ERRATUM TO: SURFACE BUNDLES OVER SURFACES WITH ARBITRARILY MANY FIBERINGS

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Abstract. This short note corrects an error appearing in the published version of the titular paper and gives a brief discussion of an example illustrating the corrected construction.

This note corrects an error that appears in the published version of the paper, appearing there as Theorem 2.9. This result describes a general construction for building a surface bundle over a surface with many fiberings (called E_X) associated to the data of a bipartite graph X with certain labeling data. In the published version, it is asserted that all of the given fiberings have a common base surface Σ. This is incorrect. Moreover, the construction given in the proof does not work as it appears - the construction as described in the published version can create "broken" fiberings where pieces of the fiber in different regions of E_X do not patch together correctly. Below we describe an amended version of the theorem. It differs from the printed version only in that now, different fiberings f_i of E_X fiber over potentially different surfaces Σ_{f_i}, each of which is finitely covered by the surface Σ of the original construction. It should be noted that in the case where the graph X is a tree, the assertions made in the original version of Theorem 2.9 are still valid.

The main results of the paper are largely unaffected. Theorem 2.9 is invoked in three subsequent places: Remark 2.10, Theorem 2.12, and the discussion of the function $N(d)$ immediately following the proof of Proposition 3.1. The example discussed in Remark 2.10 is still an output of the amended construction, and needs no further adjustment. The construction used in the discussion of $N(d)$ is based on a graph X that is a tree, and hence is likewise unaffected. The only change that needs to be made to the statement of Theorem 2.12 is as follows: the graph X in the statement should now be assumed to be a *tree* (note in particular that the subsequent Example 2.13 is unaffected).

The following result provides a corrected version of Theorem 2.9. The construction of the 4-manifold E_X given in the printed version is unchanged, and we do not reproduce it here.

Theorem 1. *Let X be a connected finite bipartite graph, possibly with multiple edges, with vertex set $V(X)$ and edge set $E(X)$ of cardinalities C, D respectively.*

(1) E_X admits 2^C fiberings as a surface bundle over a surface, pairwise-distinct up to π_1-fiberwise diffeomorphism. Precisely, for each map $f : V(X) \to \{1, 2\}$, there is a surface Σ_f such that E_X admits a fibering $p^f : E_X \to \Sigma_f$ as a surface bundle over a surface Σ_f with connected fiber.

Date: November 6, 2020.

Supported by NSF Award No. DMS-2003984.

(2) *When X is a tree we can take $\Sigma_f = \Sigma$ for each f as above. Generally, each Σ_f admits a finite-sheeted regular covering by Σ.*
(3) *The total space E_X has the structure of a graph of groups modeled on X where the vertex groups are free-by-surface group extensions Γ and the edge groups are given by $\pi_1 UT\Sigma$ (with notation as in Remark 2.5).*

Proof. Compared to the situation of Theorem 2.8, the construction of p^f is now complicated by the fact that one must take care to make the restrictions of p^f to adjacent components E_2^v match up along the identification sites. The construction is simplest in the case where X is a tree. Choose some root vertex v. Given $f : V(X) \to \{1,2\}$, define p^f on the root component E_2^v via $p^{v,f(v)}$. Next consider an adjacent vertex w. Then there is a *unique* element $\tau \in G$ for which $p^{v,f(v)}$ and $\tau \circ p^{w,f(w)}$ agree on the intersection $E_2^v \cap E_2^w$, and we define p^f on E_2^w as this $\tau \circ p^{w,f(w)}$. In this way, we can continue through the tree X defining p^f one component at a time.

For a general bipartite graph X, it is possible that the τ as above will not be unique. To rectify this, we define a subgroup $G' \leqslant G$ as follows. Choose a root vertex v as above. The procedure above assigns an element $\tau_\gamma \in G$ to each simplicial path γ from v to some vertex w. Define $G' \leqslant G$ as the group generated by elements $\tau_\gamma \tau_{\gamma'}^{-1}$ as γ, γ' range over all simplicial paths in X ending at a common vertex. Define $\Sigma_X = \Sigma/G'$, and denote the projection $\Sigma \to \Sigma_X$ by q. Defining p'^f on each component E_2^v as $q \circ p^{v,f(v)}$ yields a well-defined map $p'^f : E_X \to \Sigma_X$ giving E_X the structure of a surface bundle over a surface.

