Mapping class groups and the monodromy of some families of algebraic curves

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AG crash course

Plane curve: $\{[x:y:z] \mid x^d + y^d + z^d = 0\} \subset \mathbb{C}P^2$

Smooth: $\{[x:y:z] \mid x^d + y^d + z^d = 0\} \cong \Sigma_g$

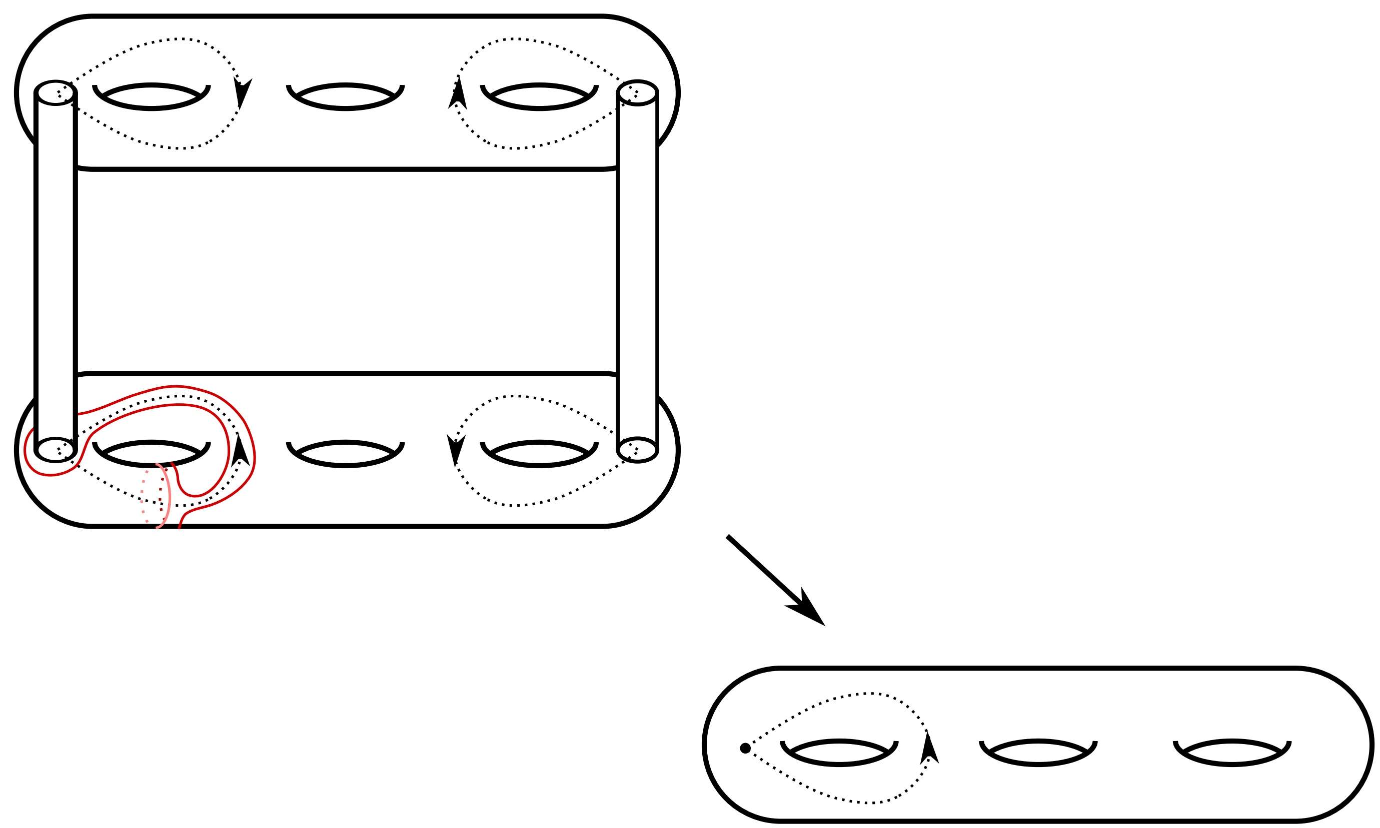
Degree-genus formula: $g = \binom{d-1}{2}$

Moduli space of plane curves: $\mathcal{P}_d = \mathbb{C}P^N \setminus \mathcal{D}_d$

Universal plane curve: $\Sigma_g \to \mathfrak{X}_d \to \mathcal{P}_d$

Surface bundles have *monodromy*:

$$\rho_d : \pi_1(\mathcal{P}_d) \to \mathrm{Mod}_g$$



Basic question: What is $\Gamma_d := \mathrm{im}(\rho_d) \subset \mathrm{Mod}_g$?

