Surface bundles, monodromy, and arithmetic groups

Nick Salter and Bena Tshishiku
Harvard University
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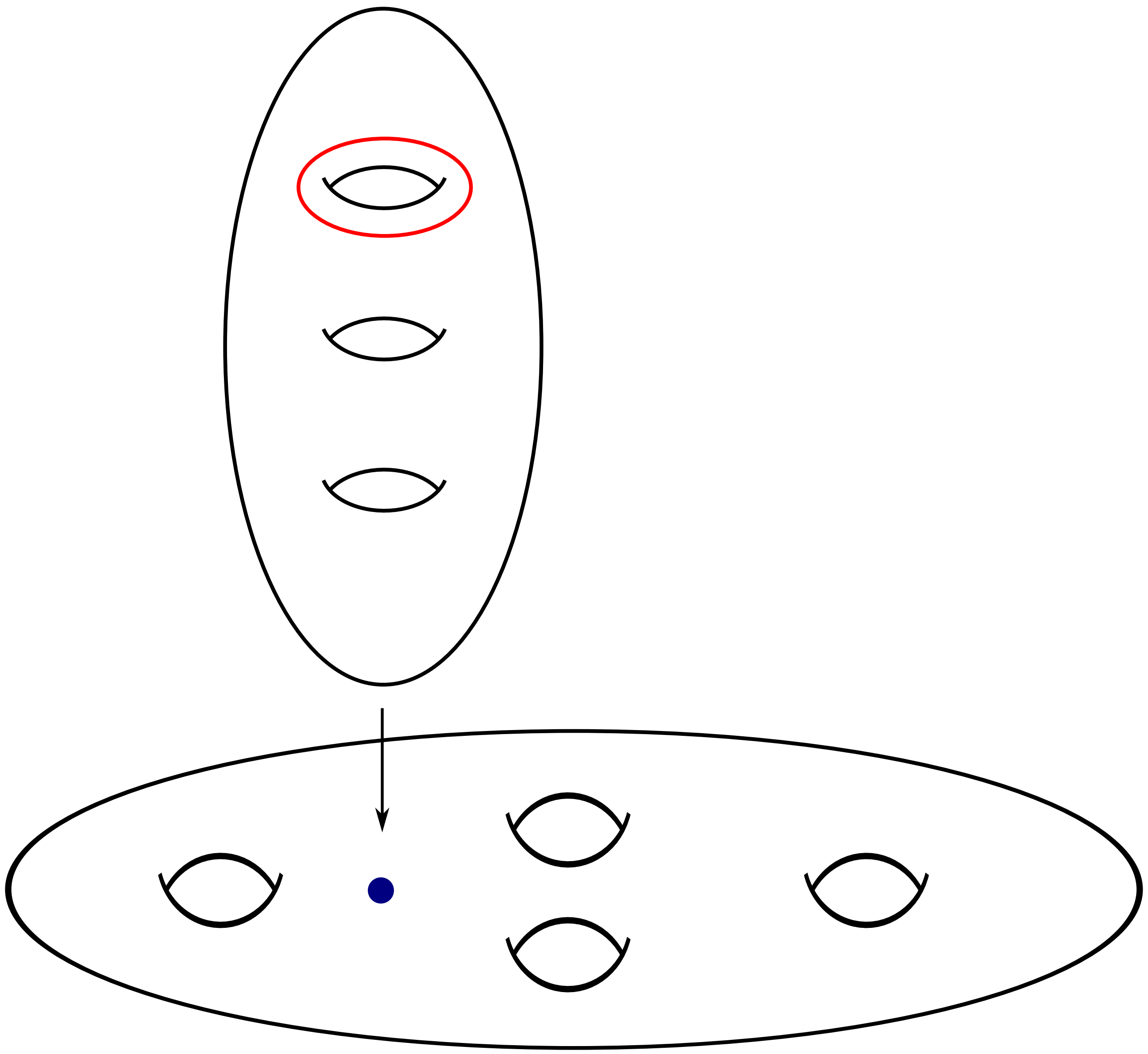
What is monodromy?



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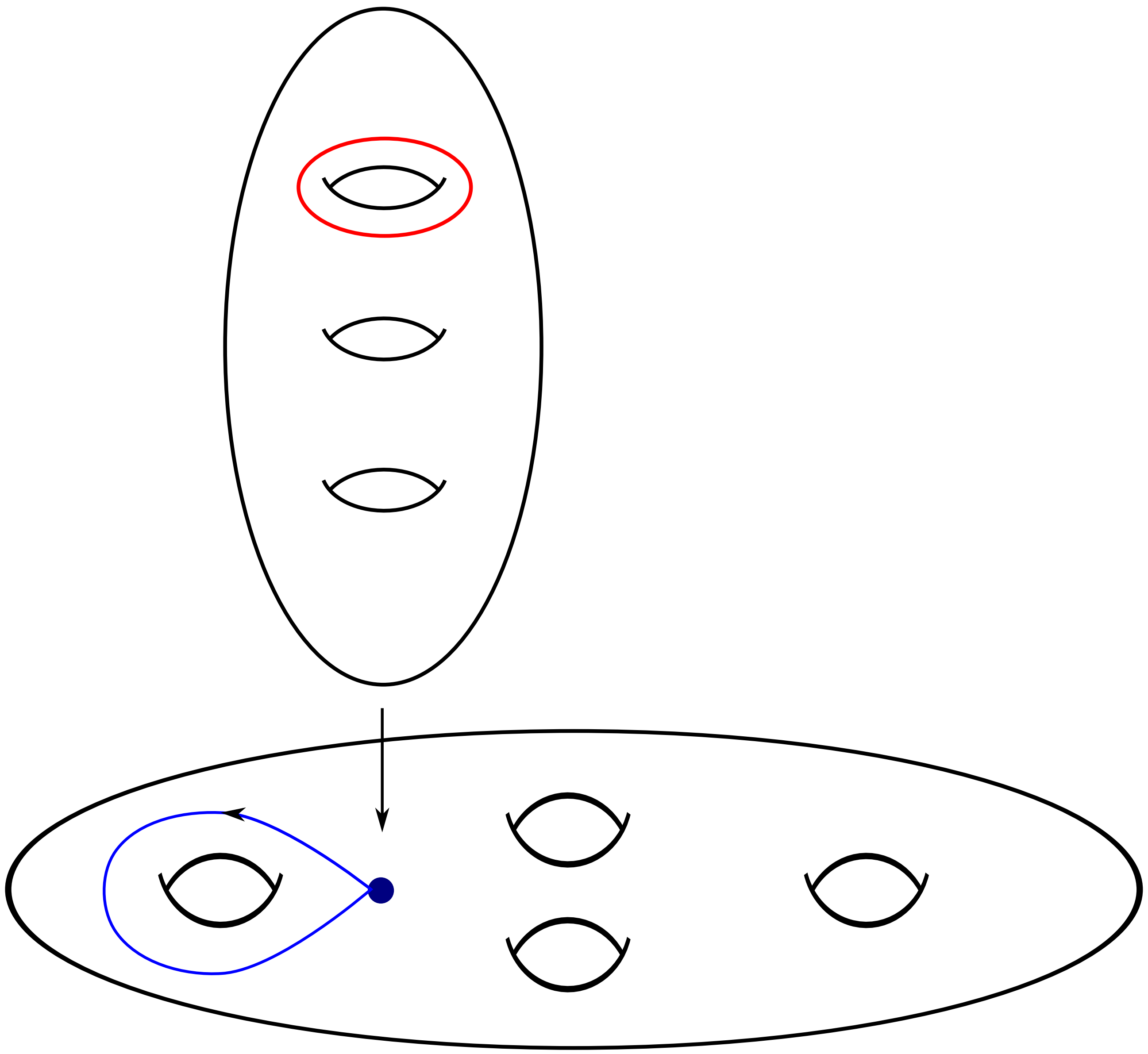

