Topology of strata of translation surfaces:
an unfortunately comprehensive survey

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Represents joint work with Aaron Calderon
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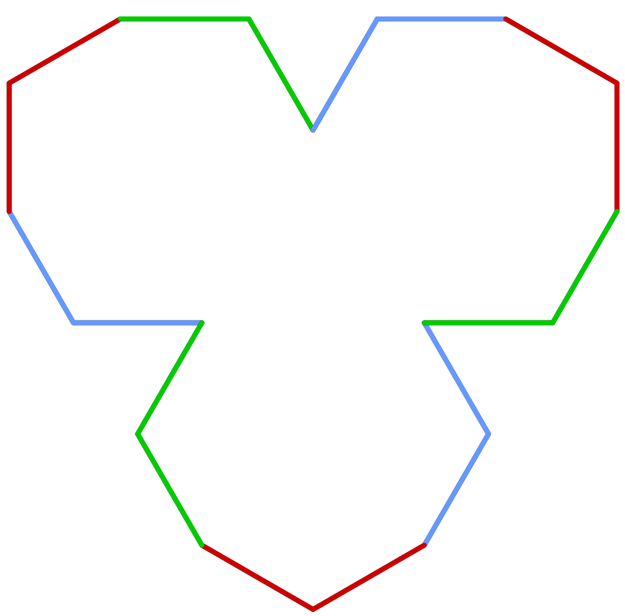
Translation surfaces

Flat geometry

Polygon with parallel edges glued by translation
(up to cut/translate/paste)



Cone angle

Algebraic geometry

Pair (X, ω) of holomorphic 1-form on a Riemann surface

dz

$$\left(x^3 + y^4 = 1,\ \frac{dx}{y^3}\right)$$

$$x \mapsto \int_{x_0}^{x} \omega$$

Order of zero

Strata

Pick a genus g and a partition $\kappa = \{\kappa_1, \dots, \kappa_n\}$ of $2g-2$

Flat geometry:

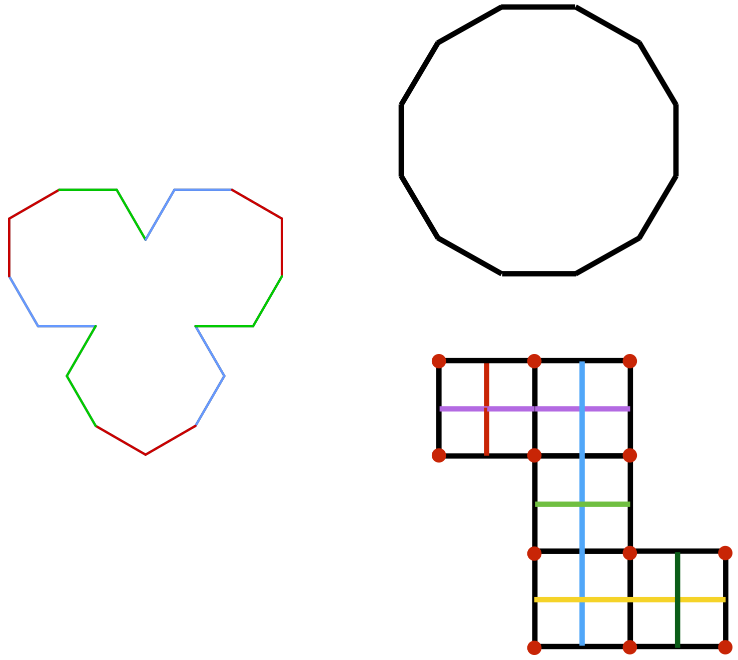
$\mathscr{H}\Omega(\kappa)$: space of translation surfaces with cone-angle set given by κ.

Algebraic geometry:

$\mathscr{H}\Omega(\kappa)$: space of pairs (X, ω) with X a genus-g Riemann surface, ω a 1-form with multiplicities of $Z(\omega)$ given by κ

Complex orbifolds of dimension 2g + n - 1 (*period coordinates*)

Virtually *quasiprojective varieties*:
- Finite CW structure
- Hence $\pi_1(\mathscr{H}\Omega(\kappa))$ finitely presented.

