

# GENERATING THE HOMOLOGY OF COVERS OF SURFACES

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ABSTRACT. Putman and Wieland conjectured that if  $\tilde{\Sigma} \rightarrow \Sigma$  is a finite branched cover between closed oriented surfaces of sufficiently high genus, then the orbits of all nonzero elements of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  under the action of lifts to  $\tilde{\Sigma}$  of mapping classes on  $\Sigma$  are infinite. We prove that this holds if  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is generated by the homology classes of lifts of simple closed curves on  $\Sigma$ . We also prove that the subspace of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  spanned by such lifts is a symplectic subspace. Finally, simple closed curves lie on subsurfaces homeomorphic to 2-holed spheres, and we prove that  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is generated by the homology classes of lifts of loops on  $\Sigma$  lying on subsurfaces homeomorphic to 3-holed spheres.

## 1. INTRODUCTION

Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces. The homology of  $\tilde{\Sigma}$  encodes subtle information about the mapping class group of  $\Sigma$ , and over the last decade has been intensely studied [5, 9, 10, 11, 12, 13, 14, 15, 17, 18, 20]. Much of this is motivated by a conjecture of Putman–Wieland [20] we discuss below. In this note, we prove this conjecture for covers  $\tilde{\Sigma}$  such that  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is generated by certain simple elements, and also prove that in general  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is generated by slightly more complicated elements.

**1.1. Putman–Wieland conjecture.** Mark  $\Sigma$  at each branch point of the branched cover  $\pi: \tilde{\Sigma} \rightarrow \Sigma$ . Let  $\text{Mod}(\Sigma)$  be the pure mapping class group of  $\Sigma$ , i.e., the group of isotopy classes of orientation-preserving homeomorphisms of  $\Sigma$  that fix each marked point. There is a finite-index subgroup  $\text{Mod}(\Sigma, \tilde{\Sigma})$  of  $\text{Mod}(\Sigma)$  that can be lifted to  $\tilde{\Sigma}$  to give a well-defined action of  $\text{Mod}(\Sigma, \tilde{\Sigma})$  on  $H_1(\tilde{\Sigma}; \mathbb{Q})$ . Putman–Wieland [20] made the following conjecture.

**Conjecture 1.1** ([20]). *Let the notation be as above, and assume that the genus of  $\Sigma$  is sufficiently large.<sup>1</sup> Consider some nonzero  $\vec{v} \in H_1(\tilde{\Sigma}; \mathbb{Q})$ . Then the  $\text{Mod}(\Sigma, \tilde{\Sigma})$ -orbit of  $\vec{v}$  is infinite.*

The main theorem of [20] says that this holds if and only if the virtual first Betti number of the mapping class group is 0 when the genus is sufficiently large, which is a well-known conjecture of Ivanov [7].

**1.2. Simple closed curve homology.** To prove Conjecture 1.1, it is natural to try to find generators for  $H_1(\tilde{\Sigma}; \mathbb{Q})$ . A first idea is that  $H_1(\tilde{\Sigma}; \mathbb{Q})$  might be generated by lifts of simple closed curves. Define the *simple closed curve homology* of  $\tilde{\Sigma}$ , denoted  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})$ , to be the subspace of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  spanned by the homology classes of loops  $\tilde{\gamma}$  on  $\tilde{\Sigma}$  that avoid the branch points and project to simple closed curves  $\gamma$  on  $\Sigma$ . The restriction of the branched covering map  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  to  $\tilde{\gamma}$  is thus a possibly nontrivial cover  $\tilde{\gamma} \rightarrow \gamma$ .

Unfortunately, this need not be all of  $H_1(\tilde{\Sigma}; \mathbb{Q})$ . For all closed surfaces  $\Sigma$  with  $\pi_1(\Sigma)$  nonabelian, Malestein–Putman [15, Theorem B] constructed finite branched covers  $\tilde{\Sigma} \rightarrow \Sigma$

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<sup>1</sup>In [20] they further conjecture that this holds if the genus is at least 2, but counterexamples in genus 2 were found by Marković [17].

such that  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) \neq H_1(\tilde{\Sigma}; \mathbb{Q})$ . More recently, Klukowski [9] constructed unbranched covers with this property.<sup>2</sup> Our first theorem is that Conjecture 1.1 does hold if  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$ .

**Theorem A.** *Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces. Consider some nonzero  $\vec{v} \in H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})$ . Then the  $\text{Mod}(\Sigma, \tilde{\Sigma})$ -orbit of  $\vec{v}$  is infinite. In particular, if  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$  then Conjecture 1.1 holds for  $\pi: \tilde{\Sigma} \rightarrow \Sigma$ .*

This suggests that the examples from [15] and [9] might be good places to look for counterexamples to Conjecture 1.1.

**1.3. Relationship to previous work.** Conjecture 1.1 has been proved in a variety of cases; see,<sup>3</sup> e.g., [5, 11, 12, 13, 14]. However, we know very little about when  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$ . The only general result we are aware of is that this holds when  $\tilde{\Sigma} \rightarrow \Sigma$  is a finite unbranched abelian cover. This is implicit in work of Looijenga [14], and we provide a self-contained proof in Proposition 5.3 below.<sup>4</sup> Beyond this, it is unclear which known cases of Conjecture 1.1 follow from Theorem A.

*Remark 1.2.* It would be interesting to extend this to prove that  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$  for finite branched abelian covers.  $\square$

Next, let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be one of the examples from [15] or [9] where  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) \neq H_1(\tilde{\Sigma}; \mathbb{Q})$ . Below in Theorem B we will prove that  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})$  is a symplectic subspace of  $H_1(\tilde{\Sigma}; \mathbb{Q})$ , so

$$H_1(\tilde{\Sigma}; \mathbb{Q}) = H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) \oplus H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})^\perp.$$

Theorem A says that the  $\text{Mod}(\Sigma, \tilde{\Sigma})$ -orbit of all nonzero  $\vec{v} \in H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})$  is infinite. It turns out that there are also nonzero  $\vec{v} \in H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})^\perp$  whose  $\text{Mod}(\Sigma, \tilde{\Sigma})$ -orbits are infinite. Indeed, letting  $D$  be the deck group of  $\pi: \tilde{\Sigma} \rightarrow \Sigma$ , it follows from the constructions in [15] and [9] that some  $D$ -isotypic subspace  $V$  of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  lies in  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})^\perp$ , and Landesman–Litt [12] proved that some nonzero  $\vec{v} \in V$  has an infinite  $\text{Mod}(\Sigma, \tilde{\Sigma})$ -orbit.

**1.4. Symplectic subspace.** We next clarify the nature of the subspace  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})$  of  $H_1(\tilde{\Sigma}; \mathbb{Q})$ . By Poincaré duality, the algebraic intersection form  $\omega$  on  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is a *symplectic form*, i.e., an alternating bilinear form that induces an isomorphism between  $H_1(\tilde{\Sigma}; \mathbb{Q})$  and its dual. A subspace  $V$  of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is a *symplectic subspace* if the restriction of  $\omega$  to  $V$  is a symplectic form. We will prove the following:

**Theorem B.** *Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces. Then  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q})$  is a symplectic subspace of  $H_1(\tilde{\Sigma}; \mathbb{Q})$ .*

In fact, we will prove something more general. A *nontrivial simple closed curve* on  $\Sigma$  is a simple closed curve  $\gamma$  that avoids the marked points and does not bound a disk containing at most one marked point. We will always consider such curves up to isotopy.<sup>5</sup> The group

<sup>2</sup>Earlier Koberda–Santharoubane [10] constructed unbranched covers of closed surfaces with  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Z}) \neq H_1(\tilde{\Sigma}; \mathbb{Z})$ . This is weaker since it is possible that in their examples  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Z})$  is a finite-index subgroup of  $H_1(\tilde{\Sigma}; \mathbb{Z})$ .

<sup>3</sup>Not all of these papers explicitly prove cases of Conjecture 1.1, but it can be deduced from their results in the cases they cover.

<sup>4</sup>Proposition 5.3 is stronger than this: it shows that  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is spanned by lifts of *nonseparating* simple closed curves.

<sup>5</sup>These are isotopies through nontrivial simple closed curves, so during the isotopies the curves cannot pass through the marked points.

