GLOBAL FIXED POINTS OF MAPPING CLASS GROUP ACTIONS AND A THEOREM OF MARKOVIC

LEI CHEN AND NICK SALTER

Abstract. We give a short and elementary proof of the non-realizability of the mapping class group via homeomorphisms. This was originally established by Markovic, resolving a conjecture of Thurston. With the tools established in this paper, we also obtain some rigidity results for actions of the mapping class group on Euclidean spaces.

1. introduction

In this paper, we discuss a new strategy to study Nielsen's realization problem. We will use this to give a new and very simple proof of the main result of [Mar07].

Let Σ_g be the surface of genus g and $\mathrm{Homeo}_+(\Sigma_g)$ the group of orientation-preserving homeomorphisms of Σ_g. Denote by $\mathrm{Mod}(\Sigma_g) := \pi_0(\mathrm{Homeo}_+(\Sigma_g))$ the mapping class group of Σ_g. The *Nielsen realization problem* asks whether the natural projection

$$p_g : \mathrm{Homeo}_+(\Sigma_g) \to \mathrm{Mod}(\Sigma_g)$$

has a group-theoretic section (this particular formulation is attributed to Thurston; c.f. [Kir78, Problem 2.6]). The realization problem can also be posed for other regularities and for arbitrary subgroups of $\mathrm{Mod}(\Sigma_g)$, and there is a rich literature on the many variants that arise. We refer the reader for the survey paper [MT18] for further discussion.

In [Mar07], Markovic shows that p_g has no section for $g > 5$ and in [MS08], Markovic and Saric extend these methods to show that p_g has no section for $g \geq 2$. The proof is very involved and uses many dynamical tools. The main result of this note gives an elementary proof of these results in the optimal range $g \geq 2$.

Theorem 1.1. *For $g \geq 2$, the projection p_g has no sections.*

We obtain this as a consequence of a rigidity theorem for actions of a closely related group. Let $\Sigma_{g,1}$ be the surface of genus g with one marked point and let $\mathrm{Homeo}_+(\Sigma_{g,1})$ denote the group of orientation-preserving homeomorphisms of $\Sigma_{g,1}$ that fix the marked point. Define $\mathrm{Mod}(\Sigma_{g,1}) := \pi_0(\mathrm{Homeo}_+(\Sigma_{g,1}))$ to be the "pointed mapping class group" of $\Sigma_{g,1}$.

Theorem 1.2. *For $g \geq 2$, any nontrivial action of $\mathrm{Mod}(\Sigma_{g,1})$ on $\mathbb{R}^2$ by homeomorphisms has a unique global fixed point.*

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The same set of ideas also leads to a rigidity theorem for mapping class group actions on $\mathbb{R}^3$ in the regime $g \geq 4$.

Theorem 1.3. *For $g \geq 4$, any nontrivial continuous action of $\mathrm{Mod}(\Sigma_{g,1})$ on $\mathbb{R}^3$ has a globally-invariant line.*

Corollary 1.4. *For $g \geq 4$, there is no nontrivial action of $\mathrm{Mod}(\Sigma_{g,1})$ on $\mathbb{R}^2$ and $\mathbb{R}^3$ by C^1 diffeomorphisms.*

All of the above results are easy consequences of the following structural result for $\mathrm{Mod}(\Sigma_{g,1})$. For an element f of a group G, we write $C(f)$ to denote the centralizer of f in G. Also recall that a homeomorphism ι is said to be *hyperelliptic* if ι has order 2 and has exactly $2g+2$ fixed points. A mapping class is said to be hyperelliptic if it admits a hyperelliptic representative.

Theorem 1.5. *For $g \geq 2$, there exists an order 6 element $\alpha_g \in \mathrm{Mod}(\Sigma_{g,1})$ such that*

$$\Gamma := \langle C(\alpha_g^2), C(\alpha_g^3)\rangle$$

is the full mapping class group:

$$\Gamma = \mathrm{Mod}(\Sigma_{g,1}).$$

If $g \neq 3$, then α_g can be constructed so that α_g^3 is not hyperelliptic.

Problem 1.6. *If $\beta \in \mathrm{Mod}(\Sigma_{g,1})$ is a torsion element of order divisible by two distinct primes p, q, then there is no* a priori *obstruction for $\mathrm{Mod}(\Sigma_{g,1})$ to be generated by $C(\beta^p)$ and $C(\beta^q)$. Does the conclusion of Theorem 1.5 hold for any torsion element β with order not a prime power?*

Organization. Sections 2 - 4 are devoted to the proof of Theorem 1.5. Sections 5 and 6 show how Theorems 1.1, 1.2, and 1.3 then follow as easy corollaries. The reader willing to take the generating set of Theorem 1.5 for granted can skip directly to Section 5.

