Simple closed curves in covers of surfaces and unitary K-theory

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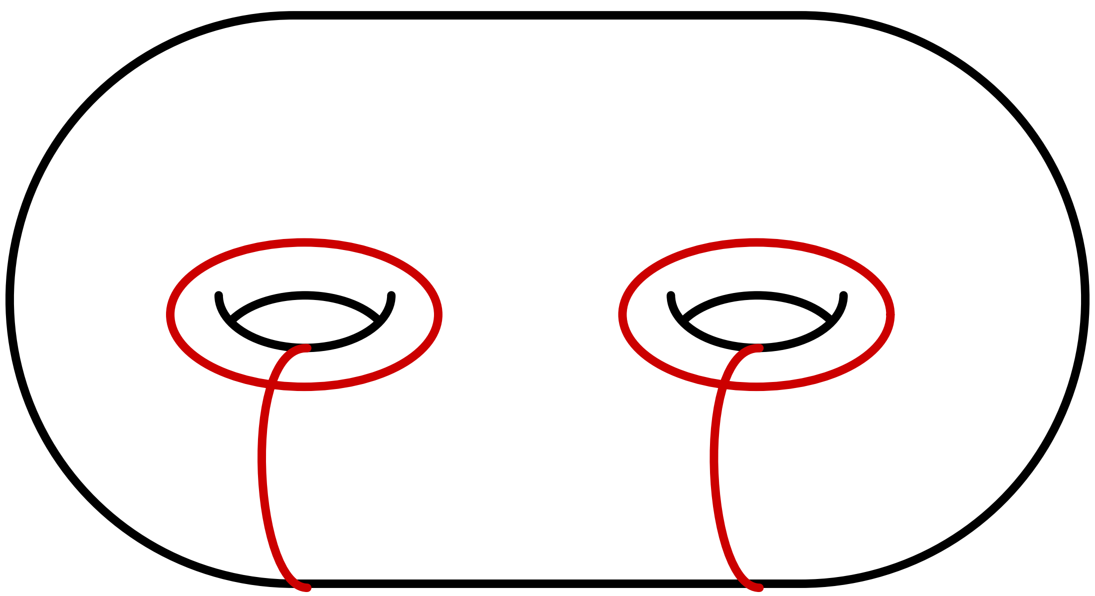
H_1 of a surface

Extremely classical fact: $H_1(\Sigma_g; \mathbb{Z})$ is generated by *geometric* classes.

A class $c \in H_1(\Sigma_g; \mathbb{Z})$ is *geometric* if
$c = [\gamma]$ for some *simple* closed curve $\gamma \subset \Sigma_g$



There is a purely algebraic criterion for geometricity:

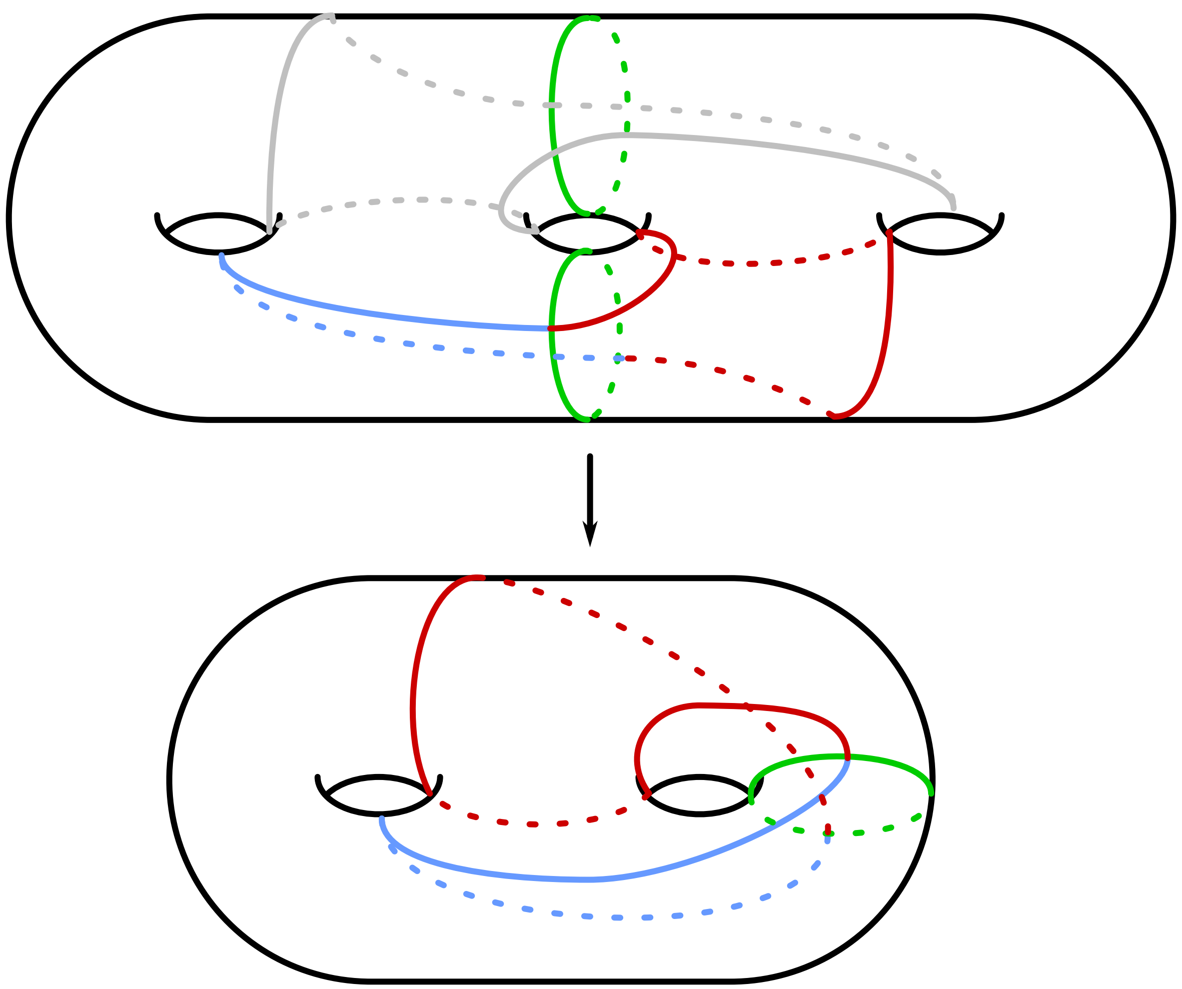
$c \in H_1(\Sigma_g; \mathbb{Z})$ is geometric if and only if
c is *primitive:* c is not a proper multiple of any other vector.
Equivalently, if the entries of c generate the unit ideal in $\mathbb{Z}$.

H_1 of a surface, relative version

Talk today: the *relative* version of this story.

Fix $f : X \to Y$ a *map* of surfaces with Y of finite type.

Typically f is a *regular* covering, possibly branched, with deck group G.
Degree not necessarily finite.



A class $c \in H_1(X; \mathbb{Z})$ is *relatively geometric* if $c = [\tilde{\gamma}]$ for $\tilde{\gamma}$ a *component of* $f^{-1}(\gamma)$, with $\gamma \subset Y$ a s.c.c.

H_1 of a surface, relative version

Basic questions:

(1) Can you describe the *subspace* $H_1^{geom}(X; R) \leq H_1(X; R)$ spanned by relatively geometric classes?

(2) Can you describe the *set* of relatively geometric classes? That is, can you give a purely algebraic characterization?

Question (1) has been studied in the last decade.

A deep and rich story we don't have time to visit.

Far from completely understood, but we now have some methods to show strict containment, and examples where this happens.

Primitive homology

(1) Can you describe the *subspace* $H_1^{geom}(X; R) \leq H_1(X; R)$ spanned by relatively geometric classes?

Question 1 has been investigated over the last decade.
Often called "primitive homology" or "scc homology".

We now know many examples of coverings $f : X \to Y$
for which $H_1^{geom}(X; R) \neq H_1(X; R)$,
both for $R = \mathbb{Z}, \mathbb{Q}$ (latter is stronger!)

Koberda-Santharoubane `16: first examples; $R = \mathbb{Z}$.

