once this is recognized, it's not hard to complete the work of the step.
- (3) This step is just lots and lots of applications of relations in the mapping class group (chain relation, lantern relation, etc.). Things get substantially more subtle in the case of *r* even (where there is an Arf invariant to worry about); in a forthcoming paper [CS20], Aaron and I finally have written down the "definitive" proof of this.

4. MONODROMY PROBLEMS (II): COMPUTATIONS AND CONSEQUENCES

Back in Lecture 1 we formulated the main problems we are interested in solving. We found two classes of surface bundles (families of Riemann surfaces) for which the monodromy

$$\rho: \pi_1(B) \to \operatorname{Mod}_g[\phi]$$

was valued in a spin structure stabilizer group. We explained how the ultimate goal was to be able to show that such ρ is surjective, for which we needed to have a criterion for a collection of elements in $Mod_g[\phi]$ to generate. This was accomplished in the previous lecture in Theorem 3.7. In this final lecture we will return to a study of the families themselves. Our task is now to find a sufficient collection of elements of $Im(\rho)$ to satisfy Theorem 3.7; to do this we will need to understand more about the families themselves.

4.1. Linear systems on toric surfaces. This will be an exposition of the work of Crétois–Lang [CL18]. They found a method to write down a large finite collection of Dehn twists in the image of the monodromy map for any linear system \mathcal{L} on a smooth toric surface X. Recall from Example 2.8 that Dehn twists arise as the monodromy associated to a nodal degeneration. Thus, find lots of nodal degenerations and you've found lots of Dehn twists. The challenge arises in keeping track of how different degenerations (different vanishing cycles/twists) interact with/ intersect each other.

The basic idea of Crétois–Lang is to use the methods of tropical geometry to find a beautiful combinatorial scheme for creating lots of degenerations in such a way that it is possible to track the relative locations of the vanishing cycles. I cannot comment on the details of their argument, and I refer the interested reader to their paper [CL18]. What I can do is explain what their theory looks like and how it interfaces with the

theory we've developed. We follow [Sal19, Sections 10,11].

Linear systems on toric surfaces: what do they look like? Answer: they look like this (see Figure 9).



FIGURE 9. Representing the line bundle $\mathcal{O}(6)$ on \mathbb{CP}^2 as a lattice polygon.

Let's explain. We start with the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. A lattice polygon Δ (i.e. vertices of Δ lie in \mathbb{Z}^2) determines a line bundle \mathcal{L}_{Δ} on a toric surface X_{Δ} as follows: let $S = \Delta \cap \mathbb{Z}^2$ be the set of lattice points in Δ ; say this has cardinality m. Enumerate the points in S as (a_i, b_i) for $i = 1, \ldots, m$. This specifies an embedding $(\mathbb{C}^{\times})^2 \to \mathbb{CP}^{m-1}$ via

$$(z,w)\mapsto [z^{a_1}w^{b_1}:\cdots:z^{a_m}w^{b_m}].$$

The toric surface X_{Δ} is defined as the closure of $(\mathbb{C}^{\times})^2$ under this embedding, and the line bundle \mathcal{L}_{Δ} is the restriction of $\mathcal{O}(1)$.

Example 4.1. Consider the polygon Δ given as the convex hull of (0,0), (1,0), (0,1). Then the embedding is

$$(z,w)\mapsto [1:z:w].$$

Thus the toric surface is \mathbb{CP}^2 with line bundle $\mathcal{O}(1)$ (compare Example 1.1).

Example 4.2. Consider now Δ given as the convex hull of (0,0), (2,0), (0,2). Then the lattice points in Δ are given as

 $S = \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,0)\}$

and the embedding is

$$(z, w) \mapsto [1: z: z^2: w: zw: w^2].$$

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Those with an algebro-geometric background will recognize this as the *Veronese* embedding, whose closure gives an embedding $\mathbb{CP}^2 \to \mathbb{CP}^5$. The line bundle $\mathcal{O}(1)$ on \mathbb{CP}^5 has sections given by linear forms, and under the Veronese embedding, these pull back to give *quadratic* forms on \mathbb{CP}^2 . That is, Δ as above gives the pair ($\mathbb{CP}^2, \mathcal{O}(2)$). More generally, applying this procedure to Δ the convex hull of (0,0), (d,0), (0,d)gives ($\mathbb{CP}^2, \mathcal{O}(d)$).