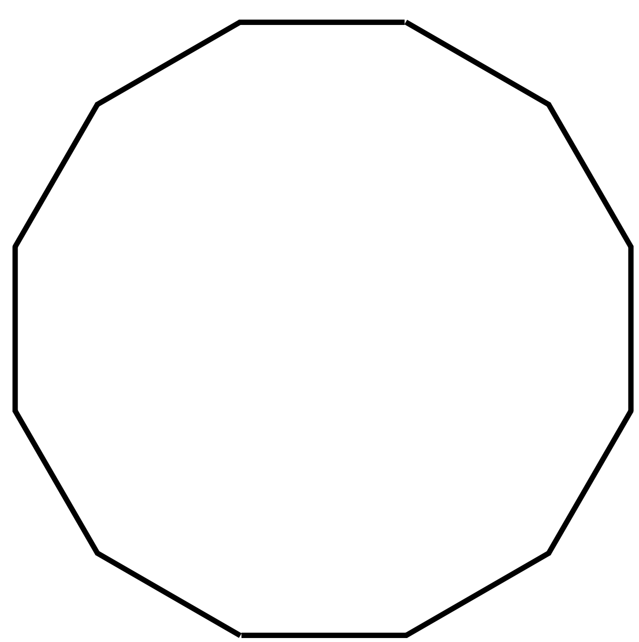


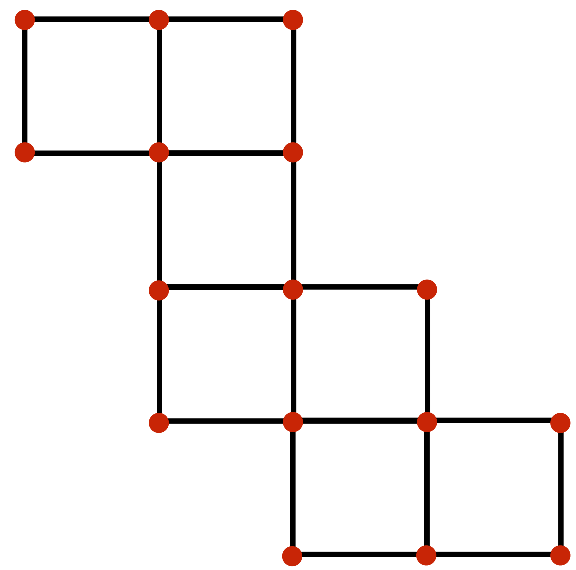
Components

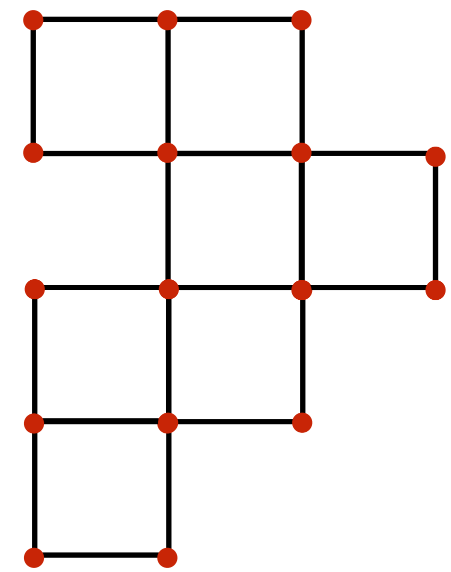
$\pi_0(\mathscr{H}\Omega(\kappa))$ determined by Kontsevich-Zorich

If $\kappa = \{2g-2\}$ or $\{g-1, g-1\}$, there is a special component of *hyperelliptic differentials*

Set $r = \gcd(\kappa)$. If r is even, then there are exactly two non-hyperelliptic components of $\mathscr{H}\Omega(\kappa)$, and one otherwise.





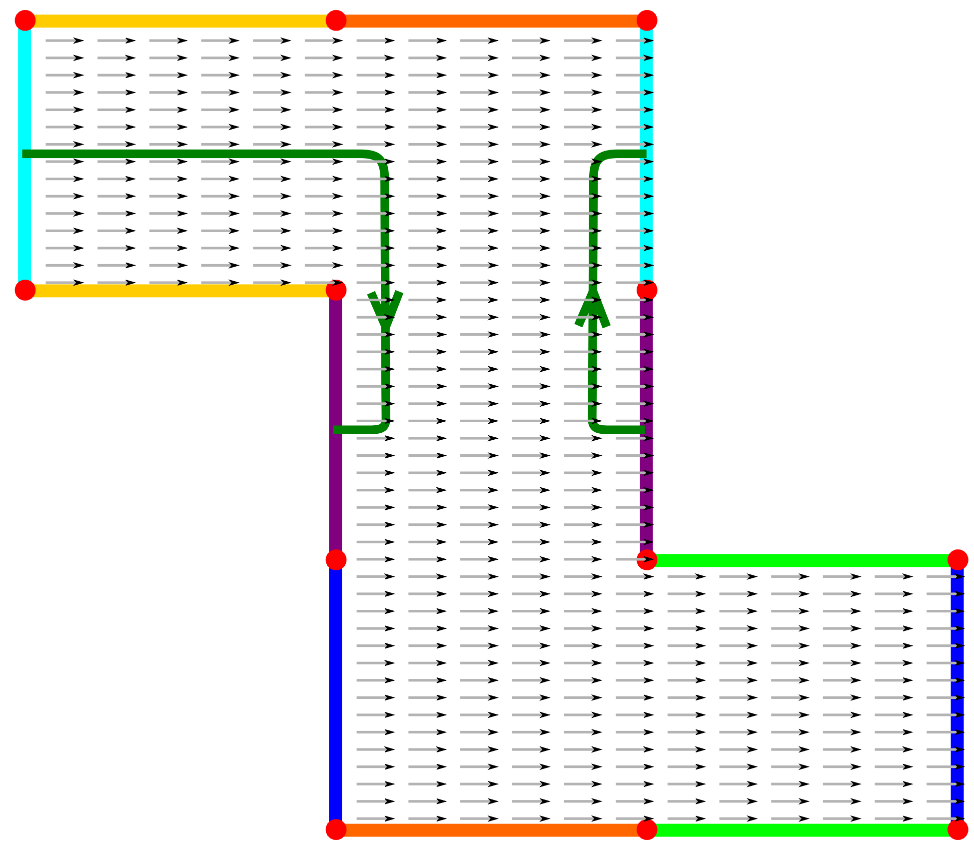


Convention: When κ is understood, will write $\mathscr{H}$ for a stratum-component