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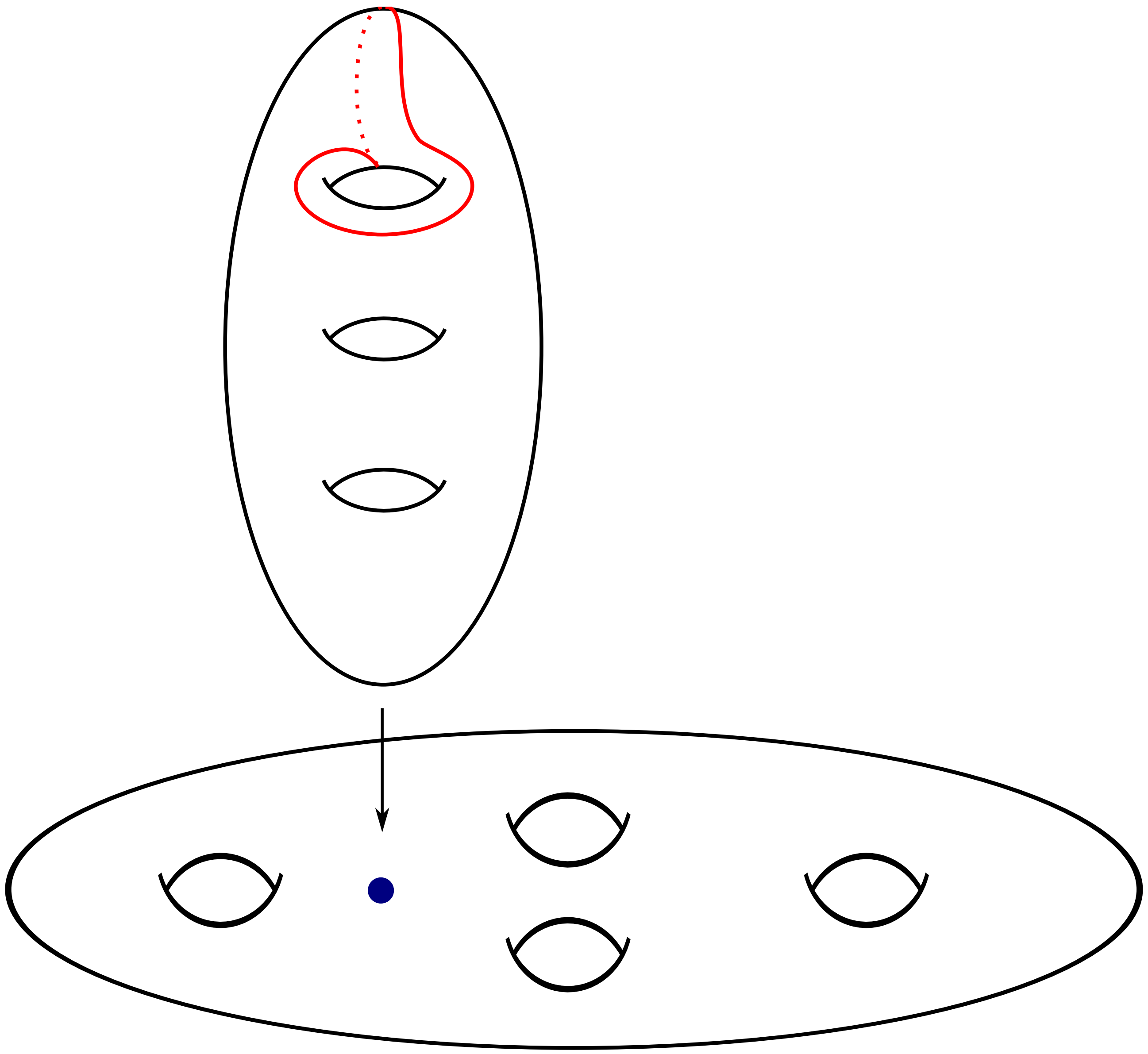

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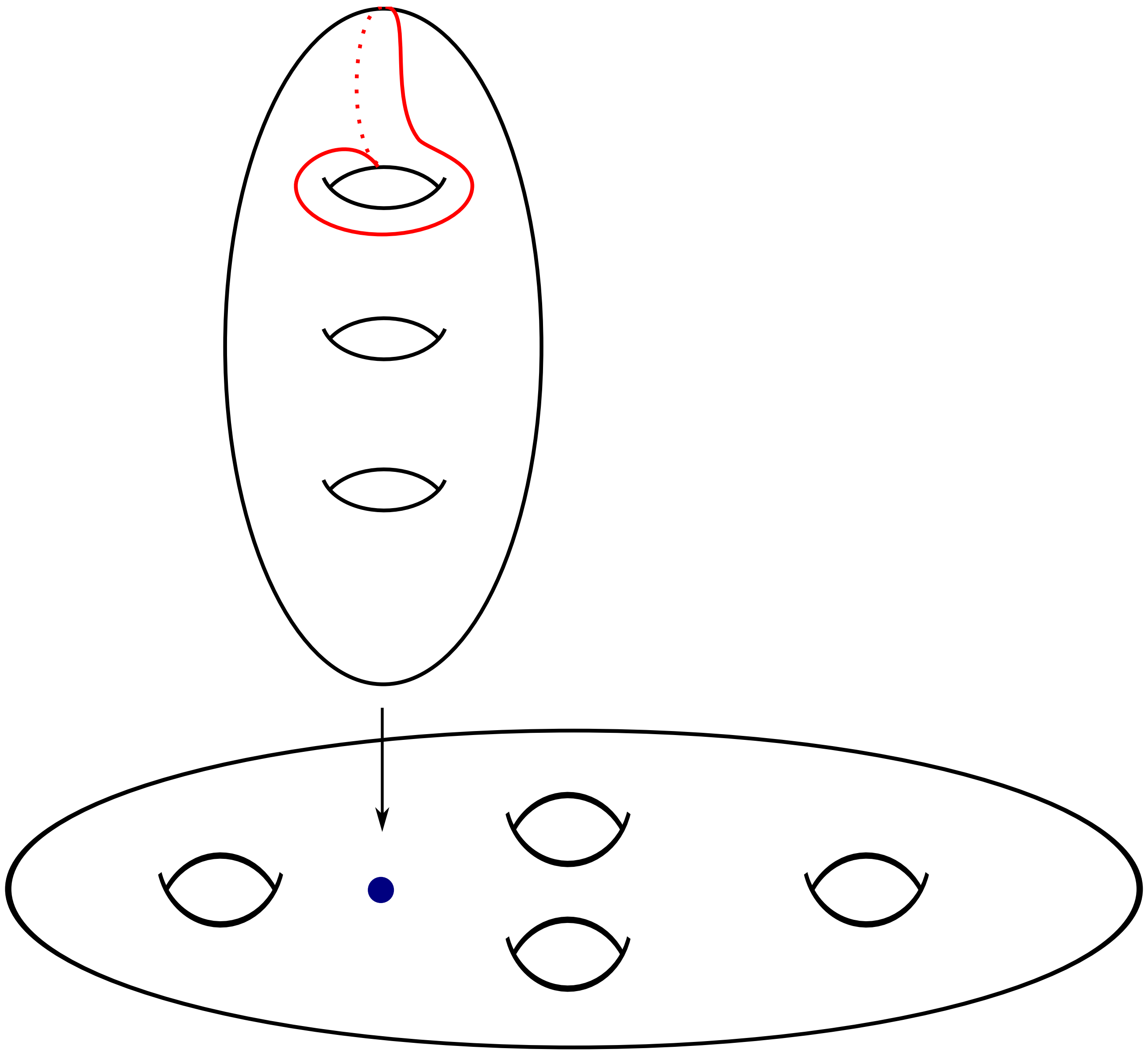

Monodromy: homomorphism $\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma_g)$