$\text{Mod}(\Sigma)$  acts on the set of nontrivial simple closed curves on  $\Sigma$ , and the orbits of this action are the *topological types* of nontrivial simple closed curves.<sup>6</sup>

If  $\sigma$  is a set of topological types of nontrivial simple closed curves on  $\Sigma$ , then denote by  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Q})$  the subspace of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  spanned by the homology classes of loops  $\tilde{\gamma}$  on  $\tilde{\Sigma}$  that avoid the branch points and project to simple closed curves  $\gamma$  on  $\Sigma$  such that the topological type of  $\gamma$  lies in  $\sigma$ . For instance, if  $\sigma$  is the set of all topological types of nontrivial simple closed curves on  $\Sigma$ , then

$$H_1^\sigma(\Sigma; \mathbb{Q}) = H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}).$$

The following therefore generalizes Theorem B:

**Theorem B'.** *Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces and  $\sigma$  be a set of topological types of nontrivial simple closed curves on  $\Sigma$ . Then  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Q})$  is a symplectic subspace of  $H_1(\tilde{\Sigma}; \mathbb{Q})$ .*

We can also define  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Z})$  and  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Z})$ , and it is natural to wonder whether Theorems B and B' hold integrally. For Theorem B', the answer is no in general:

**Theorem C.** *Let  $\Sigma$  be a closed oriented surface of genus at least 2 and let  $\sigma$  be the set of nonseparating simple closed curves on  $\Sigma$ . Then there exists a finite unbranched cover  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  such that  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Z})$  is not a symplectic subspace of  $H_1(\tilde{\Sigma}; \mathbb{Z})$ .*

Here  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Z})$  is a free abelian group, and a symplectic form on a free abelian group  $A$  is an alternating  $\mathbb{Z}$ -valued bilinear form on  $A$  that identifies  $A$  with its dual  $A^* = \text{Hom}(A, \mathbb{Z})$ . Unfortunately, our proof of Theorem C breaks down if we allow separating curves, so we cannot answer the following question:

**Question 1.3.** *Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces. Is  $H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Z})$  a symplectic subspace of  $H_1(\tilde{\Sigma}; \mathbb{Z})$ ?*

However, Theorem C suggests that the answer to this should be “no”.

*Remark 1.4.* An important ingredient in our proof of Theorem C is a theorem of Irmer [6] giving certain finite abelian covers  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  for which  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Z})$  is a proper subgroup of  $H_1(\tilde{\Sigma}; \mathbb{Z})$  (see Theorem 5.2.(ii) below). To make this paper more self-contained, we also include a simplified proof of this theorem.  $\square$

**1.5. Pants homology.** Regular neighborhoods of simple closed curves on  $\Sigma$  are homeomorphic to annuli, i.e., spheres with two boundary components. This suggests weakening the definition of simple closed curve homology as follows.

Recall that a *pair of pants* is a sphere with three holes. Define the *pants homology* of  $\tilde{\Sigma}$ , denoted  $H_1^{\text{pant}}(\tilde{\Sigma}; \mathbb{Q})$ , to be the subspace of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  spanned by the homology classes of loops  $\tilde{\gamma}$  on  $\tilde{\Sigma}$  such that there exists a subsurface  $P \subset \Sigma$  homeomorphic to a pair of pants with  $\pi(\tilde{\gamma}) \subset P$ . Since every simple closed curve on  $\Sigma$  is contained in some<sup>7</sup> such  $P$ , we have

$$H_1^{\text{scc}}(\tilde{\Sigma}; \mathbb{Q}) \subset H_1^{\text{pant}}(\tilde{\Sigma}; \mathbb{Q}).$$

Our final main theorem is as follows. It answers positively a question<sup>8</sup> of Kent [8].

<sup>6</sup>By the change of coordinates principle from [4, §1.3.2], the topological types are determined by the marked surface with boundary one gets by cutting  $\Sigma$  open along  $\gamma$ . For instance, one topological type is the set of all nonseparating  $\gamma$ .

<sup>7</sup>We do not require the boundary components of  $P$  to be non-nullhomotopic curves on  $\Sigma$ , so this even holds if  $\Sigma$  is a surface like a sphere that does not contain pairs of pants  $P$  whose boundary components are non-nullhomotopic.

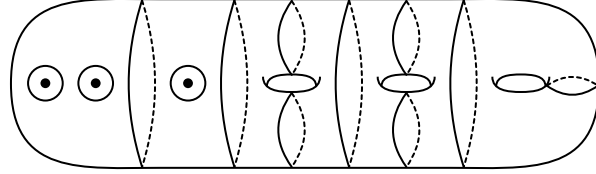
<sup>8</sup>Kent actually asked whether  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is generated by lifts of elements that do not fill  $\Sigma$ , which is much weaker than lying in a pair of pants.

**Theorem D.** *Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces. Then  $H_1^{\text{pant}}(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$ .*

*Remark 1.5.* This also holds for punctured surfaces of finite type, which can be reduced to Theorem D as follows. Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between punctured surfaces of finite type. Filling in the punctures yields a finite branched cover between closed surfaces to which one can apply Theorem D. To conclude, note that filling in the punctures has the effect of killing the homology classes in  $H_1(\tilde{\Sigma}; \mathbb{Q})$  of loops around the punctures, which lie in  $H_1^{\text{sc}}(\tilde{\Sigma}; \mathbb{Q}) \subset H_1^{\text{pant}}(\tilde{\Sigma}; \mathbb{Q})$ .  $\square$

*Remark 1.6.* Theorem D might appear to contradict [15, Theorem C] and [9, Corollary 1.1.3], which give examples of finite covers  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  such that  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is not spanned by the homology classes of loops  $\tilde{\gamma}$  such that  $\pi(\tilde{\gamma})$  is not in any given finite set of mapping class group<sup>9</sup> orbits of curves. However, in the definition of  $H_1^{\text{pant}}(\tilde{\Sigma}; \mathbb{Q})$  there is no restriction on the number of self-intersections of the projections of the curves to the  $P$ , so they do not fall into finitely many mapping class group orbits.  $\square$

The proof of Theorem D actually shows something stronger. A *pants decomposition* of  $\Sigma$  is a collection  $\mathcal{P} = \{\delta_1, \dots, \delta_n\}$  of disjoint simple closed curves on  $\Sigma$  that avoid the branch points such that each component of  $\Sigma \setminus \cup_{j=1}^n \delta_j$  is either a disk containing a single branch point or a pair of pants containing no branch points:



We will prove the following, which implies Theorem D:

**Theorem D'.** *Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces and  $\mathcal{P}$  be a pants decomposition of  $\Sigma$ . Let  $\sigma$  be the set of topological types of nontrivial curves appearing in  $\mathcal{P}$ . Then  $H_1(\tilde{\Sigma}; \mathbb{Q})$  is spanned by  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Q})$  and the set of homology classes of cycles  $\tilde{\gamma}$  on  $\tilde{\Sigma}$  such that  $\pi(\tilde{\gamma})$  is disjoint from all curves in  $\mathcal{P}$ .*

We can also define  $H_1^{\text{pant}}(\tilde{\Sigma}; \mathbb{Z})$ , and pose the following question:

**Question 1.7.** *Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces. Is  $H_1^{\text{pant}}(\tilde{\Sigma}; \mathbb{Z}) = H_1(\tilde{\Sigma}; \mathbb{Z})$ ?*

Our proof of Theorems D and D' shows that Question 1.7 has a positive answer if Question 1.3 does. Though we expect that Question 1.3 has a negative answer, we do not know what answer to expect for Question 1.7.

**1.6. Outline.** We prove Theorem A in §2, Theorems B and B' in §3, Theorems D and D' in §4, and Theorem C in §5.

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## 2. SIMPLE CLOSED CURVE HOMOLOGY AND DEHN TWISTS

This section contains the proof of Theorem A.

<sup>9</sup>or even automorphism group of free group

**2.1. Notation.** Fix a closed oriented surface  $\Sigma$  and a finite branched cover  $\pi: \tilde{\Sigma} \rightarrow \Sigma$ . The surface  $\tilde{\Sigma}$  is thus also a closed oriented surface. Let  $g \geq 0$  be its genus, let

$$H = H_1(\tilde{\Sigma}; \mathbb{Q}) \cong \mathbb{Q}^{2g},$$

and let  $\omega$  be the algebraic intersection form on  $H$ . By Poincaré duality,  $\omega$  is a symplectic form, i.e., an alternating form that induces an isomorphism between  $H$  and its dual. The symplectic group  $\mathrm{Sp}(H, \omega) \cong \mathrm{Sp}_{2g}(\mathbb{Q})$  acts on  $H$ .