Framings, spin structures, parity

Invariant: *parity of spin structure*

Translation surfaces are canonically *framed.*



Surface topology fact: when r is even, there are *two* MCG orbits of framings, one otherwise. “Arf invariant”

Kontsevich-Zorich “conjecture”

In 1997, Kontsevich and Zorich stated the following conjecture:

Each stratum-component is a $K(\pi,1)$ space for π a group commensurable with some mapping class group.

I have heard that one or both of the authors no longer believe this, and there are reasons to doubt both parts, but regardless, it is asking the right questions:

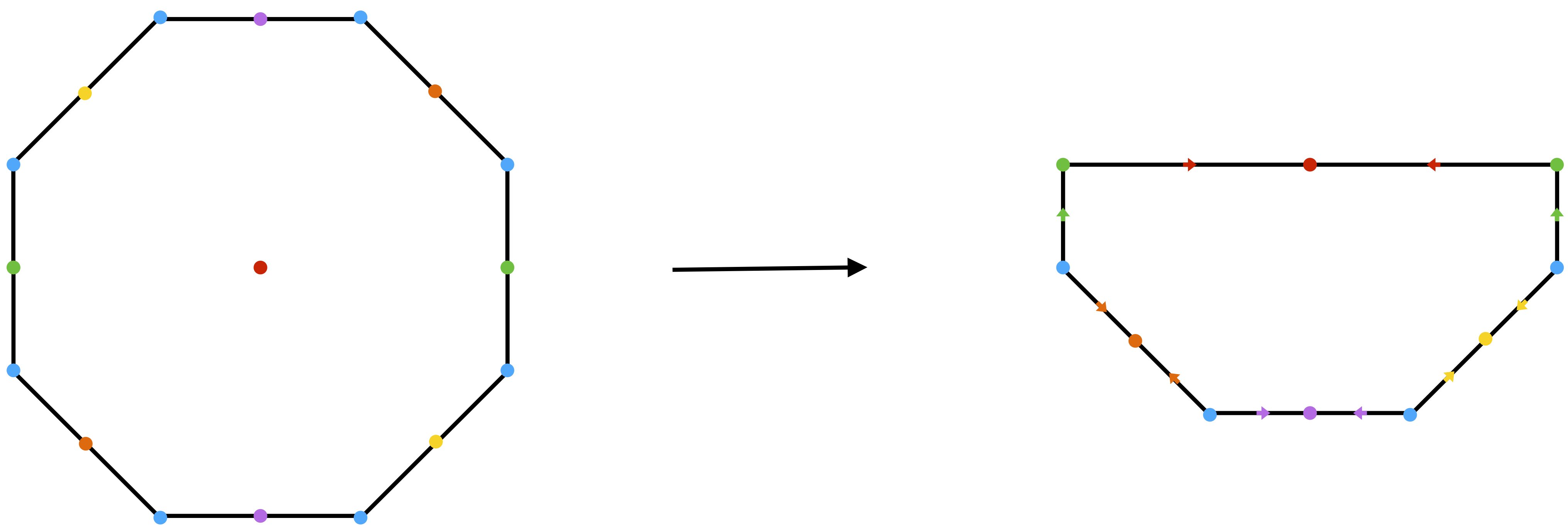
What is the homotopy type of a stratum? In particular, what kinds of groups appear as the fundamental group?

Sporadic examples: hyperelliptic

In special cases, the algebraic geometry is simple enough to understand $\mathscr{H}\Omega(\kappa)$ quite well.

Hyperelliptic component $\mathscr{H}\Omega(\{2g-2\})^{hyp}$ is basically $\mathrm{Conf}_{2g+1}(\mathbb{C})$

Idea: take cover of $\widehat{\mathbb{C}}$ branched at $\{z_1, \ldots, z_{2g+1}, \infty\}$.
Pull back *quadratic differential* with simple poles at $\{z_i\}$.



Similar story for $\mathscr{H}\Omega(\{g-1, g-1\})^{hyp}$

Sporadic examples: genus 3

In special cases, the algebraic geometry is simple enough to understand $\mathscr{H}\Omega(\kappa)$ quite well.

Genus 3 special: every curve is either hyperelliptic or *planar quartic:* solution to degree-4 polynomial $f(x, y, z)$ on $\mathbb{CP}^2$.