An approximate answer

Symplectic representation: $\Psi : \mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$

Theorem (Beauville):

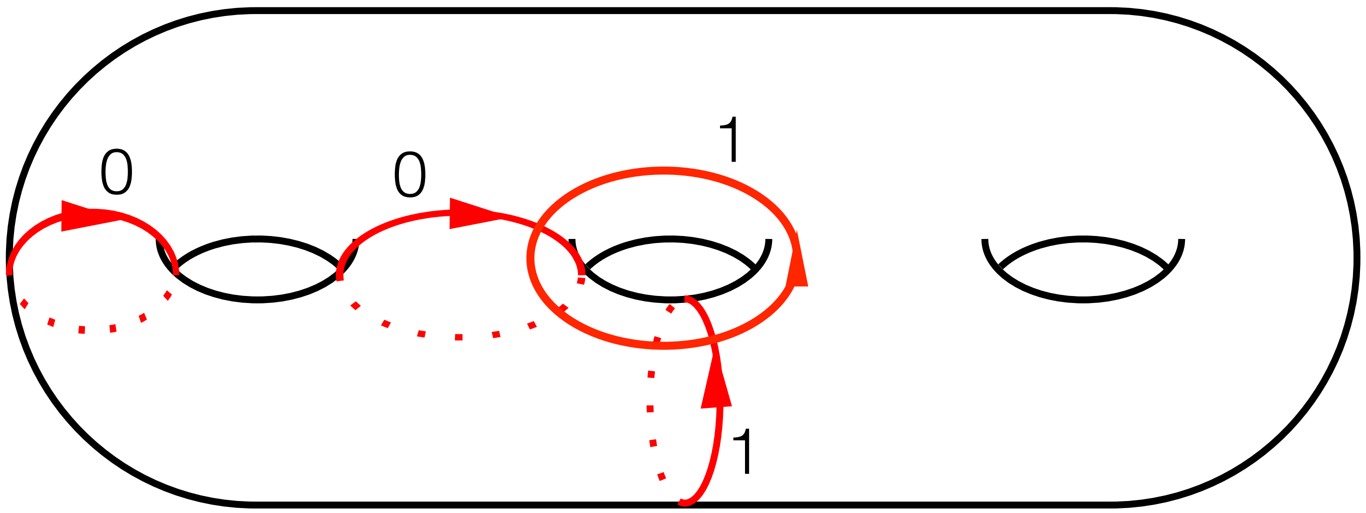
$$\Psi(\Gamma_d) = \begin{cases} \mathrm{Sp}_{2g}(\mathbb{Z}) & d \text{ even} \\ \mathrm{Sp}_{2g}(\mathbb{Z})[q] & d \text{ odd} \end{cases}$$

Here, q is a "spin structure" and $\mathrm{Sp}_{2g}(\mathbb{Z})[q]$ indicates the stabilizer

Spin structures

My favorite definition: "Winding number function"