**2.2. Lifting Dehn twists.** Recall from §1.4 that a nontrivial simple closed curve on  $\Sigma$  is a simple closed curve that avoids the marked points and does not bound a disk containing at most one marked point. Consider a nontrivial simple closed curve  $\gamma$  on  $\Sigma$ . The preimage  $\pi^{-1}(\gamma)$  is a disjoint union of simple closed curves. Enumerate them as

$$\pi^{-1}(\gamma) = \tilde{\gamma}_1 \sqcup \cdots \sqcup \tilde{\gamma}_k.$$

For each  $1 \leq j \leq k$ , the map

$$\pi|_{\tilde{\gamma}_j}: \tilde{\gamma}_j \rightarrow \gamma$$

is a finite unbranched cover. Let  $d_j$  be its degree. Set

$$(2.1) \quad d(\gamma) = \mathrm{lcm}(d_1, \dots, d_k) \quad \text{and} \quad e_j = d(\gamma)/d_j \quad \text{for } 1 \leq j \leq k.$$

If  $T_\gamma$  and  $T_{\tilde{\gamma}_j}$  denote the Dehn twists about  $\gamma$  and  $\tilde{\gamma}_j$ , then  $T_\gamma^{d(\gamma)}$  lifts to the product

$$T_{\tilde{\gamma}_1}^{e_1} \cdots T_{\tilde{\gamma}_k}^{e_k}.$$

Let  $\tilde{\tau}_\gamma$  be the image of this product of powers of Dehn twists in  $\mathrm{Sp}(H, \omega) \cong \mathrm{Sp}_{2g}(\mathbb{Q})$ . The element  $\tilde{\tau}_\gamma$  acts on  $H$  as follows:

$$\tilde{\tau}_\gamma(h) = h + \sum_{j=1}^k e_j \omega(h, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j] \quad \text{for } h \in H.$$

For a set  $\sigma$  of topological types of nontrivial simple closed curves on  $\Sigma$ , define  $D_\sigma$  to be the subgroup of  $\mathrm{Sp}(H, \omega)$  generated by the set of all  $\tilde{\tau}_\gamma$  as  $\gamma$  ranges over nontrivial simple closed curves on  $\Sigma$  whose topological type lies in  $\sigma$ .

**2.3. Fixed set of lifted twists.** As in §1.4, let  $\sigma$  be a set of topological types of nontrivial simple closed curves on  $\Sigma$  and define

$$H^\sigma = H_1^\sigma(\tilde{\Sigma}; \mathbb{Q}) \subset H_1(\tilde{\Sigma}; \mathbb{Q}) = H.$$

The following lemma will be fundamental to our paper:

**Lemma 2.1.** *Let the notation be as above. Then<sup>10</sup>  $H^{D_\sigma}$  equals the orthogonal complement  $(H^\sigma)^\perp$  of  $H^\sigma$  with respect to  $\omega$ .*

*Proof.* Let  $\gamma$  be a nontrivial simple closed curve on  $\Sigma$  whose topological type lies in  $\sigma$ . As we did in §2.2 above, write  $\pi^{-1}(\gamma)$  as a disjoint union of simple closed curves on  $\tilde{\Sigma}$ :

$$\pi^{-1}(\gamma) = \tilde{\gamma}_1 \sqcup \cdots \sqcup \tilde{\gamma}_k.$$

As in that section, there are positive integers  $e_1, \dots, e_k$  such that the generator  $\tilde{\tau}_\gamma \in D_\sigma$  acts on  $H$  as follows:

$$\tilde{\tau}_\gamma(h) = h + \sum_{j=1}^k e_j \omega(h, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j] \quad \text{for } h \in H.$$

<sup>10</sup>Here the superscript indicates that we are taking invariants:  $H^{D_\sigma} = \{h \in H \mid d \cdot h = h \text{ for all } d \in D_\sigma\}$ .

Each  $[\tilde{\gamma}_j]$  lies in  $H^\sigma$ , so it is immediate from this formula that  $(H^\sigma)^\perp \subset H^{D\sigma}$ . For the other inclusion, consider some  $h_0 \in H^{D\sigma}$ . We then know that  $\tilde{\tau}_\gamma(h_0) = h_0$ , so from the above

$$\sum_{j=1}^k e_j \omega(h_0, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j] = 0.$$

Taking the algebraic intersection with  $h_0$ , we deduce that

$$\sum_{j=1}^k e_j \omega(h_0, [\tilde{\gamma}_j])^2 = 0.$$

Since  $e_j \geq 1$  for all  $1 \leq j \leq k$ , this implies that  $\omega(h_0, [\tilde{\gamma}_j]) = 0$  for all  $1 \leq j \leq k$ . This holds for all choices of  $\gamma$  and all components of the preimage  $\pi^{-1}(\gamma)$ . These generate  $H^\sigma$ , so we conclude that  $h_0 \in (H^\sigma)^\perp$ , as desired.  $\square$

**2.4. Putman–Wieland conjecture.** We now prove Theorem A.

*Proof of Theorem A.* We start by recalling the statement. Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces. Let  $\text{Mod}(\Sigma, \tilde{\Sigma})$  be the subgroup of the mapping class group  $\text{Mod}(\Sigma)$  that lifts to  $\tilde{\Sigma}$ . Consider some nonzero  $\vec{v} \in H_1^{\text{sc}}(\tilde{\Sigma}; \mathbb{Q})$ . Our goal is to prove the  $\text{Mod}(\Sigma, \tilde{\Sigma})$ -orbit of  $\vec{v}$  is infinite.

Let  $\sigma$  be the set of all topological types of nontrivial simple closed curves on  $\Sigma$ . Since  $H_1^{\text{sc}}(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$ , Lemma 2.1 implies that  $\vec{v}$  is not fixed by the group  $D_\sigma$ , so there exists some nontrivial simple closed curve  $\gamma$  on  $\Sigma$  such that  $\tilde{\tau}_\gamma(\vec{v}) \neq \vec{v}$ . Some power of  $\tilde{\tau}_\gamma$  is the image in the symplectic group of an element of  $\text{Mod}(\Sigma, \tilde{\Sigma})$ , so it is enough to prove that the elements  $\tilde{\tau}_\gamma^n(\vec{v})$  as  $n$  ranges over  $\mathbb{Z}$  are all distinct.

Write  $\pi^{-1}(\gamma)$  as a disjoint union of simple closed curves on  $\tilde{\Sigma}$ :

$$\pi^{-1}(\gamma) = \tilde{\gamma}_1 \sqcup \cdots \sqcup \tilde{\gamma}_k.$$

There are then positive integers  $e_1, \dots, e_k$  such that

$$\tilde{\tau}_\gamma(\vec{v}) = \vec{v} + \sum_{j=1}^k e_j \omega(\vec{v}, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j].$$

Setting

$$\vec{w} = \sum_{j=1}^k e_j \omega(\vec{v}, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j],$$

the fact that  $\tilde{\tau}_\gamma(\vec{v}) \neq \vec{v}$  implies that  $\vec{w} \neq 0$ . For  $n \in \mathbb{Z}$ , we have  $\tilde{\tau}_\gamma^n(\vec{v}) = \vec{v} + n\vec{w}$ . Since  $\vec{w} \neq 0$ , the elements  $\vec{v} + n\vec{w}$  as  $n$  ranges over  $\mathbb{Z}$  are all distinct, as desired.  $\square$

### 3. THE SYMPLECTIC NATURE OF SIMPLE CLOSED CURVES

This section contains the proof of Theorem B' (which generalizes Theorem B).

**3.1. Notation.** The notation is similar to that of §2:

- $\pi: \tilde{\Sigma} \rightarrow \Sigma$  is a finite branched cover between closed oriented surfaces, and  $g$  is the genus of  $\tilde{\Sigma}$ .
- $H = H_1(\tilde{\Sigma}; \mathbb{Q}) \cong \mathbb{Q}^{2g}$ , and  $\omega$  is the algebraic intersection form on  $H$ .
- $\sigma$  is a set of topological types of nontrivial simple closed curves on  $\Sigma$ .
- $D_\sigma$  is the subgroup of  $\text{Sp}(H, \omega) \cong \text{Sp}_{2g}(\mathbb{Q})$  generated by the elements  $\tilde{\tau}_\gamma$  as  $\gamma$  ranges over nontrivial simple closed curves on  $\Sigma$  whose topological type lies in  $\sigma$ .

**3.2. Symplectic criterion.** We will need the following criterion for a subspace of  $H$  to be a symplectic subspace:

**Lemma 3.1.** *Let the notation be as above, and let  $D$  be a subgroup of  $\mathrm{Sp}(H, \omega)$ . Assume that the action of  $D$  on  $H$  is semisimple.<sup>11</sup> Then<sup>12</sup>  $H^D$  is a symplectic subspace of  $H$ .*

This is well-known; see, e.g., [3, Lemme 4.14]. For completeness, we include a proof.

*Proof of Lemma 3.1.* The symplectic form  $\omega$  induces a  $D$ -equivariant isomorphism

$$\phi: H \xrightarrow{\cong} H^*.$$

We want to prove  $\omega$  also induces an isomorphism between  $H^D$  and  $(H^D)^*$ . Letting  $\iota: H^D \hookrightarrow H$  be the inclusion and letting  $\iota^*: H^* \rightarrow (H^D)^*$  be its dual,<sup>13</sup> our goal is equivalent to proving that the composition

$$(3.1) \quad H^D \xrightarrow{\iota} H \xrightarrow{\phi} H^* \xrightarrow{\iota^*} (H^D)^*$$

is an isomorphism.