The fiber of p'^f need not be connected, however this is easily remedied. Let $\sigma_f : \pi_1(\Sigma_X) \to S_n$ be the homomorphism describing the permutation of $\pi_1(\Sigma_X)$ on the components of the fiber of p'^f, and let Σ_f be the cover of Σ_X classified by σ_f. Then p'^f admits a lift $p^f : E_X \to \Sigma_f$ with connected fiber.

To realize p^f as a smooth map, it is necessary to specify gluing maps identifying the various components of E_2, as well as appropriate collar neighborhoods. We proceed exactly as in Theorem 2.8. For each $x \in G$, there is an identification of (neighborhoods of) Δ^x with Δ^1 via the action of the diffeomorphism $\mathrm{id} \times x^{-1}$ of $\Sigma \times \Sigma$. Relative to these identifications, we will speak of identifying $\partial(N^{e,+})$ and $\partial(N^{e,-})$ via id or by h_1 as in Theorem 2.8. Likewise, we will speak of the collar neighborhoods θ_1 and θ_2 of $\partial(N^{e,\pm})$ (referred to as θ_V and θ_H respectively in Theorem 2.8).

The identifications are indexed via $E(X)$. As in Theorem 2.8, identify $\partial(N^{e,+})$ and $\partial(N^{e,-})$ via id if $f(\delta^+(e)) = f(\delta^-(e))$ and via h_1 otherwise. Then a collar neighborhood of $\partial(N^{e,\pm})$ for which p^f is smooth is given by $\theta_{f(\delta^\pm(e))}$.

The argument that each of the fiberings are distinct up to π_1-fiberwise diffeomorphism proceeds along the same lines as in Theorem 2.3. If $f_1, f_2 : V(X) \to \{1,2\}$ are distinct, then there exists at least one v for which $f_1(v) \neq f_2(v)$. Arguing as in Theorem 2.3, one produces an essential loop $\gamma \subset E_2^v$ contained in the fiber of f_1 that projects onto an essential loop under f_2.

By definition, a graph of groups on a graph X is constructed by connecting Eilenberg-Mac Lane spaces $K(\Gamma_v, 1)$ indexed by the vertices, along mapping cylinders induced from homomorphisms $\phi_e : \Gamma_e \to \Gamma_v$. In our setting, for each $v \in V(X)$, the space E_2^v is a $K(\pi_1 E_2^v, 1)$ space, since it is the total space of a fibration $\Sigma' \to E_2^v \to \Sigma$, where Σ' is obtained from Σ by removing n open disks, one for each edge incident to v. As the base and the fiber of this fibration are both aspherical, it follows from the homotopy long exact sequence that E_2^v is aspherical as well. The edge spaces are given by $\partial(N^{e,\pm})$, each of which is diffeomorphic to the aspherical space $UT\Sigma$. It follows that E_X is indeed a graph of groups. $\square$

An example of this construction may help to illuminate the essential ideas.

Example 2. Let X be the graph with vertices v, w and two edges e_1, e_2. We take $G = \mathbb{Z}/2\mathbb{Z} = \langle \tau \rangle$, and label X as shown in **??**a. This produces E_X as shown in **??**b. Denote the four fiberings of E_X as f_{ij}, with $f_{ij}(v) = i$ and $f_{ij}(w) = j$. Under the original construction described in the published paper, the fiber for f_{12} (and similarly f_{21}) does not patch together correctly, as illustrated in **??**c.

In the amended version of the construction described in **??**, $G' = G$ and hence $\Sigma_X = \Sigma/\langle \tau \rangle$. For $f = f_{11}$ and f_{22}, the fibering p'^f has disconnected fiber, and hence $\Sigma_f = \Sigma$ in these cases - it is not necessary to pass to the quotient Σ_X. For $f = f_{12}$ and f_{21}, by contrast, the fiber over Σ_X is connected. See **??**d,e.

Acknowledgements. I gratefully acknowledge Pierre Py and Christian Wieland for bringing this problem to my attention.