The proof of Theorem 1.5 is organized as follows. In Section 2 we describe our models of $\Sigma_{g,1}$ equipped with the symmetry α_g; we use a different model for each residue class g (mod 3). In Section 3, we discuss a special class of curves and subsurfaces on these model subsurfaces that feature in the proof of Theorem 1.5. We carry out the proof of Theorem 1.5 in Section 4.

The proofs of Theorems 1.1 and 1.2 are presented in Section 5. Finally, we deduce Theorem 1.3 and Corollary 1.4 from Theorem 1.5 in Section 6.

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2. Models

The aim of this section is to construct, for each $g \geq 2$, a certain symmetry $\alpha_g \in \mathrm{Mod}(\Sigma_{g,1})$ of order 6. The proof of Theorem 1.5 in Section 4 requires the existence of certain configurations of symmetric curves which are easy to find only for certain conjugacy classes of order-6 elements of $\mathrm{Mod}(\Sigma_{g,1})$; this is why we must take care in constructing our symmetries. The Riemann–Hurwitz formula (c.f. Lemma 2.1) implies that different constructions are necessary for each residue class of $g \pmod 3$. In order to give as uniform a presentation as possible, we represent each "model surface" as a disk with pairs of boundary segments identified; the rules for edge identification are specified by the data of a "monodromy tuple" to be presented below (see the table in (2)).

For $g \neq 3$, the symmetries we use (and the corresponding symmetric surfaces) are depicted in Figure 1. The case $g = 3$ requires special consideration; the model we use is shown in Figure 6. The discussion leading up to Figure 1 is not absolutely required to make sense of Figure 1, but is included so as to help orient the reader.

2.1. **Branched covers.** The symmetries we construct are realized as deck transformations of $\mathbb{Z}/6\mathbb{Z}$-branched covers of S^2. Here we recall the basic topological theory of branched coverings. Fix a group G and surfaces X and Y; we also fix the *branch locus* $B \subset Y$, a finite set of points. A branched covering $f : X \to Y$ with covering group G branched over B is then specified by a surjective homomorphism $\rho : \pi_1(Y \setminus B) \to G$. The preimage $f^{-1}(B)$ is the *ramification set* and the elements are *ramification points*. A point $x \in X$ is ramified if and only if $\mathrm{Stab}_G(x)$ is nontrivial; in this case, the *order* of x is defined to be the order of $\mathrm{Stab}_G(x)$.

When $Y = S^2$ is a sphere, this can be further combinatorialized. Enumerate $B = \{b_1, \ldots, b_n\}$, and choose an identification

$$\pi_1(S^2 \setminus B) \cong \langle a_1, \ldots, a_n \mid a_1 \ldots a_n = 1 \rangle;$$

here each a_i runs from a basepoint $p \in S^2$ to a small loop around b_i. The *local monodromy* at b_i is the corresponding element $\rho(a_i) \in G$. Without loss of generality we can assume that each $\rho(a_i) \neq 1$. The *monodromy vector* is the associated tuple $(\rho(a_1), \ldots, \rho(a_n)) \in G^n$. Note that necessarily $\rho(a_1) \ldots \rho(a_n) = 1$, and conversely, any such n-tuple gives rise to a branched G-cover.

For the purposes of this paper we will only be concerned with the case $G = \mathbb{Z}/6\mathbb{Z}$, and we adopt some further notation special to this situation. With the branch set $B \subset S^2$ fixed, we observe that each branch point b_i has corresponding order $|\rho(a_i)| \in \{2, 3, 6\}$. Define p (resp. q or r) as the number of points of order 6 (resp. 3 or 2). We define the *branching vector* as the tuple (p, q, r). The lemma below records the Riemann–Hurwitz formula specialized to this setting.