Since  $\phi$  is a  $D$ -equivariant isomorphism, it restricts to an isomorphism on  $D$ -invariants, i.e., an isomorphism

$$\phi^D: H^D \xrightarrow{\cong} (H^*)^D.$$

A linear map  $\lambda: H \rightarrow \mathbb{Q}$  in  $H^*$  is  $D$ -invariant if and only if it factors through the  $D$ -coinvariants

$$H_D = H / \langle d \cdot h - h \mid d \in D \text{ and } h \in H \rangle.$$

We thus get an isomorphism

$$\mu: (H^*)^D \xrightarrow{\cong} (H_D)^*.$$

The projection  $H \rightarrow H_D$  restricts to a map  $\eta: H^D \rightarrow H_D$ . Since the action of  $D$  is semisimple, the map  $\eta$  is an isomorphism.<sup>14</sup> Taking its dual, we get an isomorphism

$$\eta^*: (H_D)^* \xrightarrow{\cong} (H^D)^*.$$

The composition

$$(3.2) \quad H^D \xrightarrow{\phi^D} (H^*)^D \xrightarrow{\mu} (H_D)^* \xrightarrow{\eta^*} (H^D)^*$$

of isomorphisms is an isomorphism, and reflecting on the maps we see that the compositions (3.1) and (3.2) are the same. We conclude that (3.1) is an isomorphism, as desired.  $\square$

**3.3. Semisimplicity.** Our goal is to apply Lemma 3.1 to the group  $D_\sigma$  from §3.1, which requires verifying the following:

**Lemma 3.2.** *Let the notation be as above. Then the group  $D_\sigma$  acts semisimply on  $H$ .*

*Proof.* Let  $\mathbf{D}_\sigma$  be the Zariski closure of  $D_\sigma$  in  $\mathrm{Sp}(H, \omega)$ . It is enough to prove that  $\mathbf{D}_\sigma$  acts semisimply on  $H$ . For this, it is enough to prove that  $\mathbf{D}_\sigma$  is a semisimple algebraic group.<sup>15</sup>

Regard  $\Sigma$  as a closed surface with marked points at the branch points of  $\pi: \tilde{\Sigma} \rightarrow \Sigma$ . To simplify things, if there are no branch points introduce a single additional marked point on

<sup>11</sup>That is, the  $D$ -representation  $H$  decomposes as a direct sum of irreducible representations.

<sup>12</sup>Here just like in Lemma 2.1 the superscript indicates we are taking invariants.

<sup>13</sup>This dual restricts a linear map  $\lambda: H \rightarrow \mathbb{Q}$  to  $H^D$ .

<sup>14</sup>Indeed, if  $H = H^D \oplus V_1 \oplus \cdots \oplus V_n$  with the  $V_i$  nontrivial irreducible representations of  $D$ , then  $H_D = (H^D)_D \oplus (V_1)_D \oplus \cdots \oplus (V_n)_D = H^D \oplus 0 \oplus \cdots \oplus 0 = H^D$ .

<sup>15</sup>See [1, 19] for textbook references on algebraic groups. One key property of semisimple algebraic groups over  $\mathbb{Q}$  is that all of their finite-dimensional representations are semisimple [19, Proposition 22.41].

$\Sigma$ , and regard its preimage in  $\tilde{\Sigma}$  as a collection of branch points of order 1. Let  $\mathcal{M}(\Sigma)$  be the moduli space of Riemann surfaces  $S$  with marked points such that  $S \cong \Sigma$  as surfaces with marked points. The (orbifold) fundamental group of  $\mathcal{M}(\Sigma)$  is thus the mapping class group  $\text{Mod}(\Sigma)$ .

We can find a finite-index subgroup  $\Gamma$  of  $\text{Mod}(\Sigma)$  such that each element of  $\Gamma$  can be lifted to a homeomorphism of  $\tilde{\Sigma}$  fixing all the marked points. Since there is at least one marked point, these lifts are unique up to homotopy, so  $\Gamma$  acts on  $H_1(\tilde{\Sigma}; \mathbb{Q})$  in a well-defined way. Shrinking  $\Gamma$  if necessary, we can also assume that  $\Gamma$  is torsion-free. Let  $\mathcal{M}_\Gamma(\Sigma)$  be the cover of  $\mathcal{M}(\Sigma)$  corresponding to  $\Gamma$ .

Since  $\Gamma$  is torsion-free,  $\mathcal{M}_\Gamma(\Sigma)$  is a fine moduli space. It thus has a universal curve  $\mathcal{U} \rightarrow \mathcal{M}_\Gamma(\Sigma)$  whose fiber over  $S \in \mathcal{M}_\Gamma(\Sigma)$  is  $S$ . Replacing  $\Gamma$  by a deeper finite-index subgroup if necessary, we can find a fiberwise branched cover  $\tilde{\mathcal{U}} \rightarrow \mathcal{M}_\Gamma(\Sigma)$  of  $\mathcal{U} \rightarrow \mathcal{M}_\Gamma(\Sigma)$  whose fibers are the branched cover  $\tilde{\Sigma}$  of  $\Sigma$ .

The monodromy representation of  $\pi_1(\mathcal{M}_\Gamma(\Sigma)) \cong \Gamma$  on  $H_1$  of the fibers is thus exactly the action of  $\Gamma$  on  $H = H_1(\tilde{\Sigma}; \mathbb{Q})$  obtained by lifting mapping classes through the branched cover  $\tilde{\Sigma} \rightarrow \Sigma$ . The image of this representation lies in  $\text{Sp}(H, \omega)$ . Let  $\mathbf{G}$  be the Zariski closure in  $\text{Sp}(H, \omega)$  of the image of  $\Gamma$ . Deligne's semisimplicity theorem [2, Corollaire 4.2.9] implies that  $\mathbf{G}$  is a semisimple algebraic group.

From the definition (2.1) of  $d(\gamma)$  for nontrivial simple closed curves  $\gamma$  on  $\Sigma$ , it only achieves finitely many values (depending on the degree of the cover  $\tilde{\Sigma} \rightarrow \Sigma$ ). Pick some  $m \geq 1$  such that the following two properties hold for each nontrivial simple closed curve  $\gamma$  whose topological type lies in  $\sigma$ :

- $d(\gamma)$  divides  $m$ .
- $T_\gamma^m \in \Gamma$ .

Let  $E$  be the subgroup of  $\text{Mod}(\Sigma)$  generated by all the  $T_\gamma^m$  as  $\gamma$  ranges over nontrivial simple closed curves on  $\Sigma$  whose topological type lies in  $\sigma$ . For such a  $\gamma$ , we have

$$fT_\gamma^m f^{-1} = T_{f(\gamma)}^m \quad \text{for all } f \in \text{Mod}(\Sigma).$$

It follows that  $E$  is a normal subgroup of  $\text{Mod}(\Sigma)$ . By construction,  $E \subset \Gamma$ . For each nontrivial simple closed curve  $\gamma$  on  $\Sigma$  whose topological type lies in  $\sigma$ , recall that  $\tilde{\tau}_\gamma$  is the image of  $T_\gamma^{d(\gamma)}$  in  $\text{Sp}(H, \omega)$ . The Zariski closure in  $\text{Sp}(H, \omega)$  of the subgroup generated by  $\tilde{\tau}_\gamma$  is<sup>16</sup> the one-parameter subgroup  $\tilde{\tau}_{\gamma,t}$  defined by

$$\tilde{\tau}_{\gamma,t}(h) = h + \sum_{j=1}^k t e_j \omega(h, [\tilde{\gamma}_j]) \cdot [\tilde{\gamma}_j] \quad \text{for } h \in H \text{ and } t \in \mathbb{Q}.$$

The group  $\mathbf{D}_\sigma$  is generated by these one-parameter subgroups.<sup>17</sup> Since one-parameter subgroups are connected, we deduce that  $\mathbf{D}_\sigma$  is connected.

The Zariski closure in  $\text{Sp}(H, \omega)$  of the subgroup generated by

$$T_\gamma^m = \left( T_\gamma^{d(\gamma)} \right)^{m/d(\gamma)}$$

is the same one-parameter subgroup  $\tilde{\tau}_{\gamma,t}$ . It follows that the Zariski closure of the image of  $E$  in  $\text{Sp}(H, \omega)$  is also  $\mathbf{D}_\sigma$ . Since  $E$  is a normal subgroup of  $\text{Mod}(\Sigma)$ , it follows that  $\mathbf{D}_\sigma$  is a normal subgroup of  $\mathbf{G}$ . Since  $\mathbf{G}$  is semisimple, so<sup>18</sup> is  $\mathbf{D}_\sigma$ , as desired.  $\square$

<sup>16</sup>The point here is that the subgroup generated by  $\tilde{\tau}_\gamma$  is the integer points in the one-parameter subgroup  $\tilde{\tau}_{\gamma,t}$ , and the Zariski closure of  $\mathbb{Z}$  in  $\mathbb{Q}$  is  $\mathbb{Q}$ .

<sup>17</sup>This uses the fact that the subgroup of an algebraic group generated by a set of algebraic subgroups is algebraic, i.e., Zariski closed [1, Proposition 2.2].

<sup>18</sup>Every connected normal subgroup of a semisimple algebraic group is semisimple [19, Theorem 21.51].



