Totally symmetric sets

and the representation theory of mapping class groups

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Totally symmetric sets

Let G be a group. Kordek and Margalit introduced the following

Definition

A totally symmetric set (TSS) in G is a finite subset $\mathscr{A}$ such that every permutation in $\mathscr{A}$ can be induced by conjugation in G.

$$g_\sigma a_i g_\sigma^{-1} = a_{\sigma(i)}$$

Notes:

- The assignment $\sigma \mapsto g_\sigma$ need not be a homomorphism.
- Frequently add the condition that elements of $\mathscr{A}$ pairwise commute
- Can then think of $\mathscr{A}$ as an abstract "maximal torus"
- Size of maximal TSS some proxy for *rank*

Examples

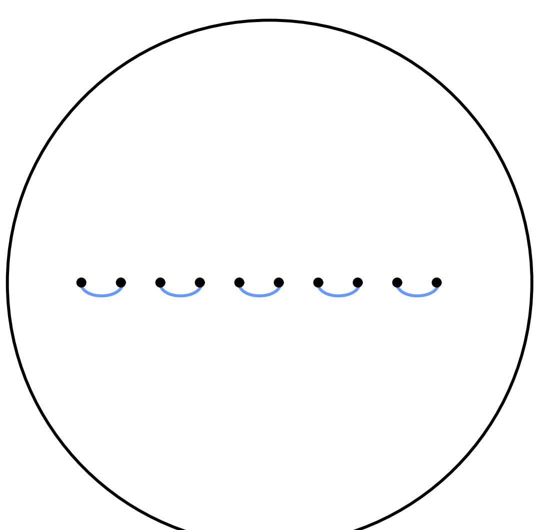
$G = S_n$

$\mathscr{A} = \{(12), (34), \ldots, (n-1\ n)\}$
Commutative TSS of size $\lfloor \frac{n}{2} \rfloor$

$\mathscr{A} = \{(12), (13), \ldots, (1n)\}$
Noncommutative TSS of size $n-1$

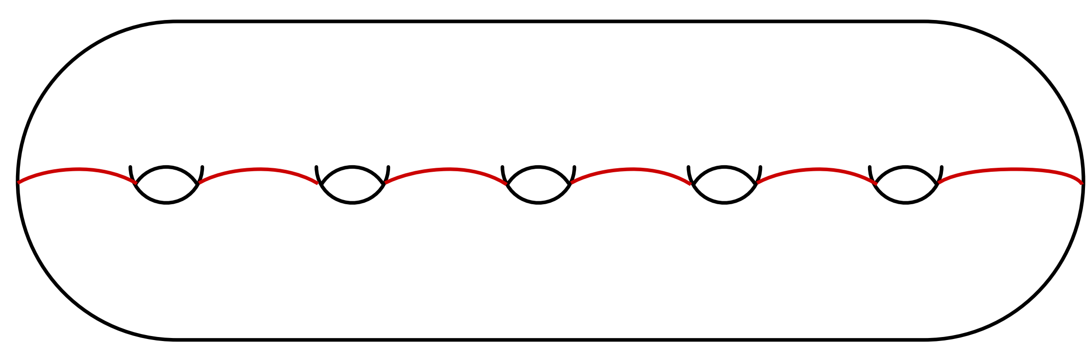
$G = B_n$

$\mathscr{A} = \{\sigma_1, \sigma_3, \ldots, \sigma_{n-1}\}$
Commutative TSS of size $\lfloor \frac{n}{2} \rfloor$



$G = \mathrm{Mod}(\Sigma_g)$

$\mathscr{A} = \{T_{c_1}, \ldots, T_{c_{g+1}}\}$
Commutative TSS of size $g+1$



The persistence lemma

The utility of TSS for studying maps between groups is due to the following lemma of Kordek - Margalit:

Lemma Let $f : G \to H$ be a homomorphism. Then $f(\mathscr{A})$ is either a TSS of size $|f(\mathscr{A})| = |\mathscr{A}|$, or else a singleton.

"Collision implies collapse"

This means that classifying TSS in G and H can, in principle, tell you about all maps $f : G \to H$.

Prior work

Kordek - Margalit: classify CTSS of size $\lfloor \frac{n}{2} \rfloor$ in B_n.
Use this to classify $f : B'_n \to B_n$.

Chen - Mukherjea: classification of $f : B_n \to \mathrm{Mod}(\Sigma_g)$ for $g \le n - 3$.

Caplinger-Kordek, Chudnovsky-Kordek-Li-Partin,
Scherich-Verberne, Kolay:

Question (Margalit): *What is the smallest non-cyclic finite quotient of B_n?*

Kolay: (essentially) always the permutation rep $B_n \to S_n$.