$$\phi : \{\text{SCC's}\} \to \mathbb{Z}/2\mathbb{Z}$$



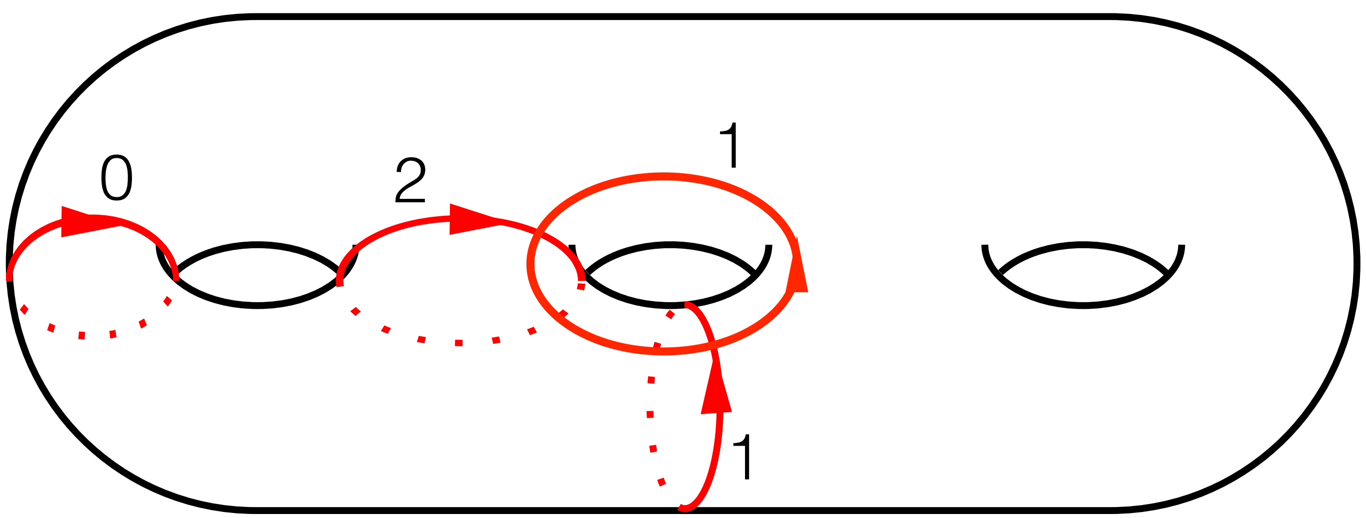
Cohomological definition:

$$\phi \in H^1(T^1\Sigma_g; \mathbb{Z}/2\mathbb{Z}); \quad \phi(\zeta) = 1$$

n-spin structures

My favorite definition: "Winding number function"

$$\phi : \{\text{SCC's}\} \to \mathbb{Z}/n\mathbb{Z}$$



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$$\phi \in H^1(T^1\Sigma_g; \mathbb{Z}/n\mathbb{Z}); \quad \phi(\zeta) = 1$$

Constraints

Observation (Folklore):

There is a natural (d-3)-spin structure ϕ_d
invariant under Γ_d

For d≥ 5, the containment

$$\Gamma_d \subset \mathrm{Mod}_g[\phi_d] \subsetneq \Psi^{-1}(\Psi(\Gamma_d))$$

is strict.

Low-degree cases

d = 3: (g = 1) Elliptic curves are all planar $(y^2 = x^3 + ax + b)$

$$\Gamma_3 = \mathrm{Mod}_1 \cong \mathrm{SL}_2(\mathbb{Z})$$

d=4: (g = 3) AG fact: “Generic” genus-3 curve planar.

Implies $\Gamma_4 = \mathrm{Mod}_3$

d=5

Theorem (S. '16): There is an equality

$$\Gamma_5 = \mathrm{Mod}(\Sigma_6)[\phi_5]$$

This ϕ_5 is a $\mathbb{Z}/2\mathbb{Z}$ spin structure.

Higher d?

Conjecture: $\Gamma_d = \mathrm{Mod}_g[\phi_d]$

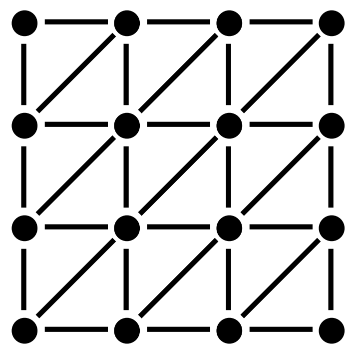
Current knowledge: Don't even know if Γ_d is finite-index!

Crétois and Lang ('17) studied a closely related problem (monodromy of linear systems on toric surfaces) and formulated the same conjecture!