Remark 4.3. The surface X_{Δ} is not necessarily smooth. This is the case exactly when at each vertex v of Δ , the "edge vectors" e_1, e_2 (the vectors connecting v to the nearest lattice point to v on each edge) generate \mathbb{Z}^2 . In such cases we say that Δ is smooth.

Given a smooth Δ , we want to understand the properties of the line bundle \mathcal{L}_{Δ} . Two things are especially important for us: (1) What is the largest r such that \mathcal{L}_{Δ} admits an r^{th} root? (2) What is the genus of a smooth section of \mathcal{L}_{Δ} ? Both of these can be read off from Δ , and both are formulated in terms of the *adjoint polygon*.

Definition 4.4. Let Δ be a lattice polygon. The *adjoint polygon* Δ_a is the convex hull of the lattice points lying in the interior of Δ .

The adjoint polygon for Δ as in Figure 9 is shown as shaded.

The terminology arises from the fact that when Δ is smooth, \mathcal{L}_{Δ_a} is the adjoint line bundle for \mathcal{L}_{Δ} : if *C* is a smooth section of \mathcal{L}_{Δ} , then

$$\mathcal{L}_{\Delta_a} \mid_C = K_C.$$

Now we can answer the two main questions. We say that a polygon Δ is a *d-fold dilate* if $\frac{1}{d}\Delta$ is also a lattice polygon.

Fact 4.5 (Dilates are roots). Let Δ_a be a lattice polygon. Then \mathcal{L}_{Δ_a} admits an r^{th} root if and only if Δ_a is an r-fold dilate, in which case

$$(\mathcal{L}_{\frac{1}{d}\Delta_a})^{\otimes d} = \mathcal{L}_{\Delta_a}.$$

In particular, a smooth section C of \mathcal{L}_{Δ} carries a canonical r-spin structure whenever Δ_a is an r-fold dilate.

The adjoint polygon Δ_a in Figure 9 is 3-divisible, hence the smooth sections of \mathcal{L}_{Δ} carry 3-spin structures. As this is a picture of $\mathcal{O}(6)$ on \mathbb{CP}^2 , we knew that already!

As for the second question, the genus of *C* also can be easily read off from Δ_a .

Fact 4.6 (Adjoint lattice points count genus). Let Δ be a lattice polygon. Let g be the number of lattice points in Δ_a . Then the genus of a smooth section C is g.

For a plane curve of degree d, this lattice point count is the triangular number $\binom{d-1}{2}$.

Vanishing cycles. Thus far, we have only seen "classical" aspects of the theory of linear systems on toric surfaces. Crétois–Lang take the polygon picture a step further, giving an explicit model for *C* built out of the polygon Δ .

Construction 4.7 (Inflation). Let Δ be a lattice polygon. The *inflation* of Δ is the topological surface S_{Δ} constructed as follows: drill out a small circle around each interior lattice point of Δ , and then double.

The inflation of the polygon Δ in Figure 9 is the genus-10 surface drawn in Figure 8.

By construction, the inflation S_{Δ} has genus equal to that of a smooth section of \mathcal{L}_{Δ} . Crétois–Lang show how to use this model to track different degenerations of C to nodal curves.

Definition 4.8. An *A-curve* on S_{Δ} is a simple closed curve that encircles an interior lattice point. A *B-curve* is a simple closed curve corresponding to doubling a line segment on Δ that does not contain any lattice points in its interior.

A-curves and *B*-curves can be seen on the inflation surface shown in Figure 8.

The theorem below says that *A*-curves and certain *B*-curves model vanishing cycles. It is an amalgamation of [CL18, Theorem 3, Propositions 7.13, 7.16]; see also [Sal19, Theorems 10.4, 10.5]. Here and throughout, r is defined to be the highest root of the adjoint line bundle.

Theorem 4.9 (Crétois–Lang). Every A-curve is a vanishing cycle for some nodal degeneration of C. A B-curve is a nodal degeneration for C if the line connecting the endpoints passes through a vertex with r-divisible coordinates.

Re-examine Figure 8: the curves shown on the inflation surface are all vanishing cycles.

Remark 4.10. Experts will recognize that I have simplified this story somewhat. I did not discuss the possibility that the curves in a linear system could all be hyperelliptic.

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In this case, the theory behaves differently and there is not a surjection onto $Mod_g[\phi]$. The hyperelliptic case was analyzed by Crétois–Lang [CL19].