Farb-Hensel `16: representation-theoretic criterion on G

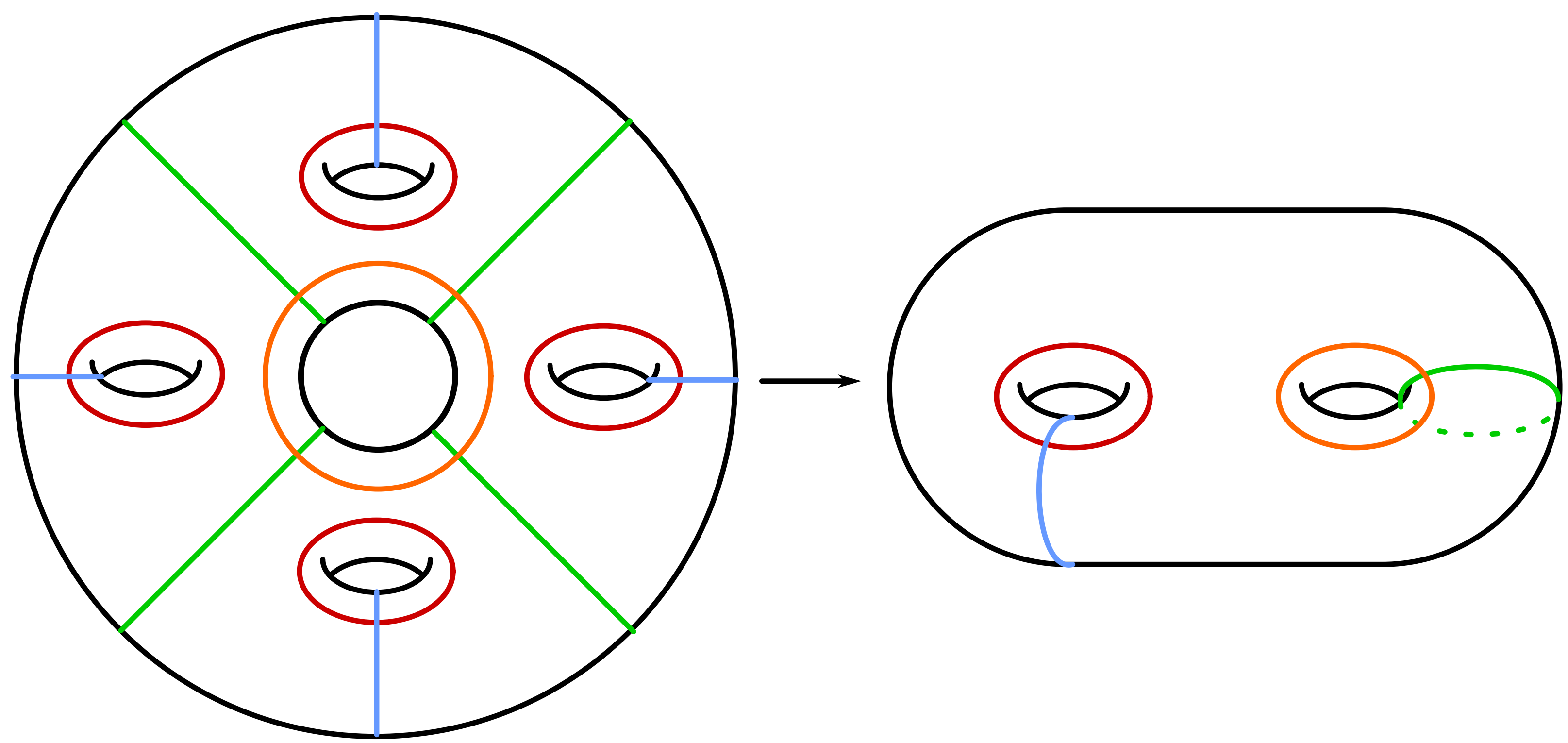
Malestein-Putman `18: infinite family of examples; $R = \mathbb{Q}$.

Lee-Rosenblum Sellers-Safin-Tenie `20: quite simple examples

(e.g. $|G| = 128$)

Our running example

Remainder of talk: will explore (2) for the $\mathbb{Z}/4\mathbb{Z}$ cover $f : \Sigma_5 \to \Sigma_2$.



Chevalley-Weyl: $H_1(\Sigma_5; \mathbb{Z}) = (\mathbb{Z}[t]/(t^4 - 1))^2 \oplus \mathbb{Z}^2$

Spanned additively by
$\tilde{x}_1, t\tilde{x}_1, t^2\tilde{x}_1, t^3\tilde{x}_1, \tilde{y}_1, t\tilde{y}_1, t^2\tilde{y}_1, t^3\tilde{y}_1, \tilde{x}_2, \tilde{y}_2$

On this basis, $f_*(t^k\tilde{z}) = z$, except $f_*(\tilde{x}_2) = 4x_2$.