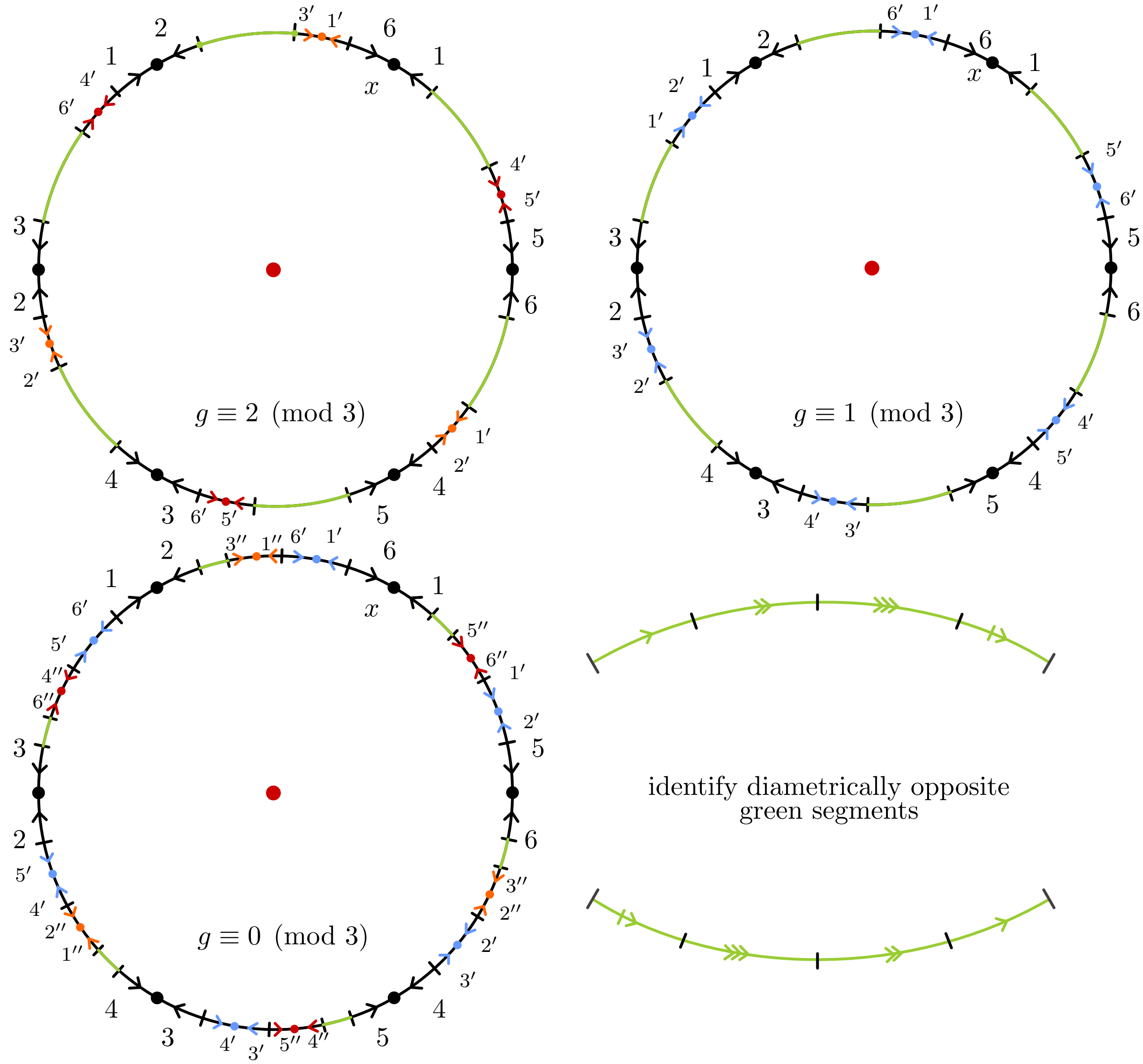

FIGURE 1. The model surfaces for $g \neq 3$: each is constructed by taking the disk D and identifying the specified edge segments of ∂D. In each case, α_g is represented as a rotation by $2\pi/6$ about the center of the disk. The lower-right portion shows the gluing instructions for the green regions. In the $g \equiv 0, 1 \pmod 3$ cases, the green regions are subdivided into $2k+1$ subintervals, and in the case $g \equiv 2 \pmod 3$, each is divided into $2k$ subintervals (completely omitted in the case $k = 0$; c.f. Figure 7). Note finally that each ramification point of order 6 or 3 appears as a vertex of the underlying CW structure, while *no* ramification point of order 2 is a vertex; these lie at the midpoints of green subintervals.