**3.4. Symplectic subspace.** We now prove Theorem B'.

*Proof of Theorem B'.* The statement we must prove is as follows. Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces and  $\sigma$  be a set of topological types of nontrivial simple closed curves on  $\Sigma$ . We must show that  $H^\sigma = H_1^\sigma(\tilde{\Sigma}; \mathbb{Q})$  is a symplectic subspace of  $H = H_1(\tilde{\Sigma}; \mathbb{Q})$ , or equivalently that  $(H^\sigma)^\perp$  is a symplectic subspace. Lemma 2.1 implies that

$$(H^\sigma)^\perp = H^{D_\sigma},$$

and Lemma 3.2 says that the group  $D_\sigma$  acts semisimply on  $H$ . The result thus follows from Lemma 3.1.  $\square$

#### 4. PANTS HOMOLOGY

In this section, we prove Theorem D' (which implies Theorem D).

*Proof of Theorem D'.* We first recall the statement. Let  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  be a finite branched cover between closed oriented surfaces and  $\mathcal{P}$  be a pants decomposition of  $\Sigma$ . Let  $\sigma$  be the set of topological types of nontrivial curves appearing in  $\mathcal{P}$ . We must prove that  $H = H_1(\tilde{\Sigma}; \mathbb{Q})$  is spanned by  $H^\sigma = H_1^\sigma(\tilde{\Sigma}; \mathbb{Q})$  and the set of homology classes of cycles  $\tilde{\gamma}$  on  $\tilde{\Sigma}$  such that  $\pi(\tilde{\gamma})$  is disjoint from all curves in  $\mathcal{P}$ .

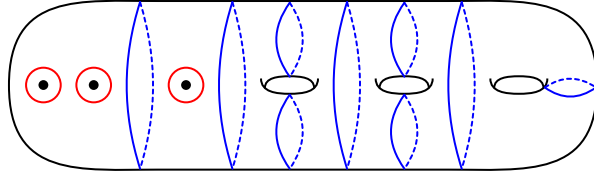
Theorem B' says that  $H^\sigma$  is a symplectic subspace of  $H$ , so

$$H = H^\sigma \oplus (H^\sigma)^\perp.$$

It is thus enough to prove that  $(H^\sigma)^\perp$  is spanned by the homology classes of cycles  $\tilde{\gamma}$  on  $\tilde{\Sigma}$  such that  $\pi(\tilde{\gamma})$  is disjoint from all the curves in  $\mathcal{P}$ .

Recall that we are working with homology with rational coefficients. Every element of  $H$  (and hence  $(H^\sigma)^\perp$ ) is a multiple of an integral class, and every integral class can be represented by an oriented multicurve. Therefore, consider an oriented multicurve  $\tilde{\gamma}$  on  $\tilde{\Sigma}$  such that  $[\tilde{\gamma}] \in (H^\sigma)^\perp$ . It is enough to prove that  $\tilde{\gamma}$  is homologous to an oriented multicurve  $\tilde{\gamma}'$  such that  $\pi(\tilde{\gamma}')$  is disjoint from all the curves in  $\mathcal{P}$ .

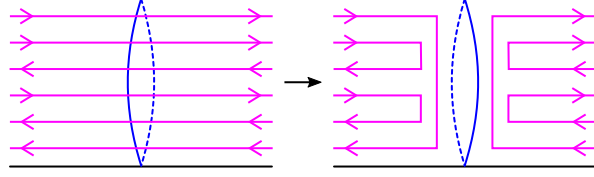
Our pants decomposition  $\mathcal{P}$  looks like the following:



Write  $\mathcal{P} = \{\delta_1, \dots, \delta_n\}$ . Call the  $\delta_j$  that bound disks containing marked points the *boundary loops* (red in the above figure) and the other  $\delta_j$  the *interior loops* (blue in the above figure). Enumerate the components of  $\pi^{-1}(\delta_j)$  as  $j$  ranges over  $1 \leq j \leq n$  as  $\{\tilde{\delta}_1, \dots, \tilde{\delta}_m\}$ . Call the  $\tilde{\delta}_j$  that project to boundary loops the *lifted boundary loops* and the  $\tilde{\delta}_j$  that project to interior loops the *lifted interior loops*.

Put the oriented multicurve  $\tilde{\gamma}$  in general position with respect to the  $\tilde{\delta}_j$ . The lifted boundary loops bound disks in  $\tilde{\Sigma}$  containing a single branch point. Isotope  $\tilde{\gamma}$  such that it is disjoint from all these disks, and in particular is disjoint from all the lifted boundary loops.

Let  $\omega$  be the algebraic intersection form. Consider a lifted interior loop  $\tilde{\delta}_j$ . Since  $[\tilde{\delta}_j] \in H^\sigma$  and  $[\tilde{\gamma}] \in (H^\sigma)^\perp$ , we have  $\omega([\tilde{\gamma}], [\tilde{\delta}_j]) = 0$ . This implies that the number of positively oriented intersection points of  $\tilde{\gamma}$  with  $\tilde{\delta}_j$  is the same as the number of negatively oriented intersection points. We can then modify  $\tilde{\gamma}$  as follows to make it disjoint from  $\tilde{\delta}_j$ :



The result is an oriented multicurve that is homologous to  $\tilde{\gamma}$ . Doing this for each lifted interior loop, we obtain an oriented multicurve  $\tilde{\gamma}'$  such that  $[\tilde{\gamma}'] = [\tilde{\gamma}]$  and such that  $\tilde{\gamma}'$  is disjoint from all the  $\tilde{\delta}_j$  and does not lie in any of the disks bounded by lifted boundary loops. This implies  $\pi(\tilde{\gamma})$  is disjoint from all the curves in  $\mathcal{P}$ , as desired.  $\square$

## 5. A NON-SYMPLECTIC EXAMPLE

This section contains the proof of Theorem C, which asserts that for all closed oriented surfaces  $\Sigma$  of genus at least 2, there exists a finite unbranched cover  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  such that for  $\sigma$  the set of nonseparating simple closed curves on  $\Sigma$ , the subspace  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Z})$  is not a symplectic subspace of  $H_1(\tilde{\Sigma}; \mathbb{Z})$ .

**5.1. Reduction.** We start with the following.

**Lemma 5.1.** *Let  $V$  be a finitely generated free abelian group equipped with a symplectic form and let  $W$  be a subgroup of  $V$ . Assume that  $W$  is a symplectic subspace of  $V$  and that  $W \otimes_{\mathbb{Z}} \mathbb{Q} = V \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then  $W = V$ .*

*Proof.* Since  $W$  is a symplectic subspace of  $V$ , we have  $V = W \oplus W^\perp$ . Since

$$W \otimes_{\mathbb{Z}} \mathbb{Q} = V \otimes_{\mathbb{Z}} \mathbb{Q} = (W \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (W^\perp \otimes_{\mathbb{Z}} \mathbb{Q}),$$

it follows that  $W^\perp \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ . We conclude that  $W^\perp = 0$  and thus that  $W = V$ .  $\square$

It is therefore enough to construct a finite unbranched cover  $\pi: \tilde{\Sigma} \rightarrow \Sigma$  such that

$$H_1^\sigma(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q}) \quad \text{but} \quad H_1^\sigma(\tilde{\Sigma}; \mathbb{Z}) \neq H_1(\tilde{\Sigma}; \mathbb{Z}).$$

For  $\ell \geq 2$ , let  $\pi: \Sigma[\ell] \rightarrow \Sigma$  be the cover corresponding to the homomorphism

$$\pi_1(\Sigma) \longrightarrow H_1(\Sigma; \mathbb{Z}/\ell).$$

By the above, it is enough to prove the following theorem, which we will do in the remainder of this section:

**Theorem 5.2.** *Let  $\Sigma$  be a closed oriented surface of genus at least 2 and  $\sigma$  be the set of nonseparating simple closed curves on  $\Sigma$ . Fix some  $\ell \geq 2$ . The following then hold:*

- (i) *We have  $H_1^\sigma(\Sigma[\ell]; \mathbb{Q}) = H_1(\Sigma[\ell]; \mathbb{Q})$ .*
- (ii) *If  $\ell \geq 3$ , then  $H_1^\sigma(\Sigma[\ell]; \mathbb{Z}) \neq H_1(\Sigma[\ell]; \mathbb{Z})$ .*

Part (ii) is a theorem of Irmer [6, Lemma 6]. We will give a simplified version of her argument below that avoids most of its complicated combinatorial group theory.