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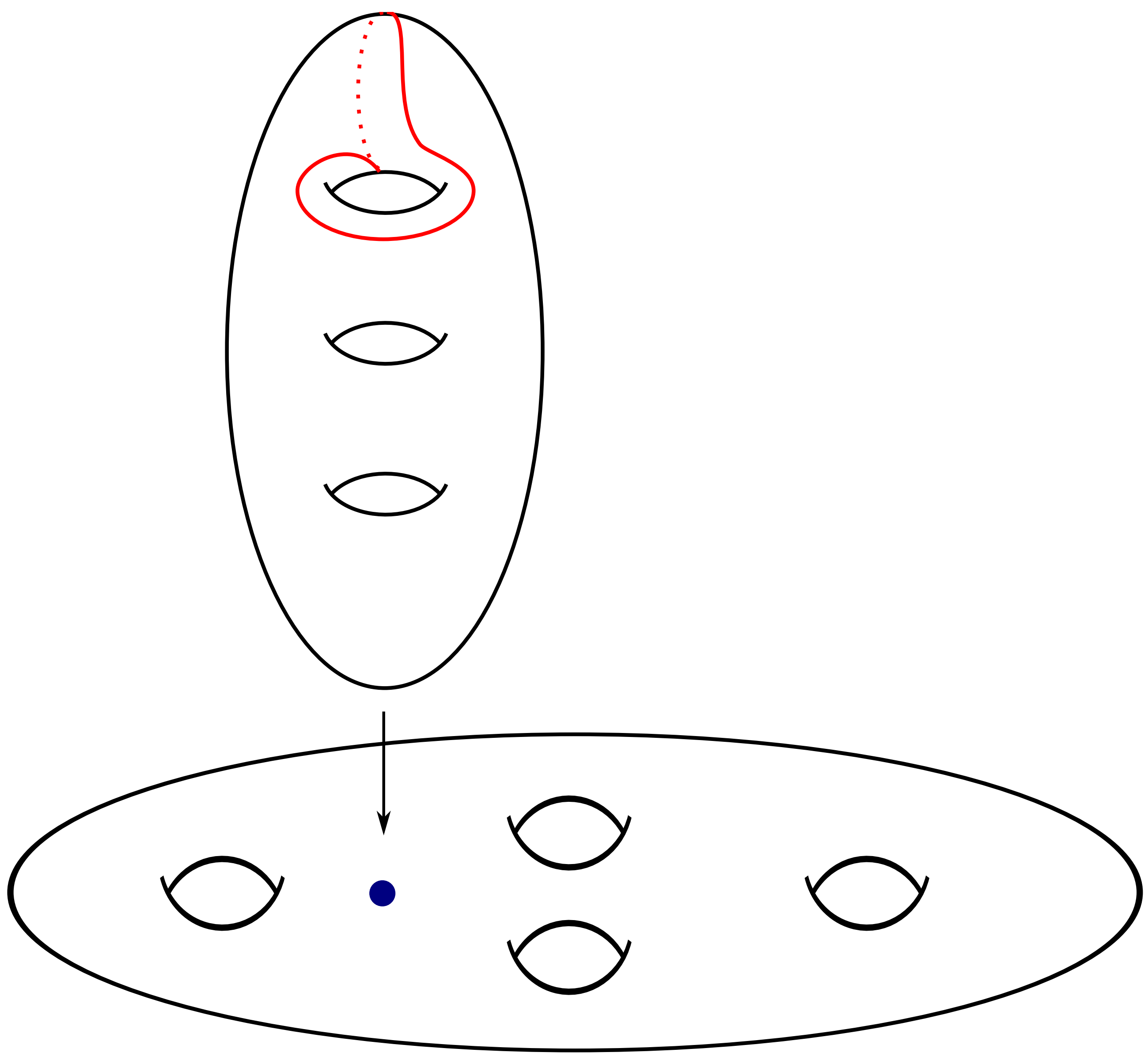


Monodromy: homomorphism $\overline{\rho} : \pi_1(B) \to \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$

Arithmeticity

Basic question: How big is monodromy of “naturally occurring” families?

What is the algebraic structure of monodromy groups?

Arithmeticity is a certificate of “bigness”

Arithmetic groups have rich algebraic/number theoretic structure

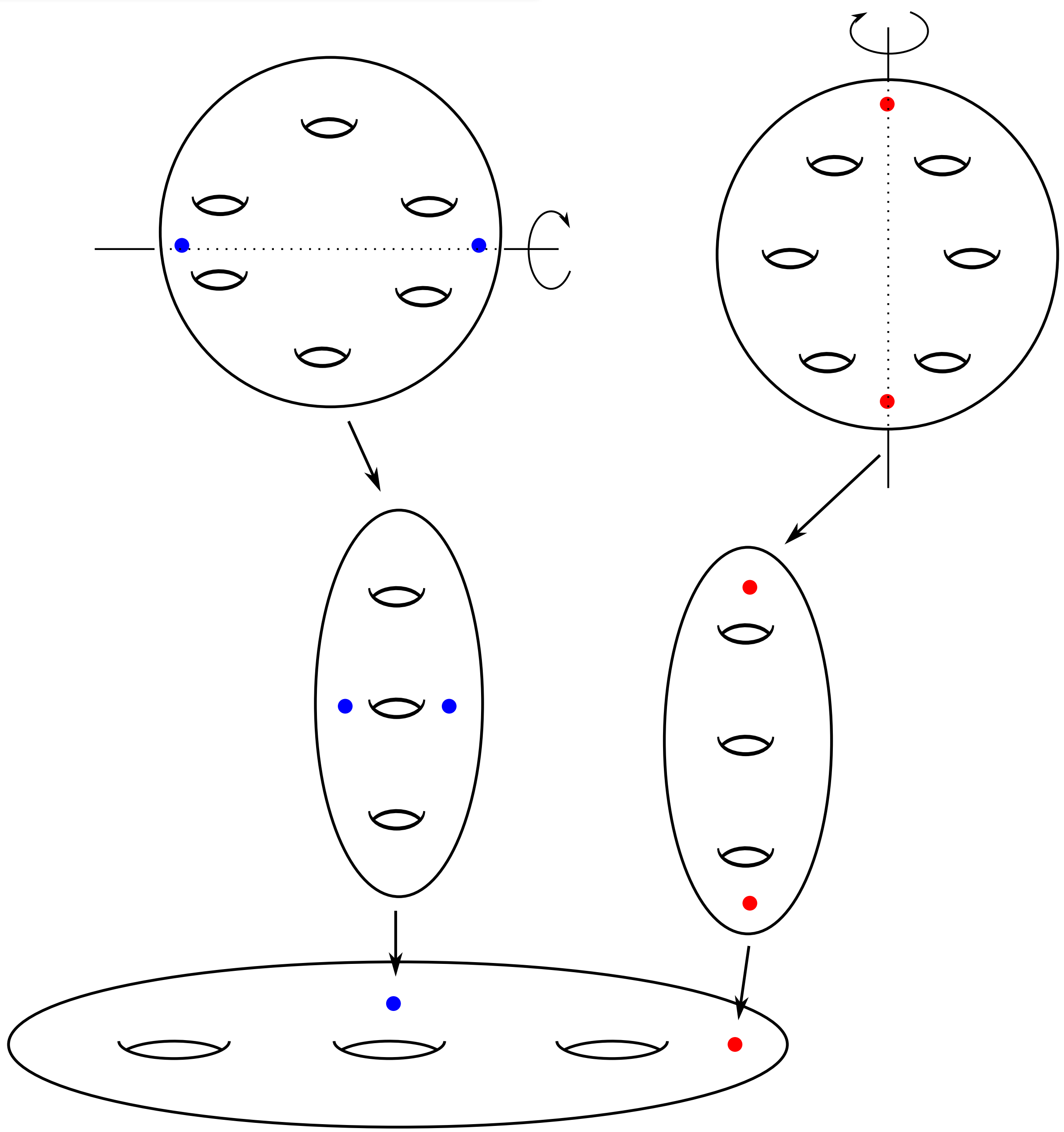
Think of $SL_n(\mathbb{Z}),\ \ Sp(2g, \mathbb{Z}),\ \ SL_2(\mathbb{Z}[\sqrt{2}])$

Technically, a group Γ is *arithmetic* if there is an ambient Lie group $G \leq SL_N(\mathbb{R})$ with $\Gamma \leq G \cap SL_N(\mathbb{Z})$ of finite index.

Some history

- Griffiths-Schmid ('75): Ask when monodromy of a family of algebraic varieties is arithmetic
- Deligne-Mostow ('86): Finds families of algebraic curves over $\mathrm{Conf}_n(\mathbb{C})$ with nonarithmetic monodromy
- Looijenga ('97): Studies abelian covers of surfaces, obtains lots of arithmetic representations of mapping class groups
- McMullen ('09): Systematic study of families over $\mathrm{Conf}_n(\mathbb{C})$, producing lots of interesting representations of braid groups
- Venkataramana ('14): Shows "most" of McMullen's representations yield arithmetic groups
- Grunewald-Larsen-Lubotzky-Malestein ('15): Studies *nonabelian* covers of surfaces, produces vast number of arithmetic representations of mapping class groups

Atiyah-Kodaira



$$E \to X \times X$$

(fiberwise cyclic branched covering of algebraic surfaces (4-manifolds))

$$\bigcup \Gamma_{\tau^i} \subset X \times X$$

(branch locus)

$$\langle \tau \rangle \cong \mathbb{Z}/m\mathbb{Z} \circlearrowright X$$

Goal: Study these monodromy groups $\Gamma_{g,m} \leq \mathrm{Sp}(2g', \mathbb{Z})$

Main Theorem

Main theorem (fake news version): For m ≥ 2, g ≥ 3, the groups $\Gamma_{g,m}$ are arithmetic lattices.