Example 1

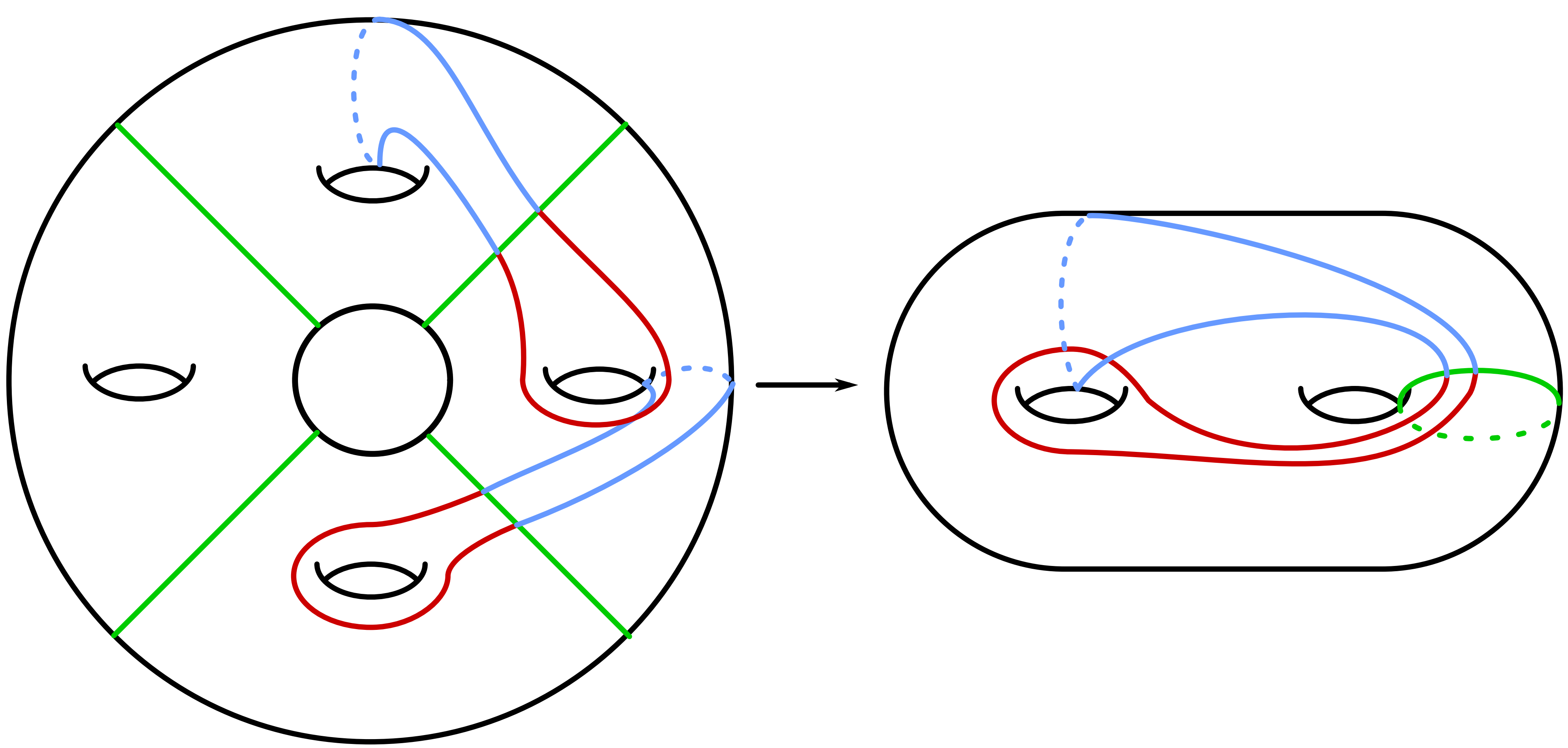
Is $v_1 = \tilde{x}_1 + t\tilde{y}_1$ relatively geometric?

Obstruction 1: isotropy

"Trivial" observation: components $\tilde{\gamma} \subset f^{-1}(\gamma)$ are *disjoint.*

So $(\tilde{\gamma}, g\tilde{\gamma}) = 0$ for any $g \in G$.