Looijenga-Mondello use this to get a good understanding of most strata in genus 3

They find a close connection between π_1 of strata and *Artin groups* (of type E, in particular)

Root systems of type E enter the picture by way of *Del Pezzo surfaces*

Summary of results: genus 3

κ	π_1^{orb} of $\mathcal{H}\Omega(\kappa)$
$\{4\}$	$A(E_6)/\text{center}$
$\{3,1\}$	$A(E_7)/\text{center}$
$\{2,1^2\}$	$A(\hat{E}_7)/\langle(\Delta_{A_7}^{-1}\Delta_{E_7})^2,$ conjugation by $(\Delta_{A_7}^{-1}\Delta_{E_7})$ nontrivial involution$\rangle$
$\{2^2\}$	$A(E_6) \rtimes \text{Aut}(\Gamma_{\hat{E}_6})/\langle \Delta_{A_5\times A_1} = \Delta_{E_6}\rangle$

$\hat{\cdot}$: *affine* Artin group
Δ: Garside element

Disclaimer: from here on out, I will suppress orbifold issues!

Quotients of $\pi_1(\mathscr{H})$

General strategy for studying mystery group: *find quotients*

Given close relationship between strata and moduli space, one obvious quotient: *monodromy representation*

$$\rho : \pi_1(\mathscr{H}) \to \mathrm{Mod}(\Sigma_{g,n})$$

Question: what is the image?

Quotients of $\pi_1(\mathscr{H})$

Question: what is the image?

Aaron Calderon and I answered this for g ≥ 5
(c.f. also Hamenstädt for certain low-genus
strata)

Theorem (Calderon - S.): For $\mathscr{H}$ non-hyperelliptic, g≥5,

$$\mathrm{im}(\rho) = \mathrm{Mod}(\Sigma_{g,n})[\phi]$$

Here, $\mathrm{Mod}(\Sigma_{g,n})[\phi]$ is the *framed mapping class group:* the stabilizer of the framing ϕ coming from flat structure.

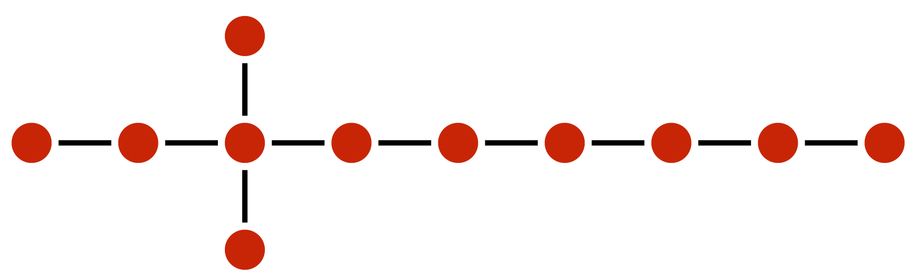
Infinite-index!

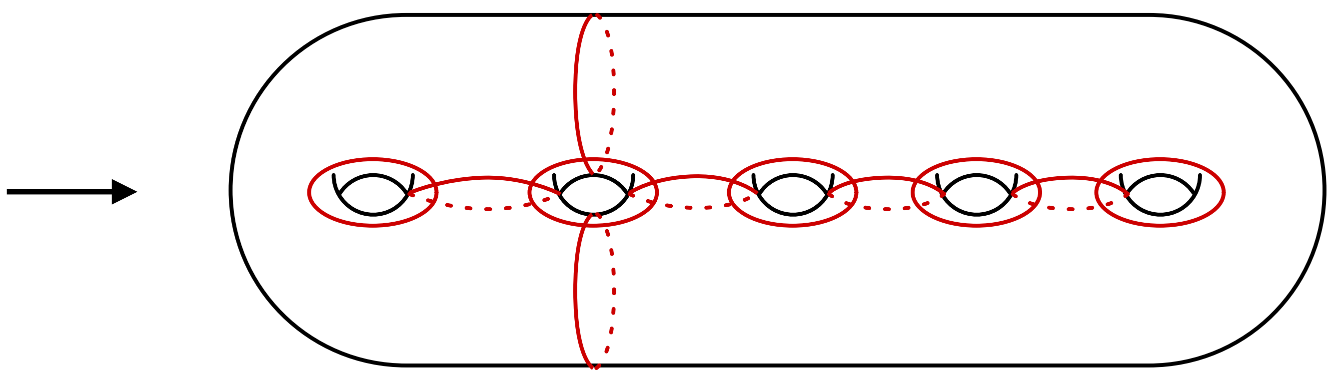
A word on the proof

All of the heavy lifting is in studying the framed mapping class group

We prove the following:

Theorem (C-S): Framed mapping class groups are *quotients of Artin groups,* with generators sent to Dehn twists.