**5.2. Rational equality.** We start by proving part (i) of Theorem 5.2. In fact, we prove a more general result:

**Proposition 5.3.** *Let  $\Sigma$  be a closed surface and  $\sigma$  be the set of nonseparating simple closed curves on  $\Sigma$ . Let  $\tilde{\Sigma} \rightarrow \Sigma$  be a finite unbranched abelian cover. Then  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Q}) = H_1(\tilde{\Sigma}; \mathbb{Q})$ .*

*Proof.* It is enough to prove that  $H_1^\sigma(\tilde{\Sigma}; \mathbb{Q})^\perp = 0$ . During the proof of Theorem B', we showed that

$$H_1^\sigma(\tilde{\Sigma}; \mathbb{Q})^\perp = H_1(\tilde{\Sigma}; \mathbb{Q})^{D_\sigma}.$$

It is thus enough to show that the group  $D_\sigma$  fixes no nonzero vectors in  $H_1(\tilde{\Sigma}; \mathbb{Q})$ . In fact, the  $D_\sigma$ -orbits of all nonzero vectors in  $H_1(\tilde{\Sigma}; \mathbb{Q})$  are infinite. This follows from work of Looijenga [14], but since it is only implicit in [14] we give a complete proof.

Let  $G$  be the deck group of the finite abelian cover  $\tilde{\Sigma} \rightarrow \Sigma$ . The actions of  $G$  and  $D_\sigma$  on  $H_1(\tilde{\Sigma}; \mathbb{Q})$  commute, so the action of  $D_\sigma$  preserves the decomposition of  $H_1(\tilde{\Sigma}; \mathbb{Q})$  into  $G$ -isotypic components. Let  $V$  be an irreducible representation of  $G$  over  $\mathbb{Q}$  and let  $W$  be the  $V$ -isotypic component of  $H_1(\tilde{\Sigma}; \mathbb{Q})$ . We must prove that the  $D_\sigma$ -orbits of all nonzero vectors in  $W$  are infinite.

Since  $V$  is an irreducible representation of the finite abelian group  $G$ , there is a finite cyclic quotient  $\phi: G \rightarrow \mathbb{Z}/d$  such that<sup>19</sup> action of  $G$  on  $V$  factors through  $\phi$ . Let

$$\Sigma[\phi] = \tilde{\Sigma}/\ker(\phi),$$

and let  $\pi: \Sigma[\phi] \rightarrow \Sigma$  be the projection, so  $\pi: \Sigma[\phi] \rightarrow \Sigma$  is the degree- $d$  cyclic cover corresponding to  $\ker(\phi)$ . The group  $\ker(\phi)$  acts trivially on  $V$  and hence on  $W$ , so  $W$  is a subrepresentation of<sup>20</sup>

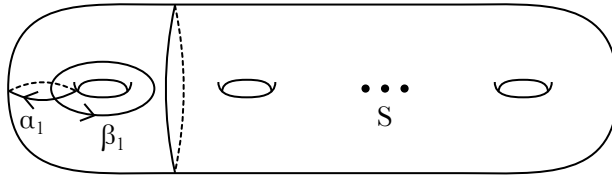
$$H_1(\tilde{\Sigma}; \mathbb{Q})_{\ker(\phi)} \cong H_1(\tilde{\Sigma}/\ker(\phi); \mathbb{Q}) \cong H_1(\Sigma[\phi]; \mathbb{Q}).$$

Letting  $v \in H_1(\Sigma[\phi]; \mathbb{Q})$  be nonzero, it is thus enough to prove that the  $D_\sigma$ -orbit of  $v$  is infinite.

Let  $\omega$  be the algebraic intersection form on  $H_1(\Sigma; \mathbb{Q})$ . There is a surjection  $H_1(\Sigma; \mathbb{Z}) \rightarrow G$ . Pick a surjection  $\tilde{\phi}: H_1(\Sigma; \mathbb{Z}) \rightarrow \mathbb{Z}$  making the diagram

$$\begin{array}{ccc} H_1(\Sigma; \mathbb{Z}) & \xrightarrow{\tilde{\phi}} & \mathbb{Z} \\ \downarrow & & \downarrow \\ G & \xrightarrow{\phi} & \mathbb{Z}/d \end{array}$$

commute. Since  $\omega$  is a symplectic form on  $H_1(\Sigma; \mathbb{Z})$ , there exists some  $a_1 \in H_1(\Sigma; \mathbb{Z})$  such that  $\tilde{\phi}(x) = \omega(a_1, x)$  for all  $x \in H_1(\Sigma; \mathbb{Z})$ . Since  $\tilde{\phi}$  is surjective,  $a_1$  is primitive<sup>21</sup> and thus there exists some oriented nonseparating simple closed curve  $\alpha_1$  such that  $[\alpha_1] = a_1$ . Let  $\beta_1$  and  $S$  be as follows:

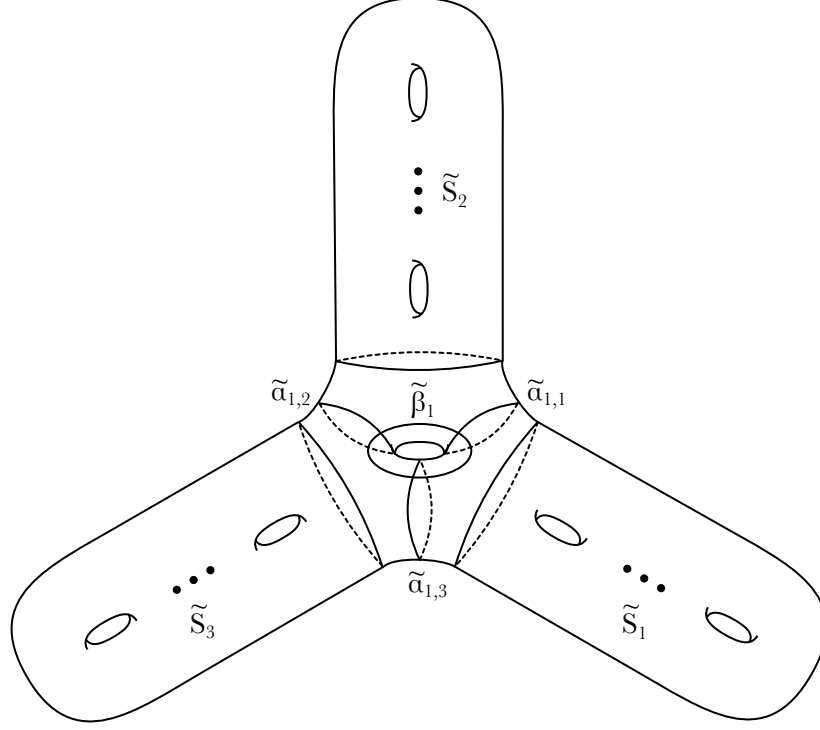


<sup>19</sup>Here is a sketch of this standard fact. Since  $G$  is abelian, the action of  $G$  on  $V$  comes from a homomorphism  $\iota: G \rightarrow \text{End}_G(V)$ . Since  $V$  is irreducible, Schur's Lemma says that  $\text{End}_G(V)$  is a division algebra over  $\mathbb{Q}$ . Let  $F$  be the  $\mathbb{Q}$ -subalgebra of  $\text{End}_G(V)$  generated by  $\text{Im}(\iota)$ . It is an easy exercise to show that for  $f \in \text{End}_G(V)$  nonzero,  $f^{-1}$  can be expressed as a polynomial in  $f$ . It follows that  $F$  is closed under taking multiplicative inverses. Since  $G$  is abelian, this implies that  $F$  is a commutative division ring, i.e., a field. The result now follows from the fact that a finite subgroup of  $F^\times$  like the image of  $\iota: G \rightarrow \text{End}_G(V)$  must be cyclic.

<sup>20</sup>Here the subscript indicates that we are taking the  $\ker(\phi)$ -coinvariants and the first isomorphism follows from the transfer map.

<sup>21</sup>That is, not divisible by any integers greater than 1.

The case  $d = 3$  of our cover  $\pi: \Sigma[\phi] \rightarrow \Sigma$  is then as follows:



More generally, we have

- $\pi^{-1}(S) = \tilde{S}_1 \sqcup \cdots \sqcup \tilde{S}_d$  with each  $\tilde{S}_i$  projecting homeomorphically to  $S$ ; and
- $\pi^{-1}(\beta_1) = \tilde{\beta}_1$ , where  $\tilde{\beta}_1$  is a simple closed curve that  $d$ -fold covers  $\beta_1$ ; and
- $\pi^{-1}(\alpha_1) = \tilde{\alpha}_{1,1} \sqcup \cdots \sqcup \tilde{\alpha}_{1,d}$ , where  $\tilde{\alpha}_{1,i}$  is a simple closed curve projecting homeomorphically to  $\alpha_1$ .

The curves  $\tilde{\alpha}_{1,i}$  are all homologous, and we have

$$H_1(\Sigma[\phi]; \mathbb{Q}) = \langle [\tilde{\alpha}_{1,1}], [\tilde{\beta}_1] \rangle \oplus \bigoplus_{i=1}^d H_1(\tilde{S}_i; \mathbb{Q}).$$

Recall that we are trying to prove that the nonzero  $v \in H_1(\Sigma[\phi]; \mathbb{Q})$  has an infinite  $D_\sigma$ -orbit. In fact, we will find some  $\tilde{\tau} \in D_\sigma$  such that the elements  $\{\tilde{\tau}^n(v) \mid n \geq 1\}$  are all distinct. Write

$$v = \lambda[\tilde{\alpha}_{1,1}] + \nu[\tilde{\beta}_1] + \sum_{i=1}^d v_i \quad \text{with } \lambda, \nu \in \mathbb{Q} \text{ and } v_i \in H_1(\tilde{S}_i; \mathbb{Q}).$$

There are three cases.