Concluding the monodromy calculation; Donaldson. To determine the monodromy, we combine Theorem 3.7 with Theorem 4.9. The precise formulation of Theorem 3.7 (c.f. [Sal19, Theorem 9.5]) implies that $Mod(C)[\phi]$ is generated by Dehn twists about the *A* curves and *B* curves of Theorem 4.9. Thus the monodromy group is the full spin structure stabilizer.

Theorem 4.11. Let \mathcal{L} be a linear system on a smooth toric surface for which the generic fiber C is not hyperelliptic. Let r be the highest root of the adjoint line bundle. Then the monodromy group $\Gamma_{\mathcal{L}}$ is computed to be

$$\Gamma_{\mathcal{L}} = \mathrm{Mod}(C)[\phi],$$

where ϕ is the *r*-spin structure corresponding to the maximal root of the adjoint line bundle.

We return now to Donaldson's question: which curves are vanishing cycles? Recall the notion of *admissible curve*: a nonseparating simple closed curve a is admissible if $\phi(a) = 0$. It is not hard to show that a curve $c \subset C$ is a vanishing cycle if and only if the corresponding twist $T_c \in \text{Im}(\rho)$. We have just shown that the Dehn twists in $\text{Im}(\rho)$ are precisely the admissible twists. Thus we have answered Donaldson's question:

Theorem 4.12. A simple closed curve $c \subset C$ is a vanishing cycle if and only if c is admissible with respect to the canonical r-spin structure on C.

Do you see now why the constraint I showed in Lecture 1 (Figure 2) holds?

4.2. Strata of abelian differentials. Let us now consider the other monodromy problem we set out to answer. We fix a partition $\kappa = \{\kappa_1, \ldots, \kappa_n\}$ of 2g - 2, and consider the stratum \mathcal{H}_{κ} of pairs (C, ω) , where *C* is a Riemann surface of genus *g* and ω is an abelian differential with zeroes of multiplicities $\kappa_1, \ldots, \kappa_n$. Setting $r = \gcd\{\kappa_1, \ldots, \kappa_n\}$, we have observed that (C, ω) carries a canonical *r*-spin structure, geometrically induced from the "horizontal vector field" $\operatorname{Re}(\omega)$.

Following the work of Lecture 3, we need to write down a family of loops of differentials for which the monodromy induces a Dehn twist about an admissible curve. We accomplish this by working with translation surfaces.

The key geometric notion we will require is that of a *cylinder*. A cylinder *A* on a translation surface *S* is just that: it is a flat-embedded annulus, i.e. a family of parallel

straight lines determining closed curves on *S*, a parallelogram with a pair of parallel sides identified, etc. We encountered cylinders in Example 2.9 above in Lecture 2. There, we saw that each cylinder determines a family of deformations called a *shear*.



FIGURE 10. The configuration at left generates $Mod_5[\phi]$ for a certain 2-spin structure ϕ (the spin structure uniquely determined by setting all indicated curves to be admissible). The configuration of curves can be converted into a translation surface as follows: convert each curve *a* into a rectangle of side lengths 1 and *k*, where *k* is the number of other curves *a* intersects, and identify the sides of length 1 with each other. If curves *a* and *b* intersect, glue a 1×1 portion of the corresponding rectangles to each other. The result is a translation surface. It lives in the following stratum: the number of zeroes corresponds to the number of components the configuration divides the surface into, and the order of each zero is (minus) the corresponding Euler characteristic. This example has two regions of Euler characteristics -2 and -6, hence the differential lives in the stratum $\{2, 6\}$ as can be seen directly.

Our task is now as follows. For each partition κ as above, we need to build a translation surface in the stratum \mathcal{H}_{κ} which is built in an explicit combinatorial way out of cylinders. We need to design these in such a way that the corresponding Dehn twists can be seen to generate $Mod(S)[\phi]$ using another criterion for generation, [CS19, Theorem B].

In [Cal19], Aaron came up with a very nice method for doing this using something called the *Thurston-Veech construction*. This allows you to start with a collection of simple closed curves on a surface (satisfying some combinatorial restrictions) and



FIGURE 11. Building a translation surface in the stratum \mathcal{H}_{κ} . Let \mathcal{C} be the configuration of curves shown in red. The curves b_1, \ldots, b_{2g-3} are shown in blue. For the partition $\kappa = \{\kappa_1, \ldots, \kappa_n\}$, we include only the curves b_{i_j} for i_j of the form $i_j = \sum_{k=1}^j \kappa_k$.

promote it to a translation surface on which all the curves become cylinders. Figure 10 shows a worked example in the case $\kappa = \{2, 6\}$ in genus 5.