The flavor

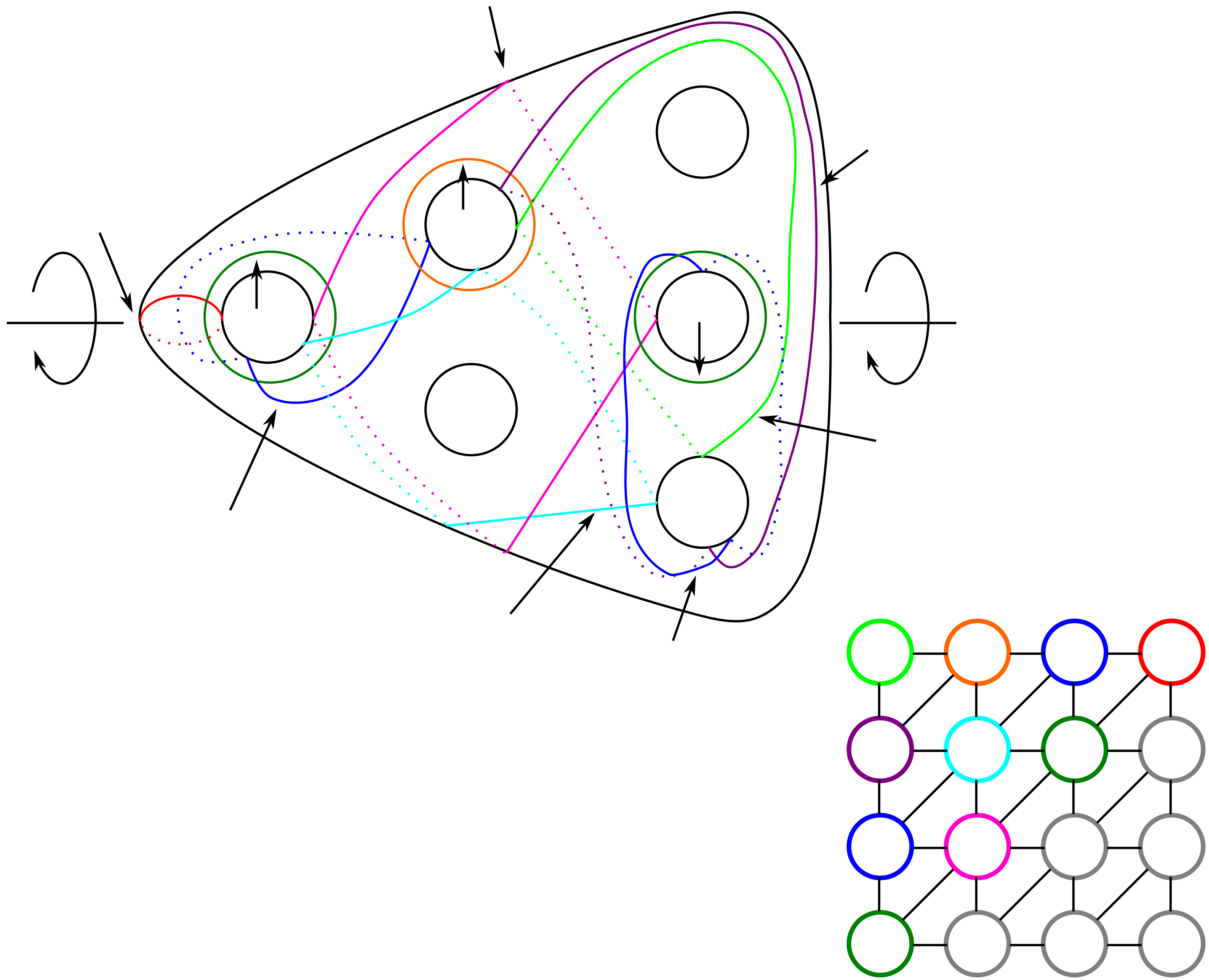
$\pi_1(\mathcal{P}_d)$ has an explicit presentation, due to Lönne.

It's a quotient of a RAAG!



"Picard-Lefschetz theory" implies that ρ_d maps generators to Dehn twists.