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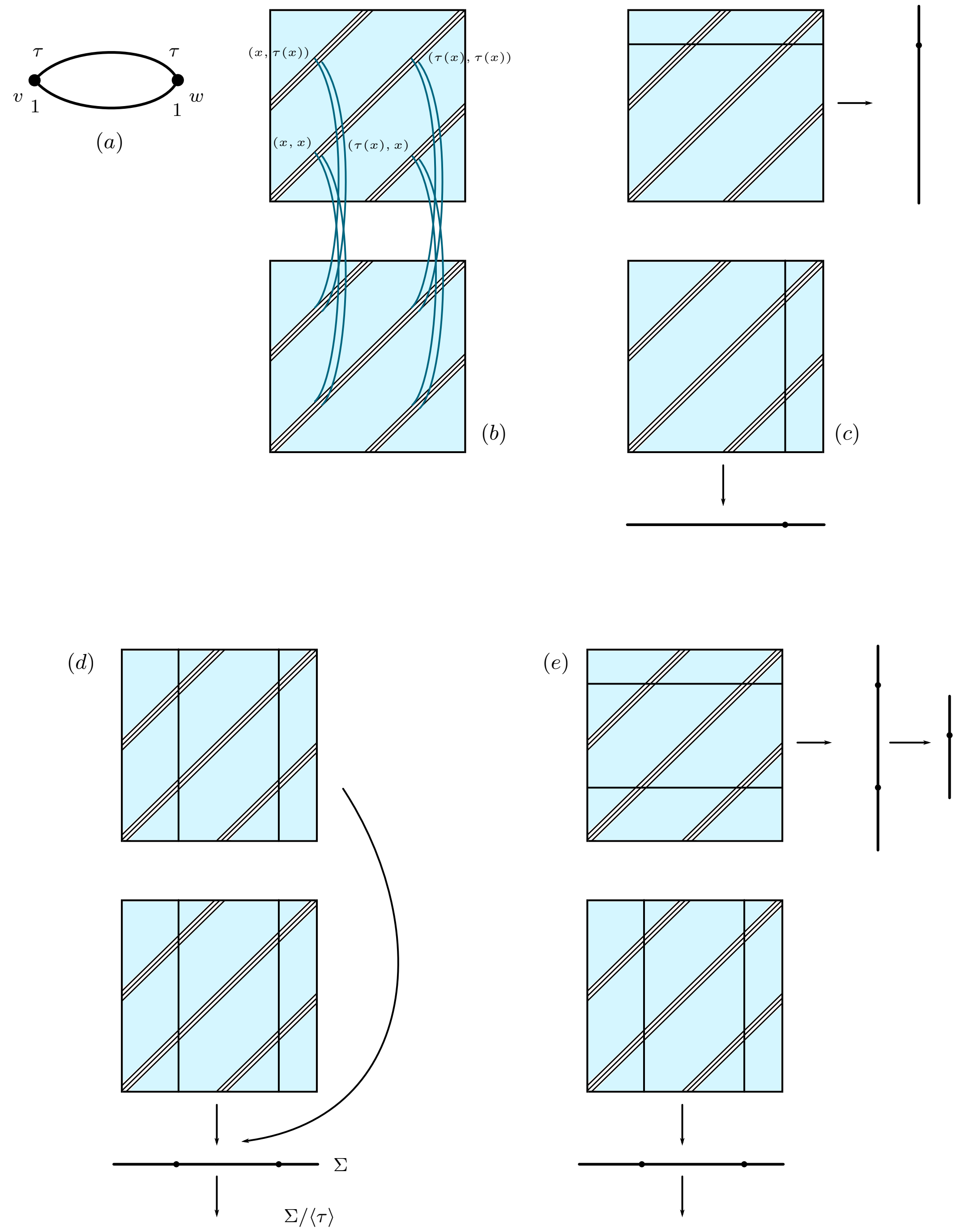

FIGURE 1. The example. (a) - the labeled graph X. (b) - the construction of E_X. (c) - The original "broken" fibering, illustrating the error in the published version. (d) - The fiberings $p'^{f_{11}} : E_X \to \Sigma/\langle\tau\rangle$ and $p^{f_{11}} : E_X \to \Sigma$; the latter has connected fiber, while the former does not. (e) - The fibering $p^{f_{12}} : E_X \to \Sigma/\langle\tau\rangle$. The fiber is connected and consists of four copies of Σ connect-summed together in a loop.