Figure 11 shows how the construction works in general. We start with the configuration C shown as red curves. We then add the curve b_{κ_1} to the configuration: observe that b_{κ_1} and C together bounds a region of Euler characteristic $-\kappa_1$, and so under the Thurston–Veech construction, this will give a translation surface with a cone point of order κ_1 as required. We then continue, adding in $b_{\kappa_2+\kappa_1}, b_{\kappa_3+\kappa_2+\kappa_1}, \ldots$ In total, we "allocate" the total Euler characteristic 2 - 2g of Σ_g among n regions of Euler characteristic $-\kappa_1, \ldots, -\kappa_n$, giving a translation surface in the stratum \mathcal{H}_{κ} , where each Dehn twist on the topological picture Figure 11 is incarnated as a cylinder.¹⁰ Note also that we get a well-defined r-spin structure (for $r = \gcd(\kappa)$ as usual) by declaring all $a \in C$ and all b_{i_1}, \ldots, b_{i_n} to be admissible.

Here is a paraphrase of [CS19, Theorem B].

Theorem 4.13 (Calderon - S.). Let $\kappa = {\kappa_1, \ldots, \kappa_n}$ be a partition of 2g - 2 for $g \ge 5$ and set $r = \text{gcd}(\kappa)$. The collection of Dehn twists about the curves $\mathcal{C} \cup {b_{i_1}, \ldots, b_{i_n}}$ as shown in Figure 11 generates $\text{Mod}_g[\phi]$, where ϕ is the r-spin structure determined by the condition that all $a \in \mathcal{C}$ and all b_{i_j} are admissible.

Applying the Thurston–Veech construction to this configuration and invoking Theorem 4.13 shows surjectivity of the monodromy for all strata!

¹⁰For the experts: when \mathcal{H}_{κ} has two components classified by parity of the Arf invariant, we use two different starting configurations $\mathcal{C}, \mathcal{C}'$, and the resulting translation surfaces will live in the different strata components.

Remark 4.14. Once again I have suppressed some aspects of the story. For the sake of honesty, here's what I've been leaving out. In [KZ03], Kontsevich–Zorich study the *components* of \mathcal{H}_{κ} , and they find that \mathcal{H}_{κ} is not always connected. They classify the components of a fixed \mathcal{H}_{κ} . For $\kappa = \{2g - 2\}$ or $\kappa = \{g - 1, g - 1\}$, they find that there is a component of \mathcal{H}_{κ} consisting entirely of hyperelliptic curves; as in the toric surface situation, this behaves very differently and was analyzed in [Cal19]. They also find that whenever r is even (and $g \ge 4$), there are two components $\mathcal{H}_{\kappa}^{even}$, $\mathcal{H}_{\kappa}^{odd}$, distinguished by the "parity of the Arf invariant". As in Remark 3.10, I do not want to discuss the Arf invariant in detail; suffice it to say that in the case where \mathcal{H}_{κ} is disconnected, we can apply the above method in each (non-hyperelliptic) component separately and compute the monodromy of each *stratum-component*.

Here is a precise statement of what we have proved.

Theorem 4.15. Let κ be a partition of 2g - 2 for $g \ge 5$ with $gcd(\kappa) = r$. Let \mathcal{H} be a non-hyperelliptic component of \mathcal{H}_{κ} with associated *r*-spin structure ϕ . Then the monodromy group $\Gamma_{\mathcal{H}}$ associated to \mathcal{H} is computed to be

$$\Gamma_{\mathcal{H}} = \operatorname{Mod}_{g}[\phi].$$

Applications. A primary reason to care about the monodromy problem for strata is that it provides the first substantial information about the highly mysterious groups $\pi_1(\mathcal{H})$ (technically we mean *orbifold* π_1). We know for abstract reasons (virtually they are fundamental groups of quasiprojective varieties) that these groups are finitely presented, but beyond that, almost nothing is known. The monodromy representation provides a homomorphism from $\pi_1(\mathcal{H})$ into the mapping class group, which is much better understood. The monodromy calculation then shows that $\pi_1(\mathcal{H})$ is rich enough to surject onto a finite-index subgroup of Mod_q .