Issue: Insufficient symmetry. $Z \to X$ regular $\mathbb{Z}/m\mathbb{Z}$ branched cover, but $Z \to X \to (X/\langle\tau\rangle)$ *is not regular.* Makes studying $\Gamma_{g,m}$ very difficult.

Fix: pass to further cover $W \to Z$ so that $W \to (X/\langle\tau\rangle)$ is regular with covering group $H(\mathbb{Z}/m\mathbb{Z})$ (Heisenberg group).

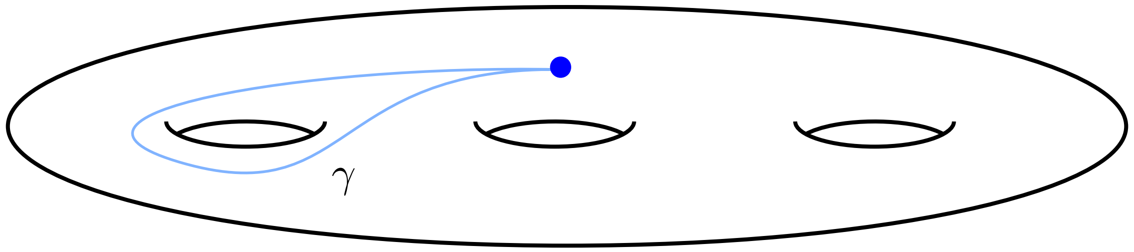
New bundle $\tilde{E}$, new monodromy group $\widetilde{\Gamma}_{g,m}$.

Main theorem (S., Tshishiku): For m ≥2, g≥3, the groups $\widetilde{\Gamma}_{g,m}$ are arithmetic lattices.

Points of interest

- Griffiths-Schmidt program. Novelty: base is *complete* (closed surface), not merely *quasiprojective* (configuration space).
- Requires developing new topological tools to study monodromy (no Picard-Lefschetz formula available)
- Corollary: number of surface bundle structures on AK manifolds (extending work of Lei Chen)
- Question: Is the original monodromy group $\Gamma_{g,m}$ arithmetic?

AK monodromy, topologically



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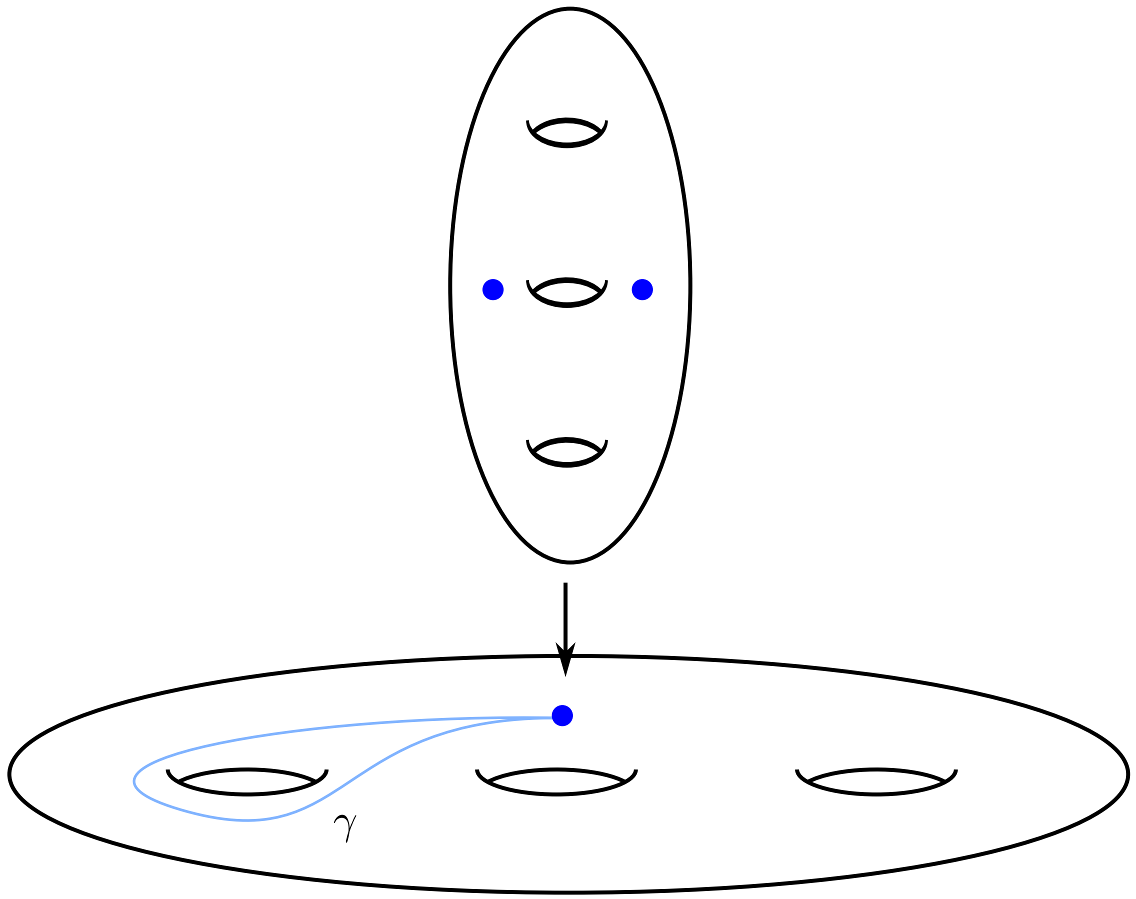

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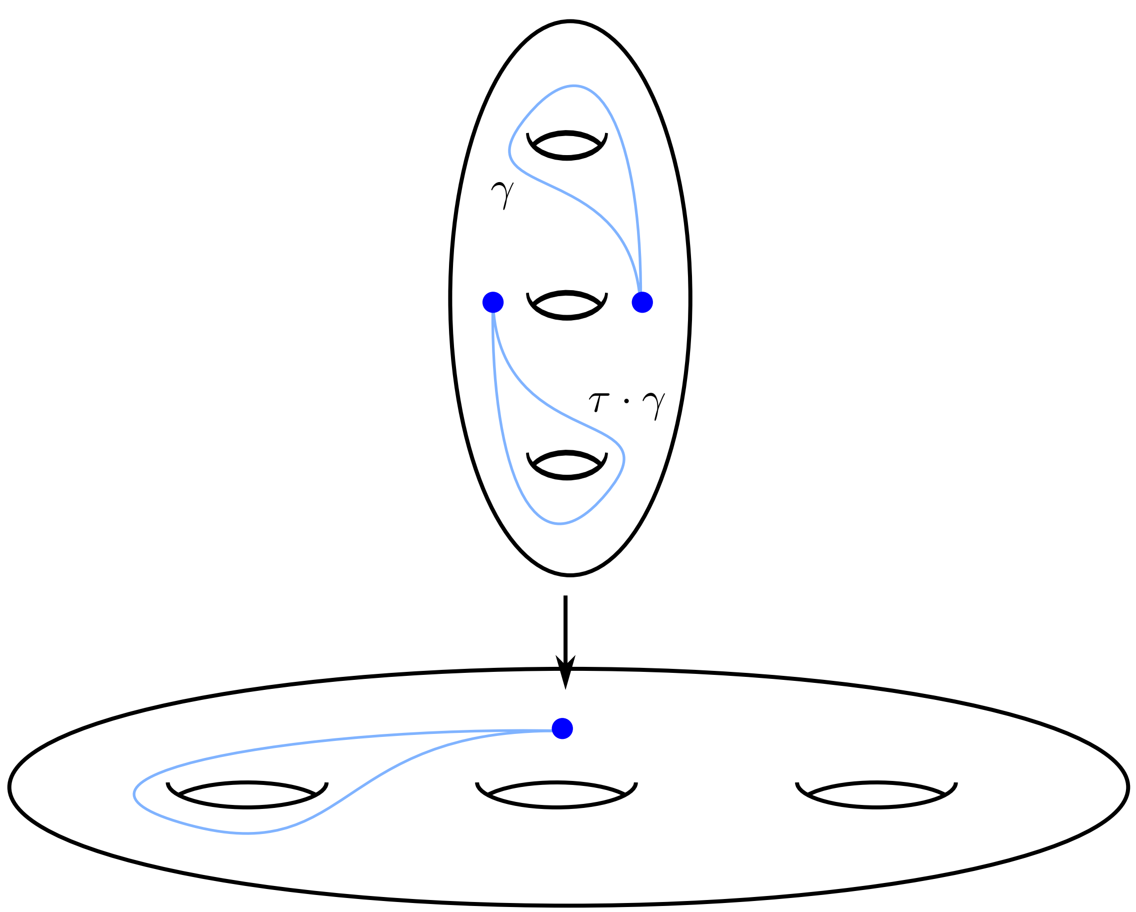