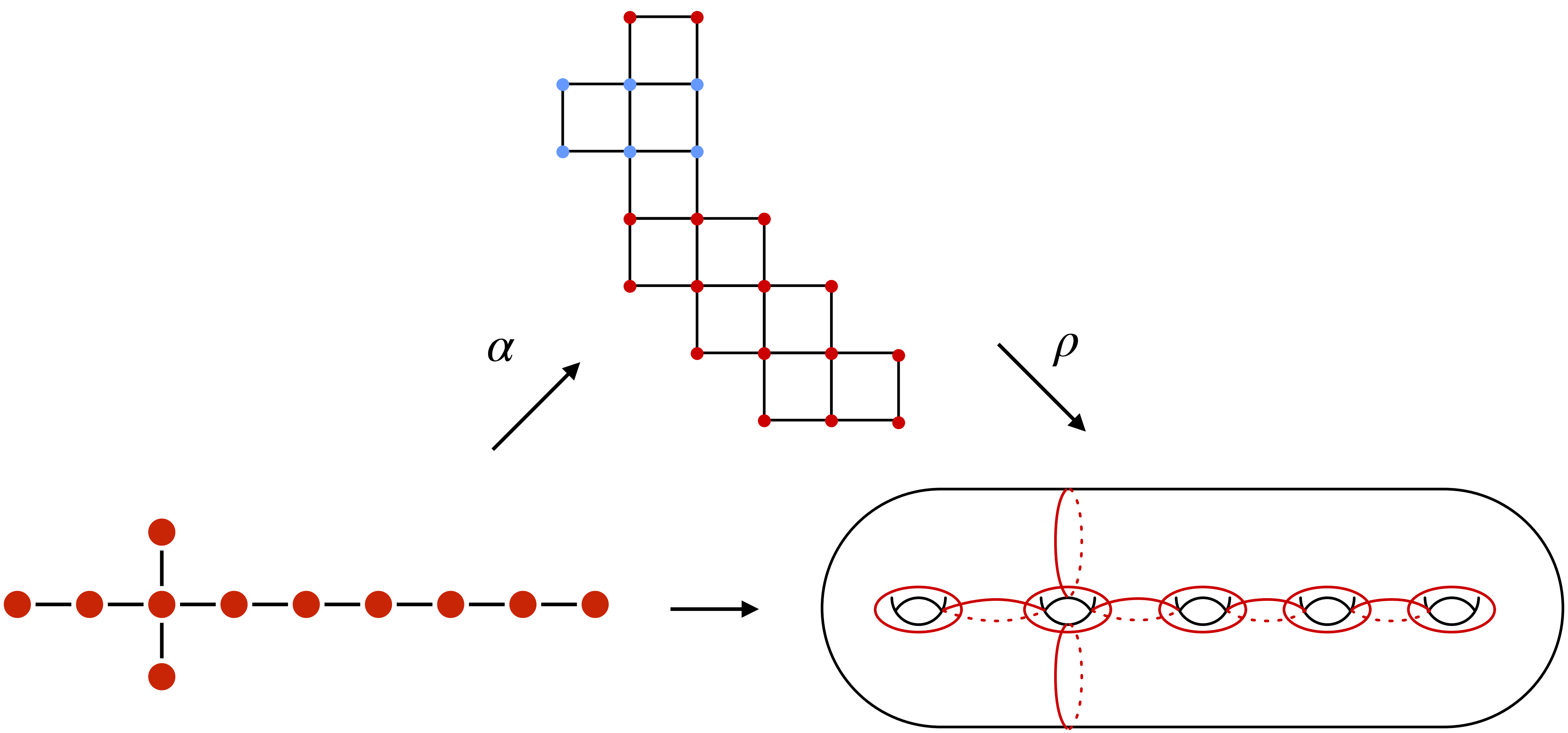




A word on the proof

To prove that $\rho : \pi_1(\mathscr{H}) \to \mathbf{Mod}(\Sigma_{g,n})[\phi]$ is a surjection, we map in the relevant Artin group!

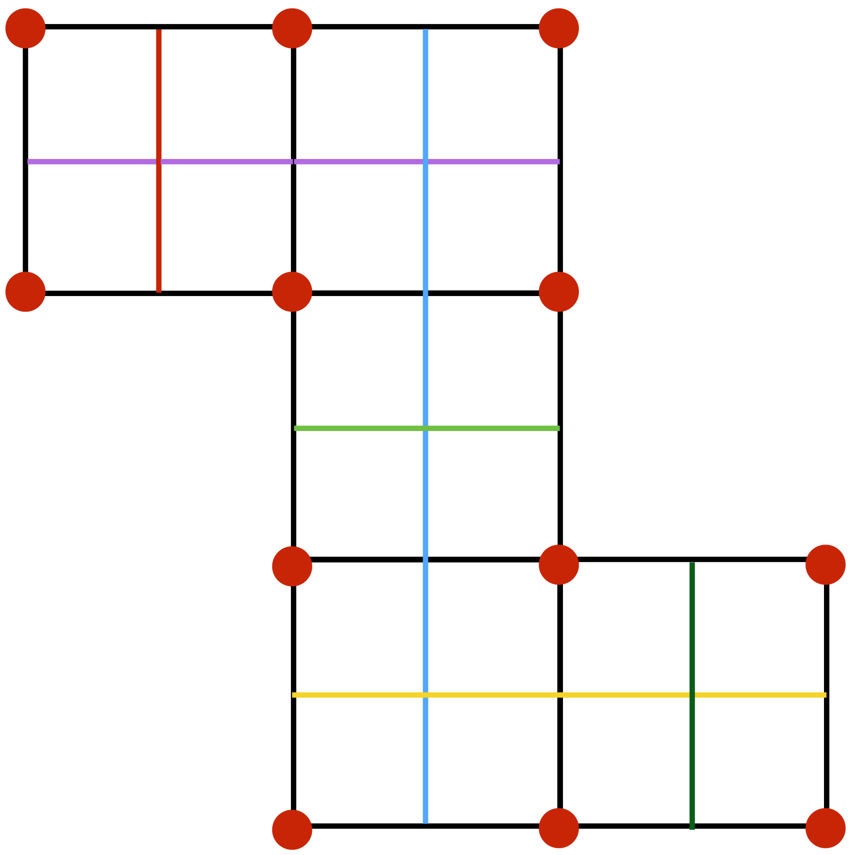
$$A(\Gamma) \to \pi_1(\mathscr{H}) \to \mathbf{Mod}(\Sigma_{g,n})[\phi]$$