Can be expressed algebraically: *relative intersection pairing.*



Relative intersection pairing

There is a $\mathbb{Z}[G]$-valued *relative intersection pairing* on $H_1(X;\mathbb{Z})$:

$$\langle x, y\rangle := \sum_{g\in G} (x, gy)g$$

Here, (x, y) denotes the ordinary pairing

Skew-Hermitian: $\langle \alpha y, x\rangle = -\alpha\overline{\langle x, y\rangle}$ for $\alpha \in \mathbb{Z}[G]$
with $\bar{\cdot} : \mathbb{Z}[G] \to \mathbb{Z}[G]$ induced from $g \mapsto g^{-1}$ on G

But e.g. $\langle \tilde{x}_1 + t\tilde{y}_1, \tilde{x}_1 + t\tilde{y}_1\rangle = t^{-1} - t$

If v is relatively geometric,
then v is *isotropic:*
$\langle v, v\rangle = 0$.

Example 2

Is $v_2 = \tilde{x}_1 + t^2\tilde{y}_1$ relatively geometric?

Obstruction 2: *super*isotropy

Problem: $\langle \tilde{x}_1 + t^2\tilde{y}_1, \tilde{x}_1 + t^2\tilde{y}_1 \rangle = t^{-2} - t^2 = 0.$ 😔

"Accidental cancellation" can't detect crossings.

Solution: lift to a further double-cover where $t^{-2} \neq t^2$.

After accounting for arbitrary choices, get a function $q : H_1(X; \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$.

Say isotropic $x \in H_1(X; \mathbb{Z})$ is *superisotropic* if $q(x) = 0$.

When G has 2-torsion, rel. geom. vectors must be *super*isotropic.

Example 3

Is $v_3 = (1+t)\tilde{x}_1 + \tilde{x}_2$ relatively geometric?

Obstruction 3: primitivity

Given $v \in H_1(X;\mathbb{Z})$, let $I_v \lhd \mathbb{Z}[G]$ denote the pairing ideal

$$\boxed{I_v = \langle H_1(X;\mathbb{Z}), v\rangle}$$

The ideal $I_{v_3} = (1+t+t^2+t^3,\ 1+t) = (1+t)$ is proper.

It turns out I_v is *tightly constrained* for rel. geom. v!

Stabilizer ideal

For $v \in H_1(X; \mathbb{Z})$ with stabilizer subgroup $G_v \leq G$, define

$$I_{G_v} := \left(\sum_{g \in G_v} g \right)$$

Can show*: If $v \in H_1(X; \mathbb{Z})$ is relatively geometric, then

$$I_v = I_{G_v}$$

$I_v \leq I_{G_v}$: easy from definitions

$I_{G_v} \leq I_v$: construct a "partner curve" w for v.