Lemma 2.1. *Let $f : \Sigma_g \to S^2$ be a $\mathbb{Z}/6\mathbb{Z}$-branched covering with branching vector (p, q, r). Then*

$$5p + 4q + 3r = 10 + 2g.$$

Monodromy tuples. As a final specialization, we can shorten our notation for the monodromy vector at the cost of possibly re-ordering the elements of B. Suppose that $1 \in \mathbb{Z}/6\mathbb{Z}$ appears a times, $2 \in \mathbb{Z}/6\mathbb{Z}$ appears b times, etc. Up to a re-ordering of B, this data can be captured in the *monodromy tuple*. To make the computation of the associated p, q, r more transparent, we order the elements of $\mathbb{Z}/6\mathbb{Z}$ according to their group-theoretic order. Thus a monodromy tuple is a symbol of the following form:

$$1^a\ 5^{p-a}\ 2^b\ 4^{q-b}\ 3^r. \tag{1}$$

2.2. **The model surfaces.** The elements α_g of Theorem 1.5 will be constructed as deck transformations associated to regular $\mathbb{Z}/6\mathbb{Z}$ covers of S^2 as in the previous subsection. We will require different constructions for the three different residue classes $g \pmod 3$ and a special construction for $g = 3$. In the table (2) below, we specify $k \geq 0$. The column (V, E, F) tracks the number of $0, 1, 2$ cells (as a 3-tuple) in the CW structure implicit in Figure 1 given by identifying portions of the boundary of the disk. As the final column shows, for $g \neq 3$, the power α_g^3 has strictly fewer than $2g + 2$ fixed points and hence is not hyperelliptic.

g	(p, q, r)	tuple	(V, E, F)	# branched points of α_g^3
$2 + 3k$	$(2, 1, 2k)$	$1^2\ 4\ 3^{2k}$	$(9, 12 + 6k, 1)$	$2k \times 3 + 2 < 2g + 2$
3	$(2, 0, 2)$	$1\ 5\ 3^2$	$(7, 12, 1)$	$8 = 2g + 2$
$3 + 3(k + 1)$	$(3, 1, 2k + 1)$	$1^2\ 5\ 2\ 3^{2k+1}$	$(10, 21 + 6k, 1)$	$(2k + 1) \times 3 + 3 < 2g + 2$
$4 + 3k$	$(3, 0, 2k + 1)$	$1^3\ 3^{2k+1}$	$(8, 15 + 6k, 1)$	$(2k + 1) \times 3 + 3 < 2g + 2$

(2)

To represent $\Sigma_{g,1}$ with its symmetry α_g as in (2), we adopt the models shown in Figure 1, where $\Sigma_{g,1}$ is given as a (marked) disk D with edge identifications. See Figure 1 and its caption for a detailed discussion. We emphasize that the marked point $x \in \Sigma_{g,1}$ is *not* the fixed point at the center, but rather one of the fixed points on ∂D (labeled, and drawn with a heavy dot). The model surface for $g = 3$ is given in Figure 6.

For the purpose of later discussion, we observe here a simple property of this construction.

Definition 2.2 (Edge type)**.** Let e be an edge of ∂D. By construction, at most one endpoint of e is a ramification point (necessarily of order 6 or 3, c.f. Figure 1). If one endpoint is a ramification point, we say that e is *type* p (resp. *type* q) if this ramification point has order 6 (resp. 3). If neither endpoint is a ramification point, then e is one of the green subintervals of Figure 1, in which case we say e is *type* r.

3. Chords and convexity

In this section we develop some language for discussing a special class of curves and subsurfaces on the model surfaces. This is based around an ad-hoc identification of the disk

D with the *hyperbolic* disk $\mathbb{D}^2$. We will find it convenient to consider representatives for curves on $\Sigma_{g,1}$ as geodesics on $\mathbb{D}^2$, and especially to consider the notion of convexity in $\mathbb{D}^2$. We emphasize here that we are using $\mathbb{D}^2$ in a nonstandard way: $\mathbb{D}^2$ is *not* playing the role of the universal cover of $\Sigma_{g,1}$. Rather, we are viewing $\Sigma_{g,1}$ as a *topological quotient* of $\mathbb{D}^2$ under a set of identifications of portions of $\partial\mathbb{D}^2$. The geometry of $\mathbb{D}^2$ will provide us with a convenient framework in which to prove Theorem 1.5.

3.1. **Chordal curves.** The first special structure inherited from imposing the hyperbolic metric on D is a privileged (finite) set of simple closed curves: those that can be represented as single geodesics on $\mathbb{D}^2$.