Shears

Where do the generators go?

To *cylinder shears:*



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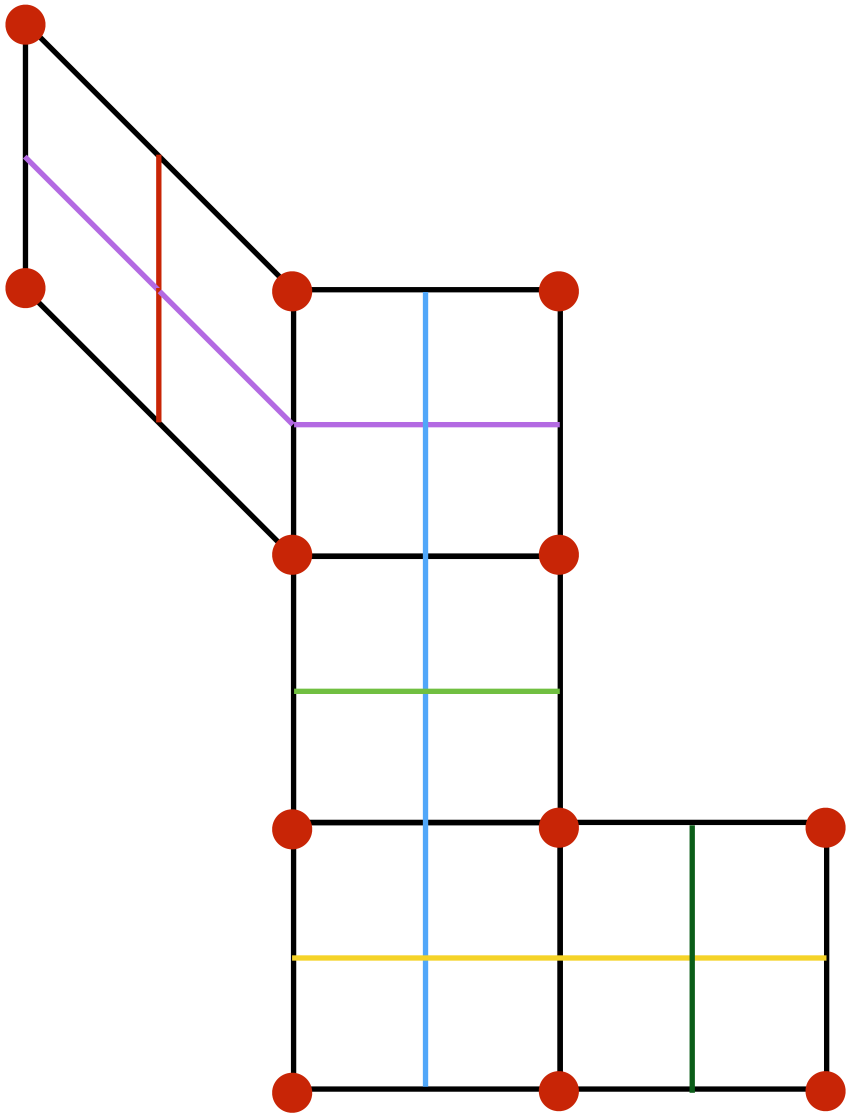