I use mapping class group techniques to determine this configuration of curves



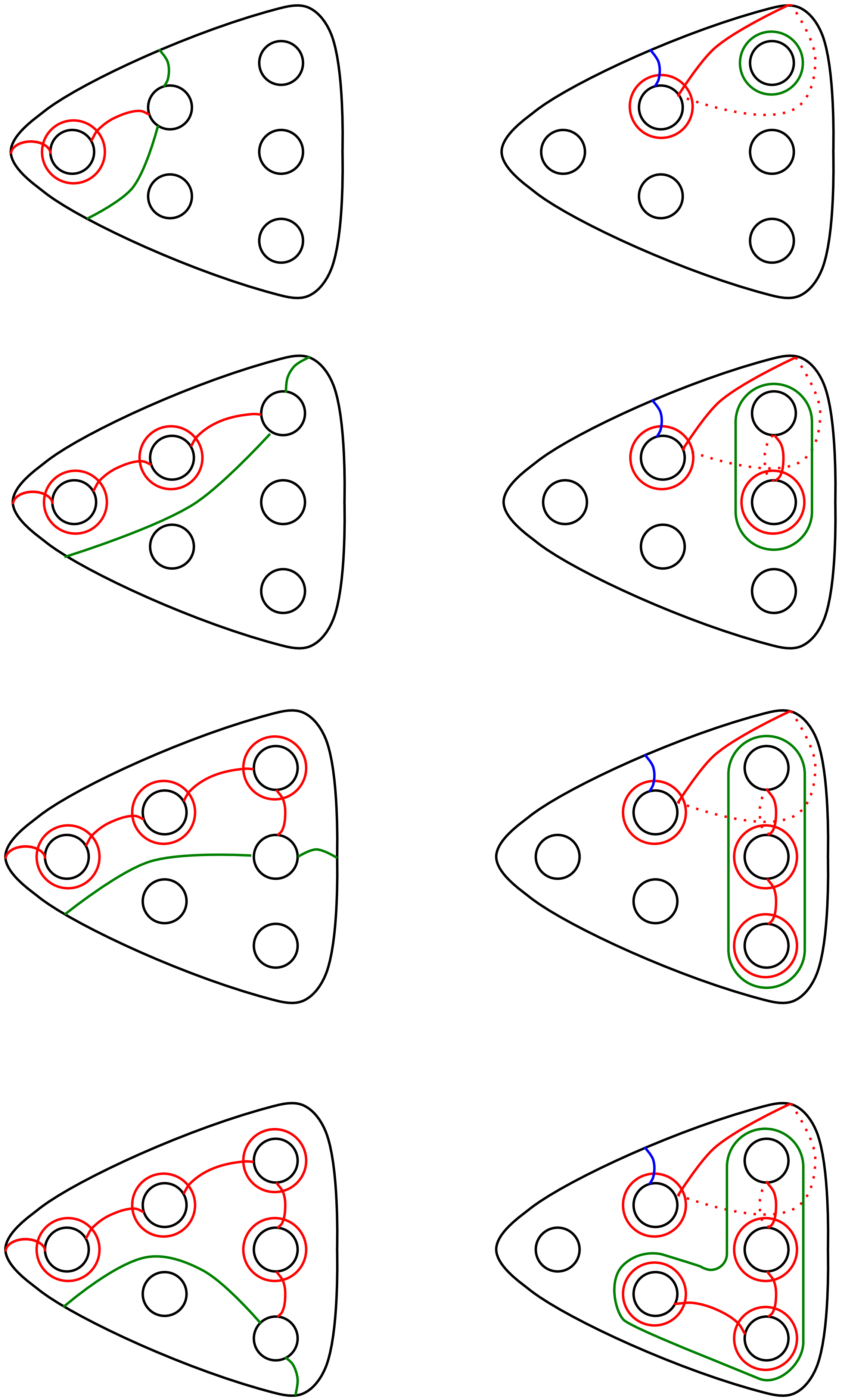
d=5?

From Beauville's result, it suffices to show

$$\Gamma_d \cap \ker(\Psi) = \mathrm{Mod}_g[\phi_d] \cap \ker(\Psi)$$

For d = 5, $\mathrm{Mod}_g[\phi_d] \cap \ker(\Psi) = \mathcal{I}_g$ (Torelli group)

Then I exhibit all of Johnson's generators for $\mathcal{I}_g$ as elements of Γ_d



d>5?

The limitation for d>5 is simply that there isn't a known set of generators for $\mathrm{Mod}_g[\phi_d] \cap \ker(\Psi)$

(or for $\mathrm{Mod}_g[\phi_d]$ itself)

How hard could this be?

(Famous last words…)