Definition 3.1 (Chordal curve)**.** Let $\Sigma_{g,1}$ be given for $g \geq 2$, and let D be the associated disk as shown in Figure 1; we identify D with the hyperbolic disk $\mathbb{D}^2$. A *chordal curve* is a simple closed curve $c \subset \Sigma_{g,1}$ that can be represented as a single geodesic on $\mathbb{D}^2$. A chordal curve is *basic* if its endpoints can be taken to lie on the *interiors* of the identified portions of ∂D. The *type* of a basic chordal curve is defined to be the type of the corresponding edge of $\partial\mathbb{D}^2$ in the sense of Definition 2.2.

See Figure 2 for some examples and non-examples of chordal curves.

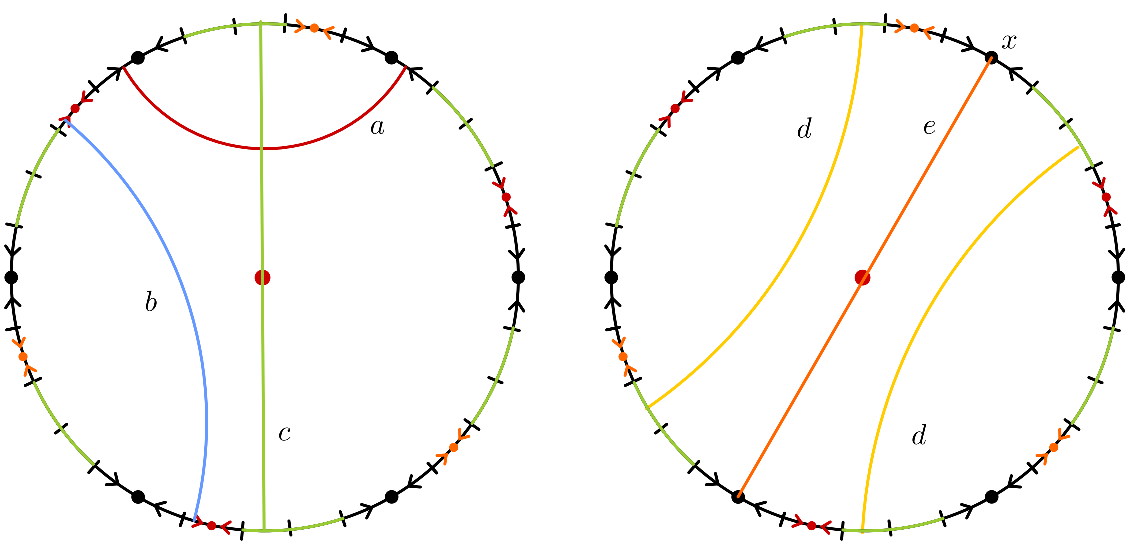

FIGURE 2. At left, three chordal curves a, b, c on $\Sigma_{5,1}$ of types p, q, r, respectively. At right, two curves d, e that are not chordal. d is not chordal because it cannot be represented as a single segment on D, and e is not chordal because it passes through the marked point x.

An individual basic chordal curve is not (in general) invariant under nontrivial powers of α_g. Lemma 3.2 shows that nevertheless, by using both $C(\alpha_g^2)$ and $C(\alpha_g^3)$, the symmetry can be broken and the associated Dehn twists can be exhibited as elements of Γ.

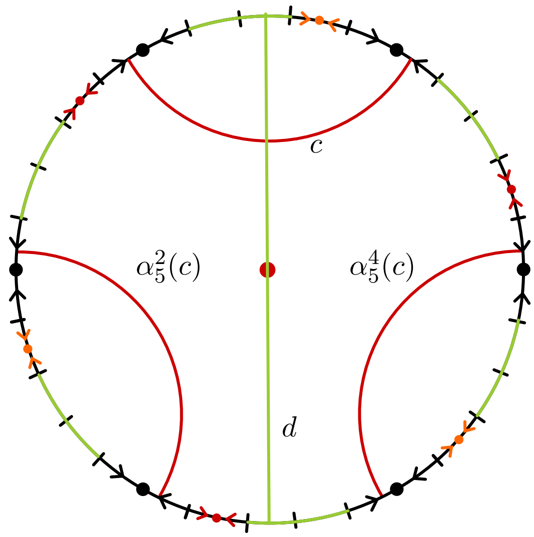

FIGURE 3. The curves $c, \alpha_g^2(c), \alpha_g^4(c), d$ of Lemma 3.2, illustrated for $g = 5$.