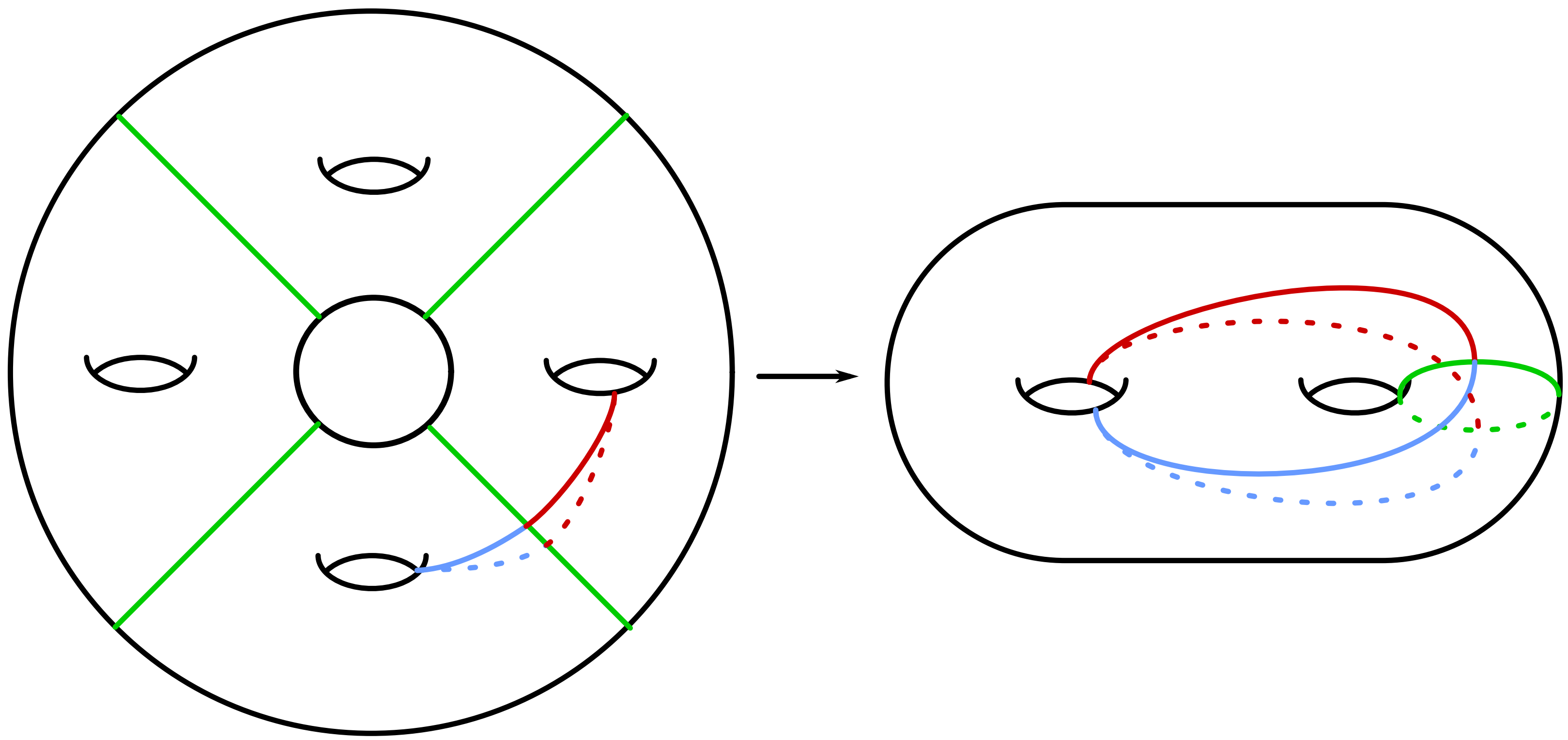
*: Actually this is a lie!

Example 4

Is $v_4 = (1-t)\tilde{y}_1$ relatively geometric?

Phenomenon: lifting separating curves

Actually, it is!



Formula $I_v = I_{G_v}$ breaks down when $f_*(v) = 0$.

Analysis of this case shows the following:

> If v is rel. geom. with $f_*(v) = 0$,
> then $v = (1-t)v'$ with v' rel. geom
> and $f_*(v') \neq 0$.

Main theorem

Summary of necessary conditions:

Let $f : X \to \Sigma_g$ be a cyclic unbranched covering of degree d (possibly $d = \infty$), and let $v \in H_1(X; \mathbb{Z})$ be relatively geometric.

(A) If $f_*(v) = 0$, then $v = (1-t)v'$ with v' rel. geom. and $f_*(v') \neq 0$.
(i.e. v' is in case (B)).

(B) If $f_*(v) \neq 0$, then the following conditions must hold:

(1) $\langle v, v \rangle = 0$	isotropy
(2) $q(v) = 0$	superisotropy (for d finite, even)
(3) $I_v = I_{G_v}$	$\mathbb{Z}[G]$-primitivity
(4) "degree-order condition"	

Theorem (S.): If $g \geq 5$, then the necessary conditions are sufficient.