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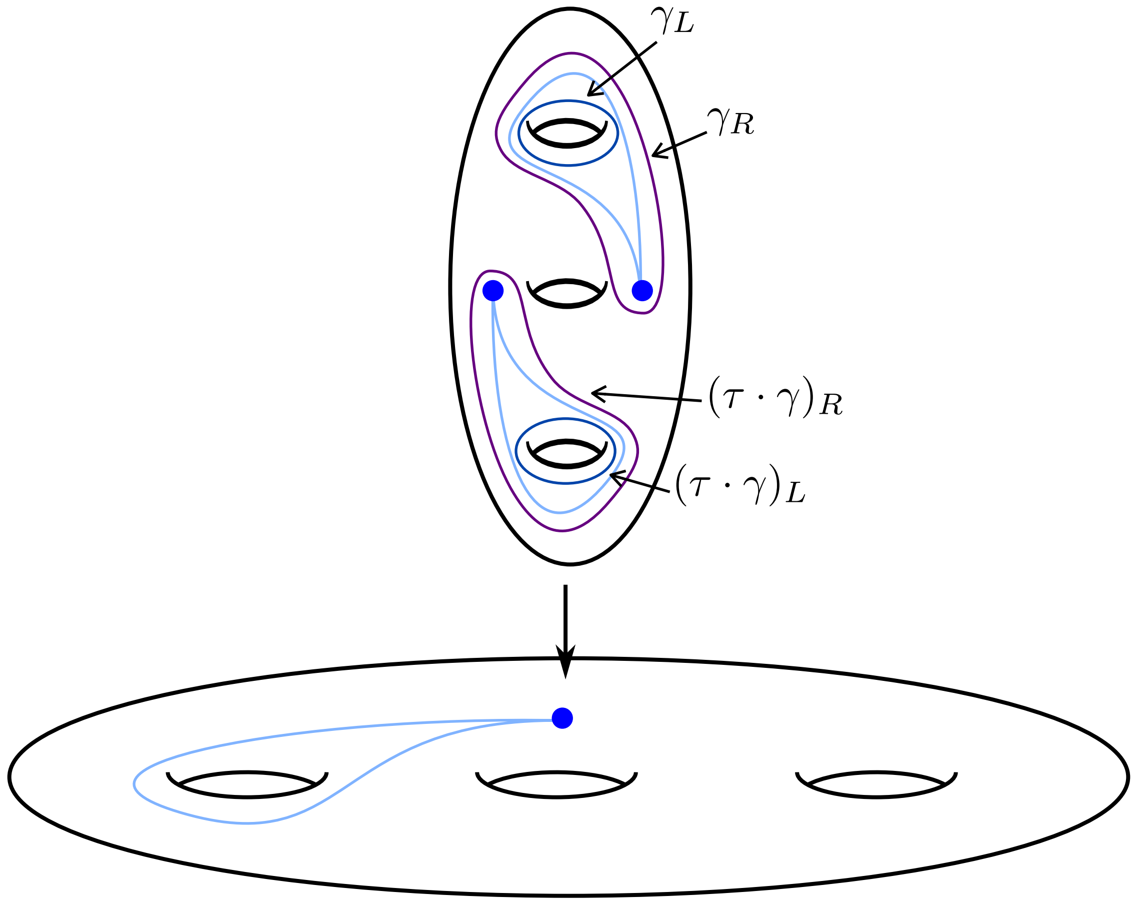

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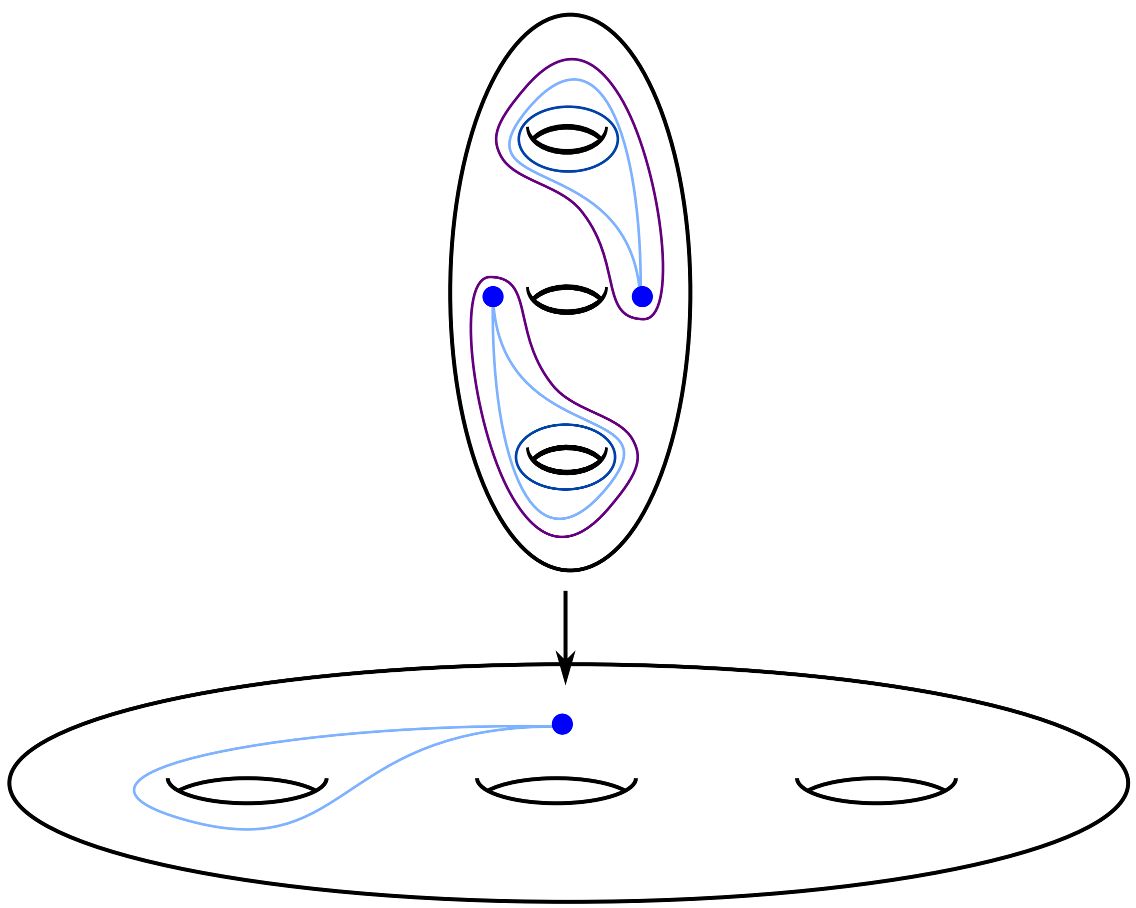