The first is  $\lambda \neq 0$ . In this case,  $T_{\beta_1}^d \in \text{Mod}(\Sigma)$  lifts to  $T_{\tilde{\beta}_1} \in \text{Mod}(\Sigma[\phi])$ . We have

$$T_{\tilde{\beta}_1}^n(v) = \lambda[\tilde{\alpha}_{1,1}] + (\nu - n\lambda)[\tilde{\beta}_1] + \sum_{i=1}^d v_i \quad \text{for } n \geq 1.$$

These are all distinct elements. Since  $D_\sigma$  is defined via the lifts to the cover<sup>22</sup>  $\tilde{\Sigma} \rightarrow \Sigma$ , there is some  $\ell \geq 1$  (necessarily divisible by  $d$ ) such that the element  $\tilde{\tau}_{\beta_1} \in D_\sigma$  is induced by the lift of  $T_{\beta_1}^\ell = (T_{\beta_1}^d)^{\ell/d}$ . We conclude that the elements  $\{\tilde{\tau}_{\beta_1}^n(v) \mid n \geq 1\}$  are all distinct.

<sup>22</sup>Remember that the degree- $d$  cyclic cover  $\Sigma[\phi] \rightarrow \Sigma$  is a subcover of  $\tilde{\Sigma} \rightarrow \Sigma$ , i.e., the map  $\tilde{\Sigma} \rightarrow \Sigma$  factors as  $\tilde{\Sigma} \rightarrow \Sigma[\phi] \rightarrow \Sigma$ .

The second is  $\nu \neq 0$ . In this case,  $T_{\alpha_1} \in \text{Mod}(\Sigma)$  lifts to  $T_{\tilde{\alpha}_{1,1}} \cdots T_{\tilde{\alpha}_{1,d}} \in \text{Mod}(\Sigma[\phi])$ . Since the  $\tilde{\alpha}_{1,i}$  are all homologous, we have

$$(T_{\tilde{\alpha}_{1,1}} \cdots T_{\tilde{\alpha}_{1,d}})^n(v) = (\lambda + dn\nu)[\tilde{\alpha}_{1,1}] + \nu[\tilde{\beta}_1] + \sum_{i=1}^d v_i \quad \text{for } n \geq 1.$$

These are all distinct elements. Just like in the previous case, we conclude that the elements  $\{\tilde{\tau}_{\alpha_1}^n(v) \mid n \geq 1\}$  are all distinct.

The third is that some  $v_i$  is nonzero. Reordering, assume that  $v_1 \neq 0$ . Let  $\bar{v}_1 \in H_1(S; \mathbb{Q})$  be the image of  $v_1 \in H_1(\tilde{S}_1; \mathbb{Q})$ . Pick an oriented simple closed curve  $\gamma$  on  $S$  with  $\omega([\gamma], \bar{v}_1)$  nonzero. Let  $\tilde{\gamma}_1 \sqcup \cdots \sqcup \tilde{\gamma}_d$  be the preimage of  $\gamma$  in  $\Sigma[\phi]$ , ordered such that  $\tilde{\gamma}_i \in \tilde{S}_i$ . By construction,  $T_\gamma \in \text{Mod}(\Sigma)$  lifts to  $T_{\tilde{\gamma}_1} \cdots T_{\tilde{\gamma}_d} \in \text{Mod}(\Sigma[\phi])$ . We have

$$(T_{\tilde{\gamma}_1} \cdots T_{\tilde{\gamma}_d})^n(v) = \lambda[\tilde{\alpha}_{1,1}] + \nu[\tilde{\beta}_1] + \sum_{i=1}^d (v_i + n\omega([\tilde{\gamma}_i], v_i)[\tilde{\gamma}_i]) \quad \text{for } n \geq 1.$$

Since  $\omega([\tilde{\gamma}_1], v_1) = \omega([\gamma], \bar{v}_1) \neq 0$ , these are all distinct. Just like before, we conclude that the elements  $\{\tilde{\tau}_\gamma^n(v) \mid n \geq 1\}$  are all distinct.  $\square$

**5.3. Nilpotent preliminaries.** Before we can prove part (ii) of Theorem 5.2, we need some preliminary results. Let  $F_n$  be the free group on  $\{x_1, \dots, x_n\}$ . Fix some  $\ell \geq 3$ . Define<sup>23</sup>

$$\widehat{\ell} = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \ell/2 & \text{if } \ell \text{ is even.} \end{cases}$$

Since  $\ell \geq 3$ , we have  $\widehat{\ell} \geq 2$ . Define  $N_n[\ell]$  to be the quotient of  $F_n$  by the normal subgroup generated by the following elements:<sup>24</sup>

- The third term  $[F_n, [F_n, F_n]]$  of the lower central series.
- The subgroup  $[F_n, F_n^{\times \widehat{\ell}}]$ , i.e., the subgroup generated by commutators  $[u, v^{\widehat{\ell}}]$  as  $u$  and  $v$  range over elements of  $F_n$ .

We will use boldface letters to denote elements of  $N_n[\ell]$ , and in particular will let  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  be the generators of  $N_n[\ell]$  coming from the generators  $\{x_1, \dots, x_n\}$  for  $F_n$ . The abelianization of  $N_n[\ell]$  is  $\mathbb{Z}^n$ , and for  $\mathbf{u} \in N_n[\ell]$  we will write  $\bar{\mathbf{u}} \in \mathbb{Z}^n$  for its image in the abelianization and  $\widehat{\mathbf{u}} \in (\mathbb{Z}/\widehat{\ell})^n$  for the image of  $\bar{\mathbf{u}}$  under the mod- $\widehat{\ell}$  reduction map.

The following lemma clarifies the nature of  $N_n[\ell]$ :

**Lemma 5.4.** *For  $n \geq 2$  and  $\ell \geq 3$ , we have a central extension*

$$1 \longrightarrow \wedge^2(\mathbb{Z}/\widehat{\ell})^n \longrightarrow N_n[\ell] \longrightarrow \mathbb{Z}^n \longrightarrow 1.$$

Here the map  $N_n[\ell] \rightarrow \mathbb{Z}^n$  is the abelianization map taking  $\mathbf{u} \in N_n[\ell]$  to  $\bar{\mathbf{u}} \in \mathbb{Z}^n$ , and for  $\mathbf{u}, \mathbf{v} \in N_n[\ell]$  the commutator  $[\mathbf{u}, \mathbf{v}] \in N_n[\ell]$  is the central element  $\widehat{\mathbf{u}} \wedge \widehat{\mathbf{v}} \in \wedge^2(\mathbb{Z}/\widehat{\ell})^n$ .

*Proof.* It is immediate from Magnus–Witt’s work on the lower central series of a free group ([16, 22]; see [21] for a textbook account) that

$$\frac{[F_n, F_n]}{[F_n, [F_n, F_n]]} \cong \wedge^2 \mathbb{Z}^n,$$

<sup>23</sup>When reading this for the first time, it might be easier to assume that  $\ell$  is odd, so  $\widehat{\ell} = \ell$ .

<sup>24</sup>Here “N” stands for “nilpotent”.

with  $[u, v] \in [F_n, F_n]$  mapping to  $\bar{u} \wedge \bar{v} \in \wedge^2 \mathbb{Z}^n$ . Here  $\bar{u}, \bar{v} \in \mathbb{Z}^n$  are the images of  $u, v \in F_n$  in its abelianization. This fits into a central extension

$$1 \longrightarrow \wedge^2 \mathbb{Z}^n \longrightarrow \frac{F_n}{[F_n, [F_n, F_n]]} \longrightarrow \mathbb{Z}^n \longrightarrow 1.$$

To get  $N_n[\ell]$  from the middle group in this extension, one quotients out the image of  $[F_n, F_n^{\times \ell}]$ , which maps to the kernel of the map

$$\wedge^2 \mathbb{Z}^n \longrightarrow \wedge^2 (\mathbb{Z}/\widehat{\ell})^n.$$

The lemma follows.  $\square$

In the rest of this section, we will identify  $\wedge^2 (\mathbb{Z}/\widehat{\ell})^n$  with the corresponding central subgroup of  $N_n[\ell]$ . The following calculation lies at the heart of our arguments:

**Lemma 5.5.** *For  $n \geq 2$  and  $\ell \geq 3$ , we have  $(\mathbf{uv})^\ell = \mathbf{u}^\ell \mathbf{v}^\ell$  for all  $\mathbf{u}, \mathbf{v} \in N_n[\ell]$ .*