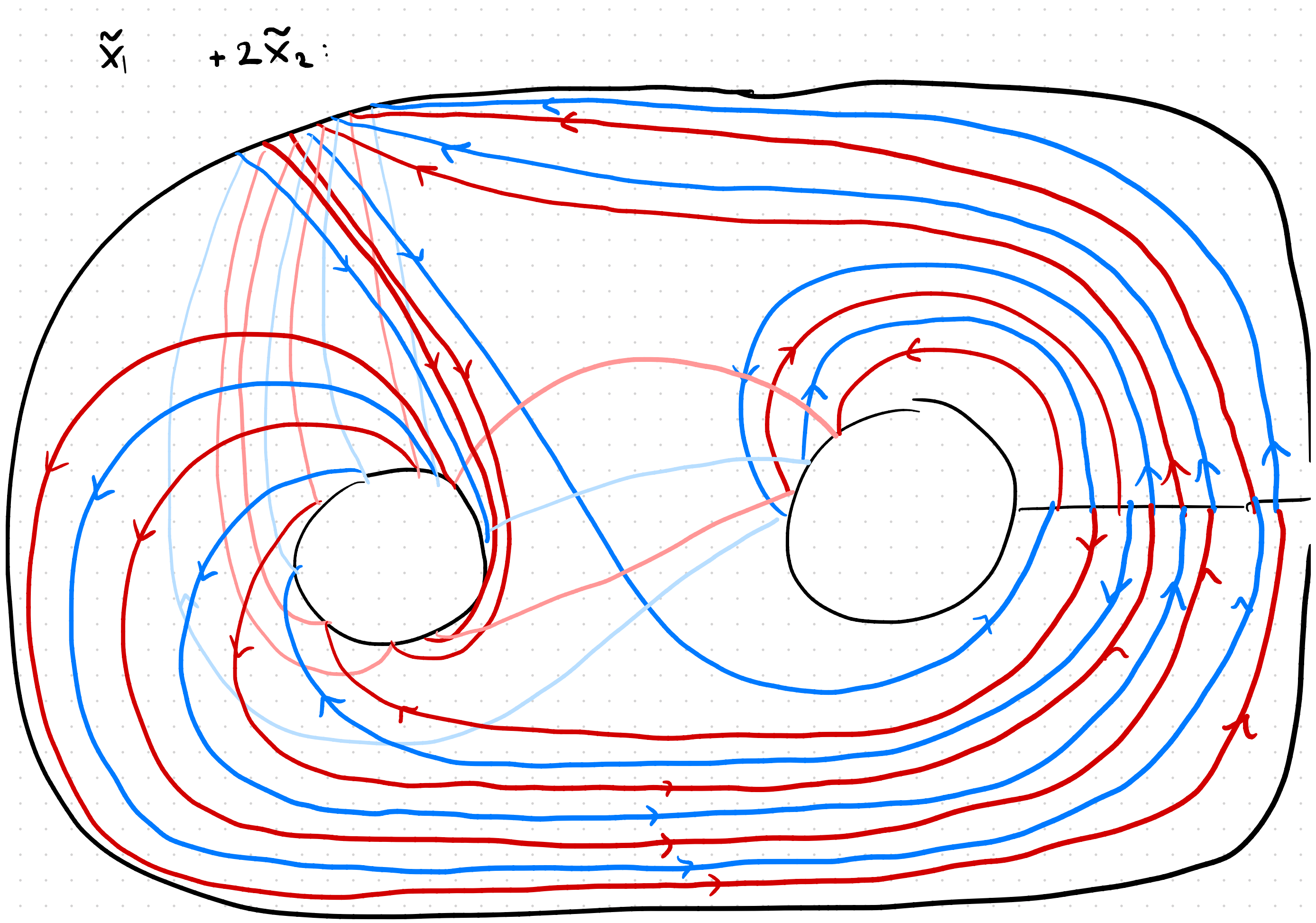
How not to prove this

What I wanted to do:

Use topology to do algebra!

Run a relative version of the “Euclidean algorithm on surfaces”: start with a cycle representing ν with many crossings/components, and resolve until ν has a relatively geometric representative.

I don’t know how to do this!



How ~~not~~ to prove this

Sadly had to resort to the other direction:

Use algebra to do topology

("Liftable subgroup" of) mapping class group $\mathrm{Mod}(Y)$ acts on $H_1(X;\mathbb{Z})$.

If you have one rel. geom. $v \in H_1(X;\mathbb{Z})$ and you completely understand the orbit of v, can understand *all* rel. geom. elements.

Lots of authors (e.g. Looijenga, McMullen, Venkataramana, Grunewald-Larsen-Lubotzky-Malestein) have investigated these representations.

Unfortunately, no result has yet been precise enough to do what I need.

Theorem (S.): Complete computation of $\mathrm{Mod}(\Sigma_g) \circlearrowright H_1(X;\mathbb{Z})$ for $f : X \to \Sigma_g$ cyclic unbranched, $g \geq 5$.

Unitary K-theory provides tools to study these sorts of matrix groups: show generation by elementary matrices.

A look ahead

Bregman and I are working on pushing this story further

Ultimate goal: *describe the image of the Burau representation*

Can be approached by understanding relative geometricity
for the Burau cover of the punctured disk.

Strange things seem to be happening.

Theorem (Bregman-S.): At least one of the following is true:

- The Burau representation for B_4 is non-injective
- The image of Burau is “much smaller image than expected”