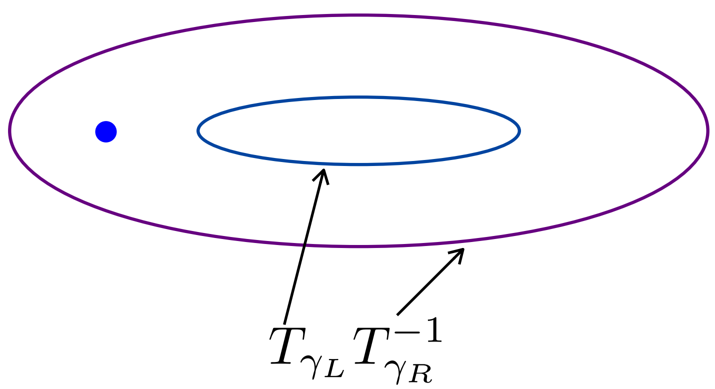

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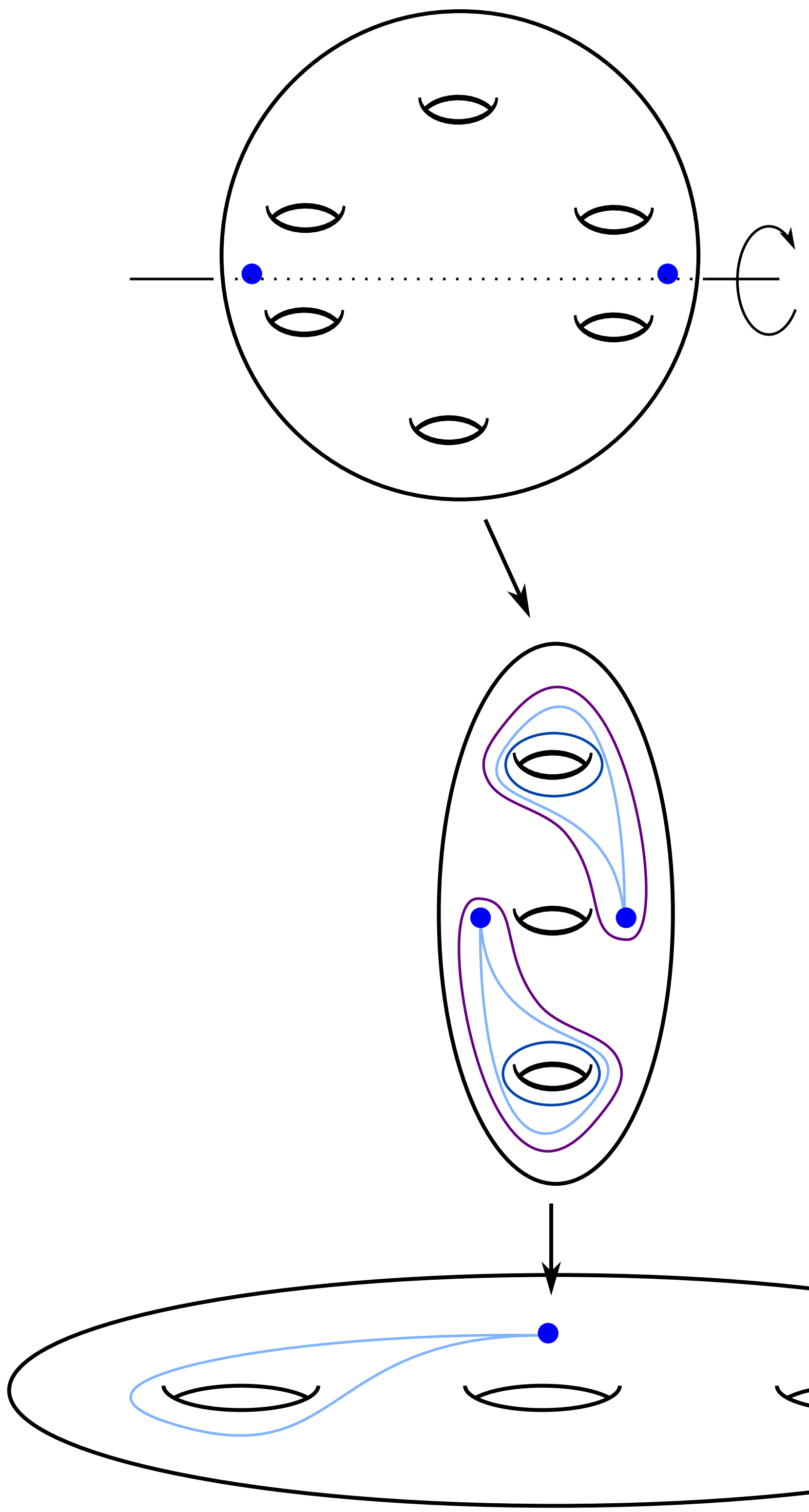

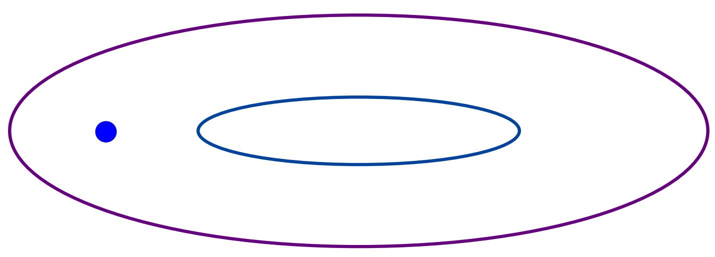

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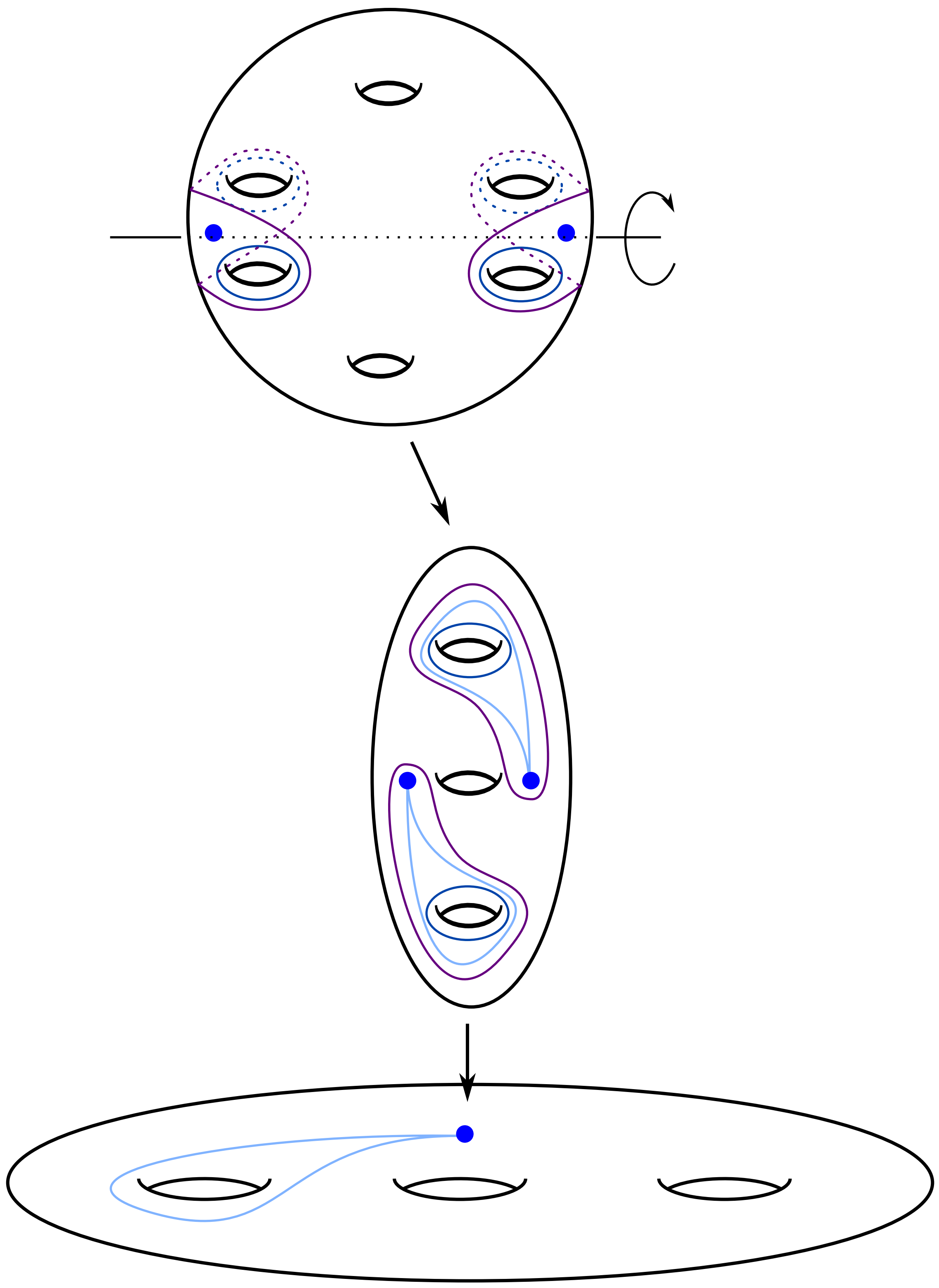