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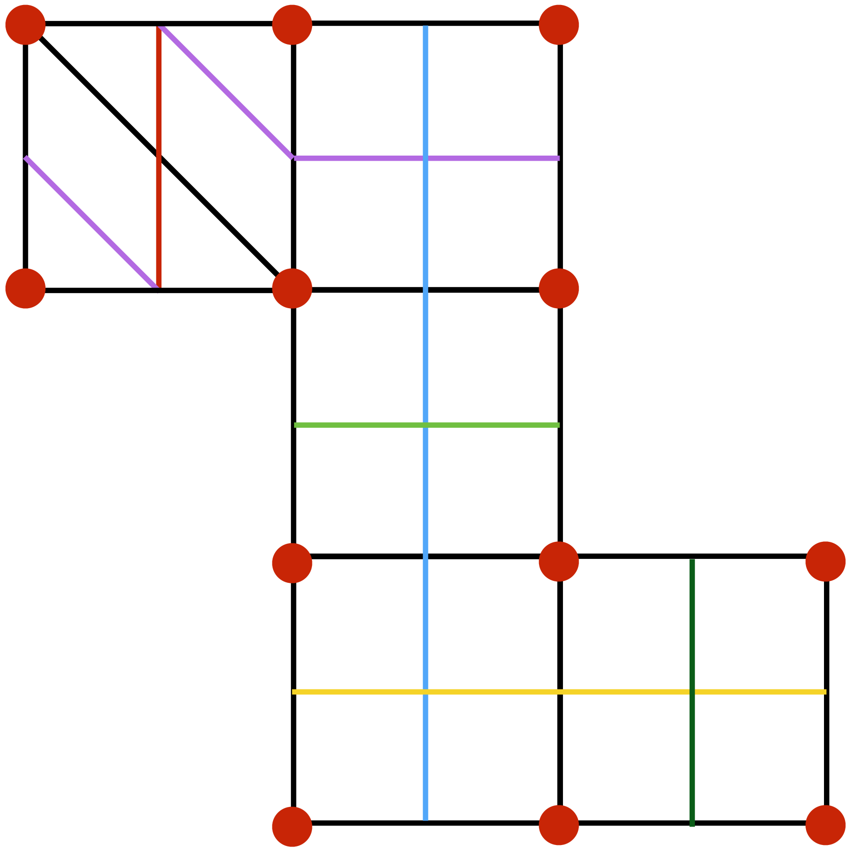


Shears

Where do the generators go?

To *cylinder shears*

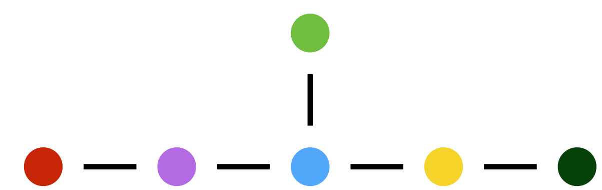
So we prove:

the monodromy quotient of $\pi_1(\mathscr{H})$ *is generated by images of shears*

Shears

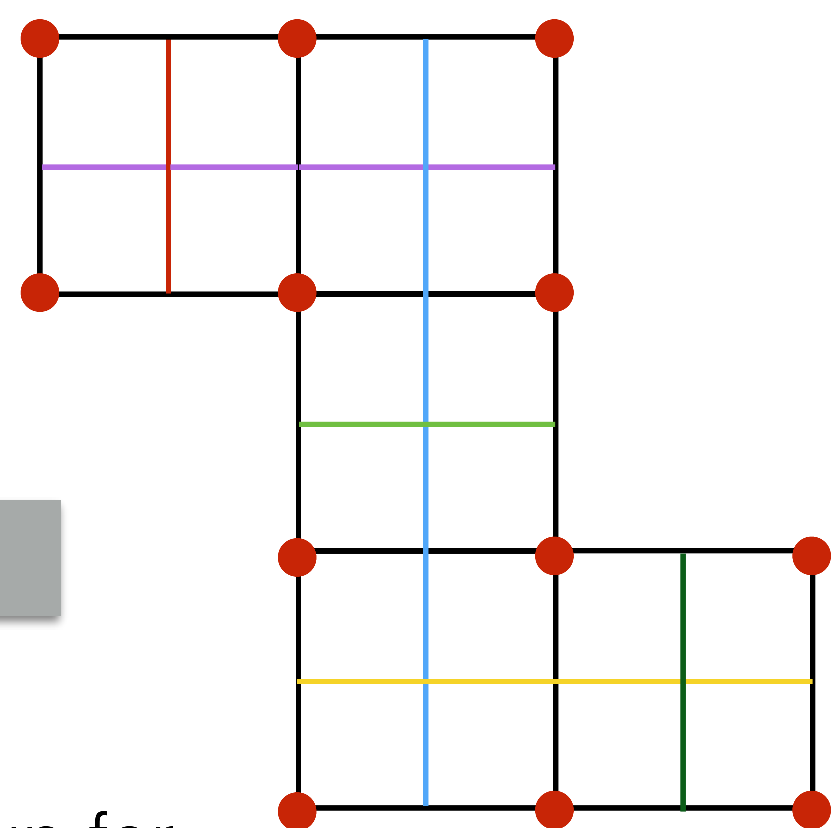
Where do the generators go?

To *cylinder shears*



Conjecture (Shear-generation)

$\pi_1(\mathscr{H})$ is generated by the finite collection of shears about cylinders on a "totally-periodic differential"



Technically: the *prong-marking cover* of $\mathscr{H}$

Implies that $\pi_1(\mathscr{H})$ is a quotient of the Artin group for the graph of cylinder intersections

Monodromy kernel

Natural question:

What is the kernel of
$\rho : \pi_1(\mathscr{H}) \to \mathrm{Mod}(\Sigma_{g,n})[\phi]$?

For hyperelliptic, ρ injective.

For non-hyperelliptic strata, we know *exactly one* kernel element!