Overarching theme: *rigidity*

Examples in $\mathrm{GL}_n(\mathbb{C})$

$$A_1 = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix} \quad A_2 = \begin{pmatrix} \mu & & \\ & \lambda & \\ & & \mu \end{pmatrix} \quad A_3 = \begin{pmatrix} \mu & & \\ & \mu & \\ & & \lambda \end{pmatrix}$$

"Standard" construction: k elements in $\mathrm{GL}_k(\mathbb{C})$.

$$A_1 = \begin{pmatrix} \lambda & & 1 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix} \quad A_2 = \begin{pmatrix} \lambda & & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \quad A_3 = \begin{pmatrix} \lambda & & -1 \\ & \lambda & -1 \\ & & \lambda \end{pmatrix},$$

"Simplex" construction: k elements in $\mathrm{GL}_k(\mathbb{C})$.

Both commutative. Standard is diagonalizable, simplex is not.

Basic questions

Can you classify all TSS in $\mathbf{GL}_n(\mathbb{C})$?

Can you bound the size of a TSS in $\mathbf{GL}_n(\mathbb{C})$?

Irreducibility

Can you classify all TSS in $\mathrm{GL}_n(\mathbb{C})$?

Idea: borrow from representation theory.

Definition:

A TSS $\mathscr{A} \subset \mathrm{GL}(V)$ is *reducible* if there is a proper subspace $W \subset V$ invariant under both $\mathscr{A}$ and a set of permutations.

$\mathscr{A}$ then restricts to a TSS in $GL(W)$, and induces a TSS on $GL(V/W)$.

Non-semi-simplicity

The standard construction is irreducible (not obvious).

The simplex construction is reducible:

$$A_1 = \begin{pmatrix} \lambda & & 1 \\ & \lambda & 0 \\ & & \lambda \end{pmatrix}, A_2 = \begin{pmatrix} \lambda & & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}, A_3 = \begin{pmatrix} \lambda & & -1 \\ & \lambda & -1 \\ & & \lambda \end{pmatrix},$$

Span of e_1, e_2 is such a W.

This illustrates an important structural feature:

The irreducible factors of a TSS do not determine the TSS.
There is an *extension problem* to solve!

TSS of partition type

General construction: choose $\kappa = \{\kappa_1, \ldots, \kappa_p\}$ a partition of k.

Choose $\lambda_1, \ldots, \lambda_p \in \mathbb{C}^\times$ distinct.

Let V be the space spanned by functions $\vec{\lambda} : [k] \to \mathbb{C}^\times$ for which $\left| \vec{\lambda}^{-1}(\lambda_i) \right| = \kappa_i$.

Let $\mathscr{A} = \{A_1, \ldots, A_k\}$ be the TSS acting diagonally on V by $A_i(f) = f(i)\, f$

$$\kappa = \{2,2\}$$

$$\lambda_1 = 1,\ \lambda_2 = 2$$

A_1	1	1	1	2	2	2
A_2	1	2	2	1	1	2
A_3	2	1	2	1	2	1
A_4	2	2	1	2	1	1

$$\{A_1, A_2, A_3, A_4\} \subset GL_6(\mathbb{C})$$

Main theorem I: irreducibles

Denote such a TSS $\mathscr{A}(\overrightarrow{\lambda})$.

Call the function $\overrightarrow{\lambda}: [k] \to \mathbb{C}$ a *weight*.

Theorem (Caplinger - S.):

Every irreducible commutative TSS is of the form $\mathscr{A}(\overrightarrow{\lambda})$.

Main theorem II: size bounds

Can you bound the size of a TSS in $\mathrm{GL}_n(\mathbb{C})$?

Theorem (Caplinger - S.):

A commutative TSS in $\mathrm{GL}_n(\mathbb{C})$ has at most n elements, and a noncommutative TSS has at most $n+1$.

Remark:

Unfortunately, it's not enough to just study irreducibles, because of the failure of semisimplicity.

Main theorem III: maximal size

Theorem (Caplinger - S.):

For $k \neq 4$, there are exactly two classes of k-element commutative TSS in $\mathrm{GL}_k(\mathbb{C})$: the standard construction and the simplex construction.

There is one additional sporadic example for $k = 4$.

There is exactly one class of $k+1$-element noncommutative TSS in $\mathrm{GL}_k(\mathbb{C})$.