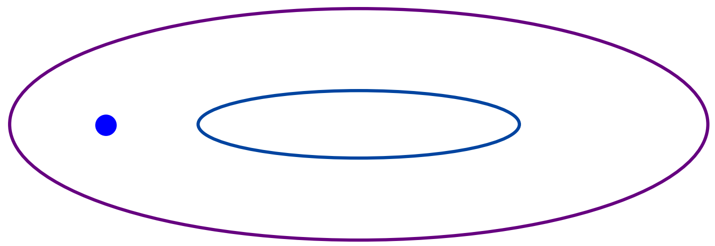

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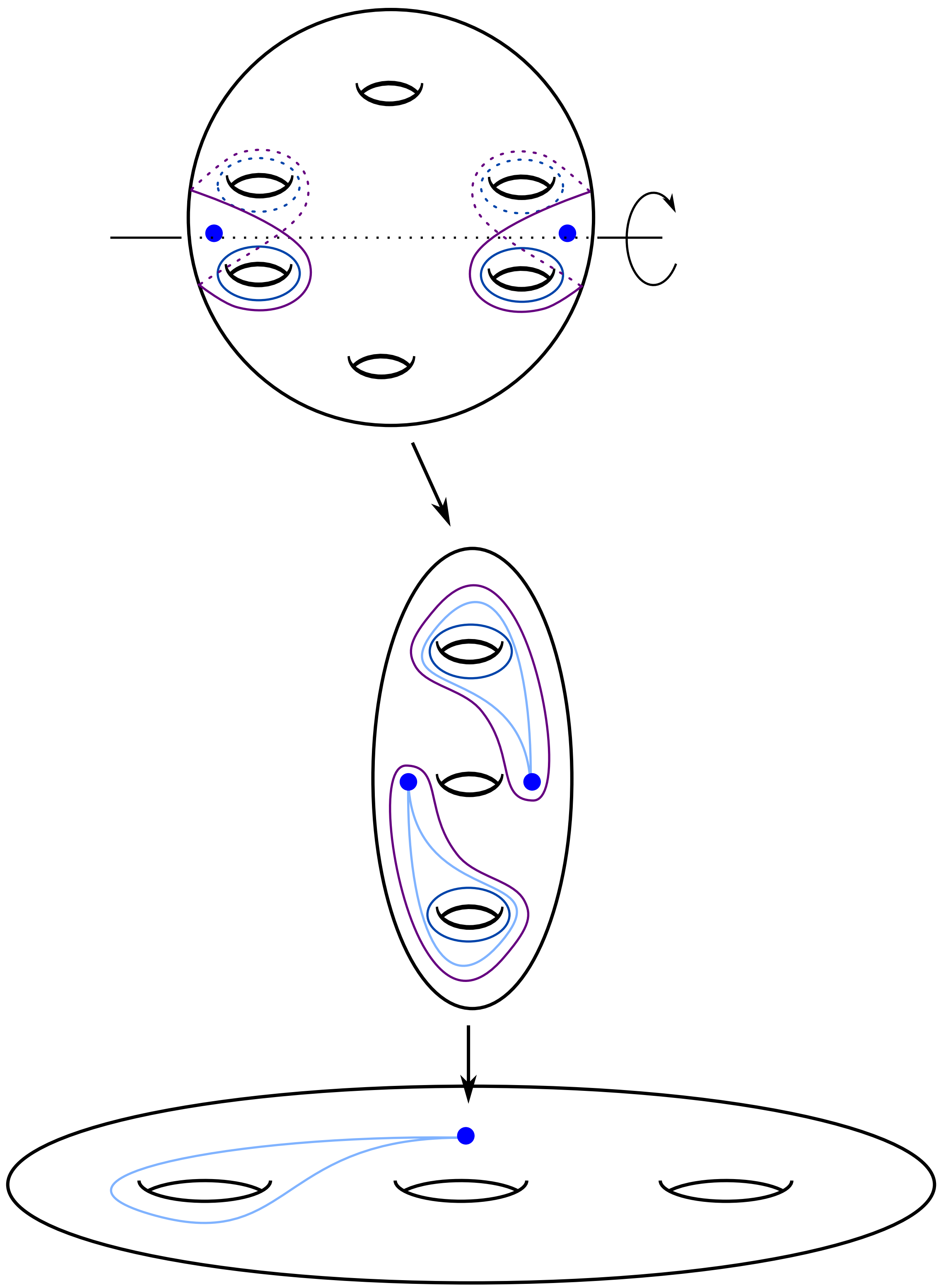