*Proof.* To transform  $(\mathbf{uv})^\ell$  into  $\mathbf{u}^\ell \mathbf{v}^\ell$ , we must commute each  $\mathbf{u}$  past all the  $\mathbf{v}$  terms to its left. Each time we commute a  $\mathbf{u}$  past a  $\mathbf{v}$ , we must introduce a commutator  $[\mathbf{v}, \mathbf{u}] = \widehat{\mathbf{v}} \wedge \widehat{\mathbf{u}}$ . This commutator is central, so it can be moved all the way to the right. The first  $\mathbf{u}$  must be commuted with 0 copies of  $\mathbf{v}$ , the second with 1 copy of  $\mathbf{v}$ , the third with 2 copies of  $\mathbf{v}$ , etc. In the end, we see that

$$(\mathbf{uv})^\ell = \mathbf{u}^\ell \mathbf{v}^\ell [\mathbf{v}, \mathbf{u}]^{0+1+2+\dots+(\ell-1)} = \mathbf{u}^\ell \mathbf{v}^\ell [\mathbf{v}, \mathbf{u}]^{\ell(\ell-1)/2}.$$

Whether  $\ell$  is even or odd,<sup>25</sup> the integer  $\ell(\ell-1)/2$  is divisible by  $\widehat{\ell}$ . Since  $[\mathbf{v}, \mathbf{u}] \in \wedge^2 (\mathbb{Z}/\widehat{\ell})^n$ , this implies that  $[\mathbf{v}, \mathbf{u}]^{\ell(\ell-1)/2} = 1$ . The lemma follows.  $\square$

Define  $P_n[\ell]$  to be<sup>26</sup> the subgroup of  $N_n[\ell]$  generated by  $\{\mathbf{u}^\ell \mid \mathbf{u} \in N_n[\ell]\}$  and define  $A_n[\ell]$  to be the subgroup<sup>27</sup> of  $N_n[\ell]$  generated by  $P_n[\ell]$  and  $\wedge^2 (\mathbb{Z}/\widehat{\ell})^n$ . We then have:

**Lemma 5.6.** *For  $n \geq 2$  and  $\ell \geq 3$ , the subgroup  $P_n[\ell]$  is a central subgroup of  $N_n[\ell]$  with  $P_n[\ell] \cong \mathbb{Z}^n$ , and  $A_n[\ell] = P_n[\ell] \times \wedge^2 (\mathbb{Z}/\widehat{\ell})^n$ .*

*Proof.* The fact that  $P_n[\ell]$  is a central subgroup follows from the fact that

$$[\mathbf{u}^\ell, \mathbf{v}] = \widehat{\mathbf{u}}^\ell \wedge \widehat{\mathbf{v}} = \ell \left( \widehat{\mathbf{u}} \wedge \widehat{\mathbf{v}} \right) = 0 \quad \text{for all } \mathbf{u}, \mathbf{v} \in N_n[\ell].$$

Recall that  $N_n[\ell]$  is generated by the elements  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , which map to a basis for the abelianization  $\mathbb{Z}^n$ . The elements  $\mathbf{x}_i^\ell \in N_n[\ell]$  are central and map to linearly independent elements in the abelianization, so

$$P'_n[\ell] = \left\{ \mathbf{x}_1^{\ell k_1} \cdots \mathbf{x}_n^{\ell k_n} \mid k_1, \dots, k_n \in \mathbb{Z} \right\}$$

is a central subgroup satisfying  $P'_n[\ell] \cong \mathbb{Z}^n$ . Moreover, letting  $A'_n[\ell]$  be the subgroup of  $N_n[\ell]$  generated by  $P'_n[\ell]$  and  $\wedge^2 (\mathbb{Z}/\widehat{\ell})^n$ , we clearly have  $A'_n[\ell] = P'_n[\ell] \times \wedge^2 (\mathbb{Z}/\widehat{\ell})^n$ .

To prove the lemma, it is therefore enough to prove that  $P_n[\ell] = P'_n[\ell]$ . Since  $\mathbf{x}_i^\ell \in P_n[\ell]$  for all  $1 \leq i \leq n$ , we have  $P'_n[\ell] \subset P_n[\ell]$ . For the reverse inclusion, consider some  $\mathbf{u} \in N_n[\ell]$ . We must prove that  $\mathbf{u}^\ell \in P'_n[\ell]$ . We can find  $k_1, \dots, k_n \in \mathbb{Z}$  and

$$\mathbf{c} \in [P_n[\ell], P_n[\ell]] = \wedge^2 (\mathbb{Z}/\widehat{\ell})^n$$

such that  $\mathbf{u} = \mathbf{x}_1^{k_1} \cdots \mathbf{x}_n^{k_n} \mathbf{c}$ . Applying Lemma 5.5 repeatedly, we deduce that

$$\mathbf{u}^\ell = \mathbf{x}_1^{\ell k_1} \cdots \mathbf{x}_n^{\ell k_n} \mathbf{c}^\ell = \mathbf{x}_1^{\ell k_1} \cdots \mathbf{x}_n^{\ell k_n} \in P'_n[\ell]. \quad \square$$

<sup>25</sup>The purpose of using  $\widehat{\ell}$  is to ensure this.

<sup>26</sup>Here ‘‘P’’ stands for ‘‘power subgroup’’.

<sup>27</sup>Here ‘‘A’’ stands for ‘‘abelian subgroup’’; see Lemma 5.6.

**5.4. Integral inequality.** We now prove part (ii) of Theorem 5.2

*Proof of Theorem 5.2, part (ii).* We first recall the statement. Let  $\Sigma$  be a closed oriented surface of genus  $g \geq 2$  and  $\sigma$  be the set of nonseparating simple closed curves on  $\Sigma$ . Fix some  $\ell \geq 3$ , and as above let

$$\widehat{\ell} = \begin{cases} \ell & \text{if } \ell \text{ is odd,} \\ \ell/2 & \text{if } \ell \text{ is even.} \end{cases}$$

Since  $\ell \geq 3$ , we have  $\widehat{\ell} \geq 2$ . We must prove that  $H_1^\sigma(\Sigma[\ell]; \mathbb{Z}) \neq H_1(\Sigma[\ell]; \mathbb{Z})$ .

Recall that  $\Sigma[\ell]$  is the cover corresponding to the homomorphism

$$\pi_1(\Sigma) \rightarrow H_1(\Sigma; \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{2g}.$$

It follows that  $\pi_1(\Sigma[\ell])$  is the kernel of this map, so  $\pi_1(\Sigma[\ell])$  is the subgroup of  $\pi_1(\Sigma)$  generated by the following two subgroups:

- The commutator subgroup  $[\pi_1(\Sigma), \pi_1(\Sigma)]$ .
- The subgroup  $P$  generated by  $\{x^\ell \mid x \in \pi_1(\Sigma)\}$ .

Each nonseparating simple closed curve  $x \in \pi_1(\Sigma)$  maps to a primitive<sup>28</sup> element of  $H_1(\Sigma; \mathbb{Z})$ , so the minimal power of  $x$  that lies in  $\pi_1(\Sigma[\ell])$  is  $x^\ell$ . It follows that the image  $\overline{P}$  of  $P$  in  $H_1(\Sigma[\ell]; \mathbb{Z})$  contains  $H_1^\sigma(\Sigma[\ell]; \mathbb{Z})$ . It is enough therefore to prove that  $\overline{P} \neq H_1(\Sigma[\ell]; \mathbb{Z})$ .

Let  $\{a_1, b_1, \dots, a_g, b_g\}$  be the standard generating set for  $\pi_1(\Sigma)$  satisfying the surface relation  $[a_1, b_1] \cdots [a_g, b_g] = 1$ . We can then define a homomorphism  $\phi: \pi_1(\Sigma) \rightarrow N_g[\ell]$  via the formulas

$$\phi(a_i) = \mathbf{x}_i \quad \text{and} \quad \phi(b_i) = 1 \quad \text{for } 1 \leq i \leq g.$$

The map  $\phi$  takes  $[\pi_1(\Sigma), \pi_1(\Sigma)]$  to the central subgroup  $\wedge^2(\mathbb{Z}/\widehat{\ell})^g$  and  $P$  to the central subgroup  $P_g[\ell]$  (see Lemma 5.6). It follows that  $\phi$  takes  $\pi_1(\Sigma[\ell])$  surjectively onto the abelian subgroup  $A_g[\ell] = P_g[\ell] \times \wedge^2(\mathbb{Z}/\widehat{\ell})^g$  identified by Lemma 5.6. The restriction of  $\phi$  to  $\pi_1(\Sigma[\ell])$  thus factors through  $H_1(\Sigma[\ell]; \mathbb{Z})$ , and takes  $\overline{P} \subset H_1(\Sigma[\ell]; \mathbb{Z})$  to the proper subgroup  $P_g[\ell]$  of  $A_g[\ell]$ . The theorem follows.  $\square$

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<sup>28</sup>That is, not divisible by any integers except  $\pm 1$ .

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