Theorem (Wajnryb '99): There is a non-central element $w \in A(E_6)$ such that $f(w) = 1 \in \mathrm{Mod}(\Sigma_{3,1})$

Recall Looijenga—Mondello: $\pi_1(\mathscr{H}\Omega(4)^{odd}) = A(E_6)/\mathrm{center}$

Implies:

$\rho : \pi_1(\mathscr{H}\Omega(4)^{odd}) \to \mathrm{Mod}(\Sigma_{3,1})$
has infinite kernel

The Wajnryb element

Wajnryb proves his theorem by using normal-form theory for $A(E_6)$

Gives no understanding of *why* $(\rho \circ \alpha)(w) = 1$.

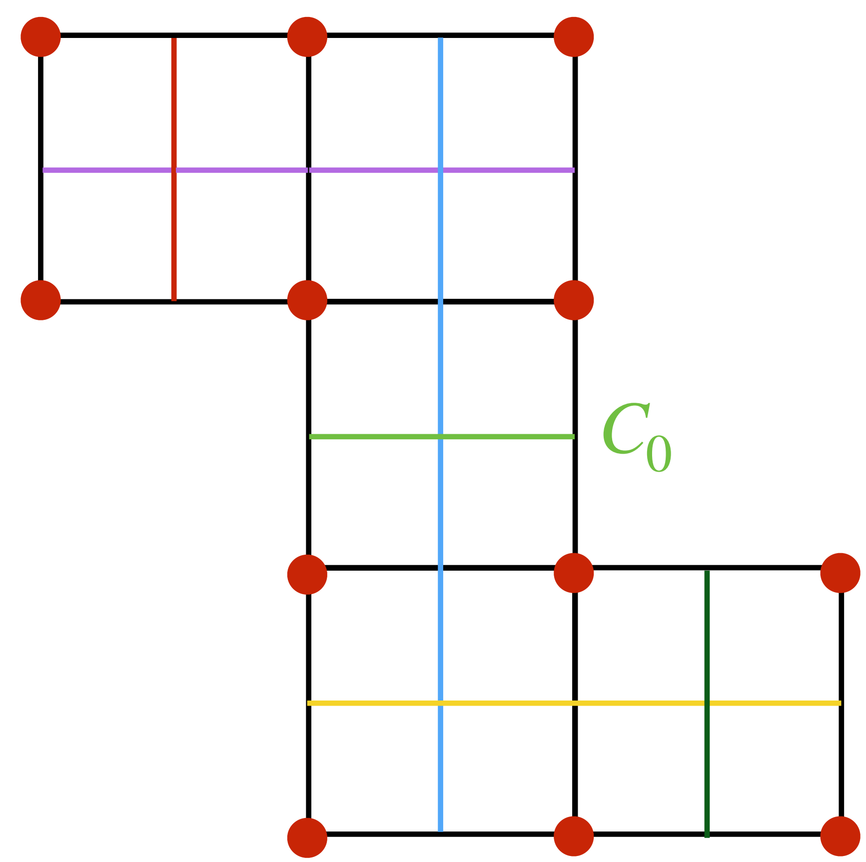
Some speculation:

$w = [a_0, x]$ for some $x \in A(E_6)$

So $\alpha(w) = [S_{C_0}, \alpha(x)]$
(here S_{C_0} is a cylinder shear)

As mapping class, $(\rho \circ \alpha)(x)$ *fixes* curve C_0

Nontriviality of $\alpha(w)$ means that
C_0 *cannot remain a cylinder all along loop* $\alpha(x)$



Monodromy kernel: prospectus

Challenge:

Give an *intrinsic* proof of the nontriviality of Wajnryb's element $\alpha(w) \in \pi_1(\mathscr{H}\Omega(4)^{odd})$. Can you find a way to show that certain cylinders *must break* along a given path?

Problem:

Develop some kind of secondary invariant ("Johnson homomorphism"?) to study $\ker(\rho)$

Conjecture:

$\ker(\rho)$ contains a nonabelian free group for every non-hyperelliptic stratum-component

GGT for stratum groups

If the Shear Generation Conjecture holds, then stratum groups $\pi_1(\mathscr{H})$ are sandwiched between Artin groups, (framed) mapping class groups.

Begs for geometric group theorists to study them!

Unlike Mod, stratum groups aren't given as automorphism groups

Unlike Artin groups, no simple presentation known

Challenge:

Find graphs/complexes/vector spaces/etc.
that $\pi_1(\mathscr{H})$ acts on *faithfully*,
or at least without factoring through ρ.

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or at least without factoring through ρ.

Natural candidate: analogues of the curve/arc graph

Cylinder graph? *Saddle* graph?

Big issue: these are too *rigid.*

Theorem (Disarlo-Randecker-Tang):

If X, Y are translation surfaces with isomorphic saddle connection complexes, then $Y = A \cdot X$ for some affine map A.

Companion issue: Flat things (cylinders, saddles) will *break* as you deform along a loop.
How do you set up an action?

Further topics

Costatini-Möller-Zachhuber compute the Chern character of the cotangent bundle of strata.

Leads to a formula for Euler characteristic.

Bell-Delecroix-Gadre-Gutierrez Romo-Schleimer,
and separately Hamenstädt,
proved recently that *flow loops* (orbit closures) generate $\pi_1(\mathscr{H})$

Alas doesn't give anything explicit.

Thank you!