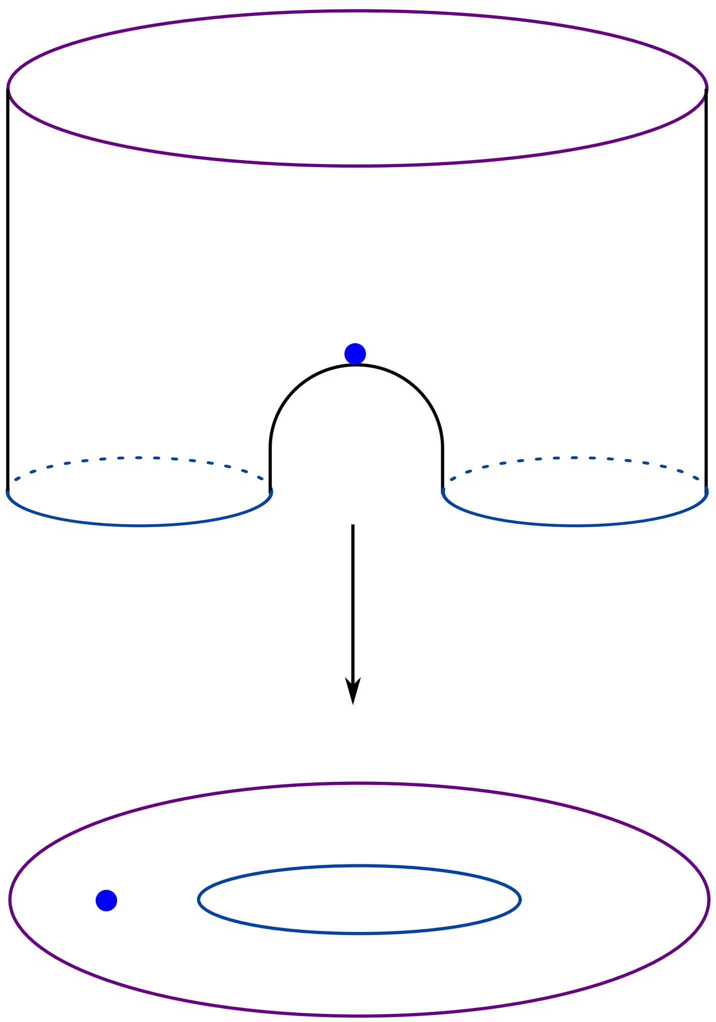

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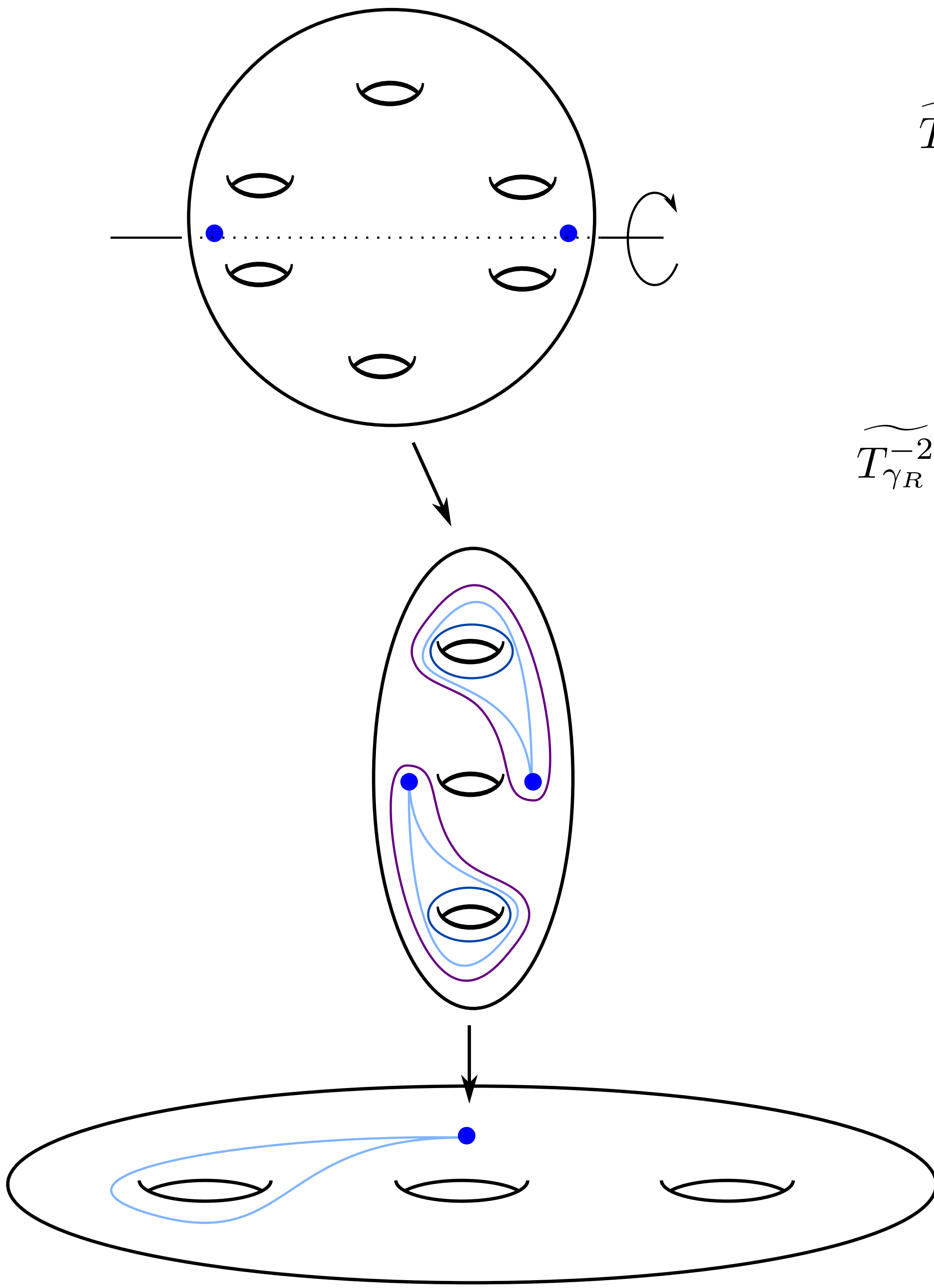
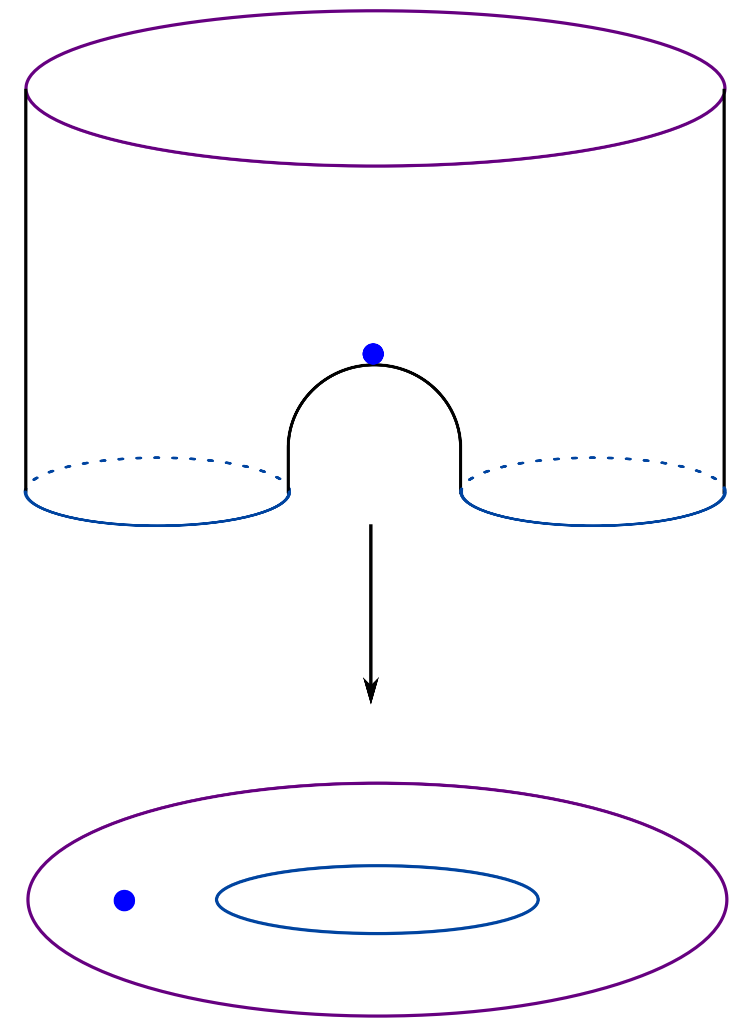


$$\widetilde{T^2_{\gamma_L}} = T_{\tilde{\gamma}_L}$$

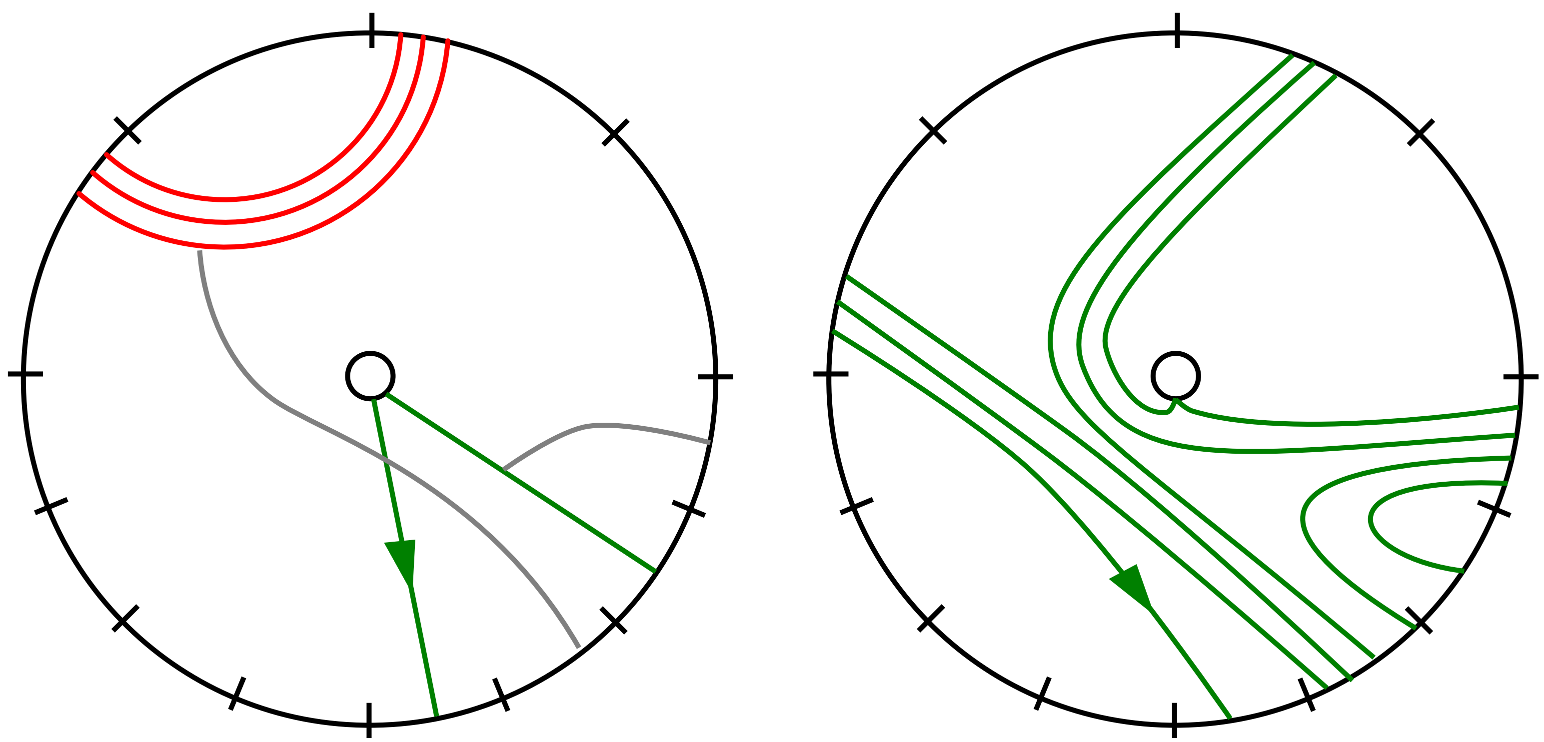
$$\widetilde{T^{-2}_{\gamma_R}} = T^{-2}_{\tilde{\gamma}_R} T^{-2}_{\tau\cdot\gamma_R}$$



Structure of the argument

- Show that simple point pushes “act like Dehn twists”

- Develop geometric tools to produce lots of simple point pushes



- Feed these into machines to certify arithmeticity