Which curves are cylinders? Another more concrete application is to answer the counterpart of Donaldson's question for translation surfaces. We fix a *marking* $f : \Sigma_g \to S$ between a topological surface Σ_g and a chosen translation surface S. Once we have fixed a marking, we can move around inside \mathcal{H} and identify simple closed curves on translation surfaces with simple closed curves on Σ_g . We ask:

Question 4.16. For which curves $c \subset \Sigma_g$ does there exist a path $\gamma \subset \mathcal{H}$ such that c is a cylinder on $\gamma(1)$?

As we did with Donaldson's question, we answer this using the monodromy calculation. We identify the set of Dehn twists in the monodromy with the set of "cylinder curves" on Σ_g . As before, the calculation $\Gamma_{\mathcal{H}} = \text{Mod}_g[\phi]$ tells us the answer:

Theorem 4.17. For $g \ge 5$, a nonseparating curve $a \subset \Sigma_g$ is the core curve of a cylinder on some non-hyperelliptic translation surface $S \in \mathcal{H}$ if and only if $\phi(a) = 0$.

If someone hands you a curve *a* on a translation surface, the winding number $\phi(a)$ can be measured explicitly. Recall from Remark 3.3 that we can compute $\phi(c)$ by computing the winding number of *c* with respect to the horizontal vector field on the translation surface *S*. Winding number is a *signed* count of the number of times a curve crosses a fixed direction, and so just because a curve *c* has $\phi(c) = 0$ doesn't mean that it literally runs "straight", i.e. lies on a cylinder. But what Theorem 4.17 is telling us is that if we have any *c* with $\phi(c) = 0$, then there is some sequence of shears that we can perform to systematically eliminate cancelling pairs of crossings and eventually take *c* to a cylinder.

5. SINS OF OMISSION

In the interest of bibliographic completeness, we briefly mention here some closely related work which didn't make its way into the body of the lectures.

Spin mapping class groups. To the author's knowledge, higher spin mapping class groups first appear in the literature in the work of Sipe [Sip82, Sip86]; she credits Mumford with the suggestion to study these groups. In [RW14], Randal-Williams proves a homological stability result for *r*-spin mapping class groups. Both he and Kawazumi [Kaw17] study versions of the problem of determining the orbit structure of the action of the mapping class group on the set of *r*-spin structures.

Monodromy of linear systems. Because the results are now superseded by [Sal19], we did not discuss some prior work on special cases of the monodromy problem for linear systems on toric surfaces. We would especially like to draw attention to a second paper [CL19] by Crétois–Lang. There, they solve the monodromy problem in the case where the highest root r of the adjoint line bundle is 2. We also did not mention our own early paper [Sal16] on the monodromy problem for smooth plane curves. Our main result there treats the case d = 5, which also corresponds to the case r = 2. While the result is very limited in scope, the method is perhaps worth mentioning. Appearing as it did before the work of Crétois–Lang, we take a different

approach to the constructive portion of the argument, where we exhibit a range of Dehn twists corresponding to different nodal degenerations. Our technique is special to plane curves, and relies on the fascinating work of Lönne [Lön09]. Lönne finds an explicit and tractable presentation for the fundamental group of the space of smooth hypersurfaces of arbitrary degree d in an arbitrary projective space \mathbb{CP}^N . We use a version of the change–of–coordinates principle to show that the generators of Lönne's presentation determine a unique configuration of Dehn twists (up to diffeomorphism). We can then directly study the subgroup of Mod_6 generated by this collection of twists and prove that it generates the stabilizer of a 2-spin structure.

More classical algebraic geometry was also interested in the homological monodromy group of the family of smooth plane curves (i.e. the action on $H_1(\Sigma_g; \mathbb{Z})$). The final result here was obtained by Beauville [Bea86], building off of earlier work of Janssen [Jan83] and Chmutov [Chm82].

Strata of abelian differentials. Walker's papers [Wal10, Wal09] are closely related to the problem of computing monodromy for strata of Abelian differentials. She studies *quadratic differentials* (holomorphic sections of the *square* of the cotangent bundle) which have a geometric incarnation as *half-translation surfaces*. She is interested in the problem of classifying components of these strata when the differentials are equipped with the data of a "marking". This essentially amounts to a monodromy computation; she is able to obtain results under the assumption that many zeroes have the same order.

Hamenstädt has also investigated the problem of computing monodromy of strata of abelian differentials. In her preprint [Ham18], she obtains a geometric characterization of the monodromy group which provides an interesting counterpart to the main results of the forthcoming paper [CS20].

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