Summary

How close does this come to a full classification?

$$\begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

Irreducibility

$$\begin{pmatrix} ? & ? & ? \\ 0 & ? & ? \\ 0 & 0 & ? \end{pmatrix}$$

Main Theorem I

$$\begin{pmatrix} \mathscr{A}(\overrightarrow{\lambda}) & ? & ? \\ 0 & \mathscr{A}(\overrightarrow{\mu}) & ? \\ 0 & 0 & \mathscr{A}(\overrightarrow{\nu}) \end{pmatrix}$$

Final step: solve the *extension problem*: classify off-diagonal blocks.

Application: symmetric group representations

Known that S_n never* admits noncyclic representations below dimension $n-1$.

* except n = 4

Standard proof cumbersome: first classify all irreps, then use *hook length formula* to compute dimensions; observe gap between $1, n-1$.

TSS provides a structural explanation:

Take $\rho : S_n \to GL_d(\mathbb{C})$

Where is the TSS $\{(1\ i)\}$ *sent?*

Noncommutative of size $n-1$, *so* $d \geq n-2$.

Unique TSS of size $n-2$ *in* $GL_{n-2}(\mathbb{C})$ *-*
check this works only for $n=4$.

Prospectus: representation theory

Broad goal: understand representations of braid and mapping class groups

Careful: residually-finite groups have *a lot* of representations!

One place where braid, mapping class groups diverge:
braid groups seem to have more representations.

For $\mathrm{Mod}(\Sigma)$, only two mechanisms known:

Action on homology of covers

Residual finiteness

(TQFT reps are *projective*)

Can we use TSS to explore the landscape of representations of $\mathrm{Mod}(\Sigma)$?

Dimension gaps

As for the symmetric group, both braid and mapping class groups have a *dimension gap* in their rep theory:

Theorems (Formanek, Sysoeva):

For n large, B_n admits no nonabelian reps of dimension $< n-2$. Up to dimension n, all irreps are classified: Burau and TYM.

Theorem (Korkmaz):

For $g \geq 3$, the unique rep of $\mathrm{Mod}(\Sigma_g)$ of dimension $\leq 2g$ is the symplectic rep.

Some questions/problems

Do there exist representations of $\mathrm{Mod}(\Sigma_g)$ with infinite image that do not arise via acting on the homology of a cover?

Increase the N for which we have a complete classification of irreps of $\mathrm{Mod}(\Sigma_g)$ in dimension $\leq N$. (Currently $N = 2g + 1$; c.f. Kasahara).

Which other groups of geometric/topological interest have large TSS? Such groups should be *rigid* in the same ways braid/mapping class groups are.

Thank you!

Bonus: more examples

The sporadic example:

$$A_1 = \begin{pmatrix} \nu & & 1 & 0 \\ & \nu & 0 & 1 \\ & & \nu & \\ & & & \nu \end{pmatrix} \qquad A_2 = \begin{pmatrix} \nu & & -\mu - \frac{2}{3} & \mu + \frac{1}{3} \\ & \nu & 0 & \mu \\ & & \nu & \\ & & & \nu \end{pmatrix}$$

$$A_3 = \begin{pmatrix} \nu & & \mu & 0 \\ & \nu & \mu + \frac{1}{3} & -\mu - \frac{2}{3} \\ & & \nu & \\ & & & \nu \end{pmatrix} \qquad A_4 = \begin{pmatrix} \nu & & \frac{-1}{3} & -\mu - \frac{1}{3} \\ & \nu & -\mu - \frac{1}{3} & \frac{-1}{3} \\ & & \nu & \\ & & & \nu \end{pmatrix}$$

Here ν can be arbitrary, *but μ must satisfy* $3\mu^2 + 2\mu + 3 = 0$*!*

Bonus: more examples

The $k+1$-element noncommutative TSS in $\mathbf{GL}_k(\mathbb{C})$:

$$A_1 = \begin{pmatrix} \lambda & \frac{\mu-\lambda}{2} \\ 0 & \mu \end{pmatrix} \qquad A_2 = \begin{pmatrix} \mu & 0 \\ \frac{\mu-\lambda}{2} & \lambda \end{pmatrix} \qquad A_3 = \begin{pmatrix} \frac{\lambda+\mu}{2} & \frac{\lambda-\mu}{2} \\ \frac{\lambda-\mu}{2} & \frac{\lambda+\mu}{2} \end{pmatrix}$$

Here λ and μ can be arbitrary (but distinct).

Conceptually:

Take $V_k^{std} = \mathbb{C}^{k-1}$ the standard rep for S_k.

To make A_i, decompose V_k^{std} as a $\mathbf{Stab}(i)$ - rep:

$$\boxed{V_k^{std} = (V_{k-1}^{std})_i \oplus \mathbb{C}_i}$$

Let A_i act by λ on $(V_{k-1}^{std})_i$ and by μ on $\mathbb{C}_i$.