Lemma 3.2. *Let c be a basic chordal curve of type p or r. If $g \geq 3$, then $T_c \in \Gamma$.*

Proof. If c is of type r, then c can be represented as a diameter of D and hence $c \in C(\alpha_g^3) \leq \Gamma$. Suppose now that c is of type p, connecting edges e_1, e_2 of ∂D of type p. In each of the model surfaces, if $g \geq 3$, then between e_1 and e_2 is an edge e_3 of type r (see Figure 3). Consider the associated basic chordal curve d of type r. As discussed above, $T_d \in \Gamma$, and also

$$T_c T_{\alpha_g^2(c)} T_{\alpha_g^4(c)} \in C(\alpha_g^2) \leq \Gamma.$$

By construction, the geometric intersection $i(c, d) = 1$, while also

$$i(\alpha_g^2(c), d) = i(\alpha_g^4(c), d) = 0.$$

Thus,

$$(T_c T_{\alpha_g^2(c)} T_{\alpha_g^4(c)}) T_d (T_c T_{\alpha_g^2(c)} T_{\alpha_g^4(c)})^{-1} = T_c T_d T_c^{-1} \in \Gamma.$$

On the other hand, since $i(c, d) = 1$, the braid relation implies that

$$T_c T_d T_c^{-1} = T_d^{-1} T_c T_d \in \Gamma,$$

and hence $T_c \in \Gamma$ as well. □

3.2. ***D*-convexity.** The second piece of hyperbolic geometry we borrow is the notion of convexity. In the proof of Theorem 1.5, we will proceed inductively, showing that Γ contains the mapping class groups for an increasing union of subsurfaces. In the inductive step, we will need to control the topology of the enlarged subsurface relative to the original; we accomplish this by restricting our attention to subsurfaces that are *convex* from the point of view of the hyperbolic metric on $\mathbb{D}^2$.

Definition 3.3 (D-convex hull, D-convexity)**.** Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be a collection of simple closed curves on $\Sigma_{g,1}$. Represent each c_i as a union of chords on D, i.e. as a union of geodesics on the hyperbolic disk $\mathbb{D}^2$. The *D-convex hull* of $\mathcal{C}$ is the subsurface $\mathrm{Hull}(\mathcal{C}) \subseteq \Sigma_{g,1}$ constructed as follows: first, take a closed ϵ-neighborhood of $\bigcup c_i$ (viewed as a subset of D), take the convex hull of this set in the hyperbolic metric on $\mathbb{D}^2$, project onto $\Sigma_{g,1}$, and then fill in any inessential boundary components.

A subsurface $S \subset \Sigma_{g,1}$ is said to be *D-convex* if it can be represented as a convex region on D with respect to the hyperbolic metric on $\mathbb{D}^2$.

Lemma 3.4. *Let $\mathcal{C}_g$ denote the set of basic chordal curves of type p and r on $\Sigma_{g,1}$. Then $\mathrm{Hull}(\mathcal{C}_g) = \Sigma_{g,1}$ for all $g \geq 2$.*

Proof. For $g \equiv 1 \pmod 3$ this is clear from inspection of Figure 1, since there are no edges of type q at all; the case $g = 3$ similarly follows by inspection of Figure 6. For $g \equiv 2 \pmod 3$, this is best seen by inspecting Figure 4. Here one must observe that the remaining boundary components d_1 and d_2 are in fact also both inessential and hence are filled in when constructing $\mathrm{Hull}(\mathcal{C}_g)$. For $g > 3$ and $g \equiv 0 \pmod 3$, there is also exactly one family of edges of type q, and the same considerations as in the case $g \equiv 2 \pmod 3$ apply here as well. □

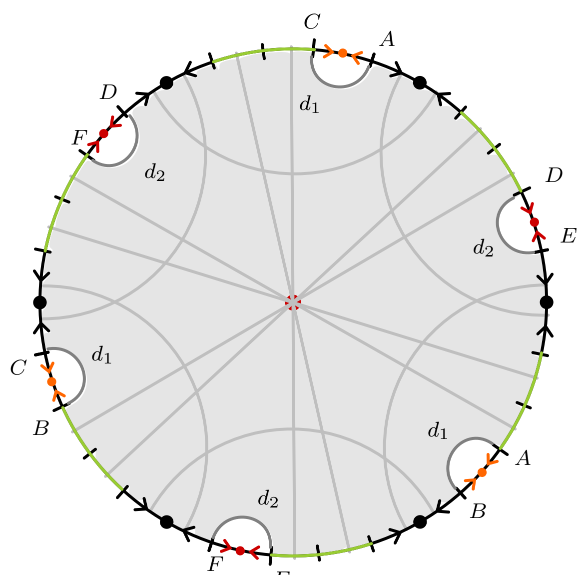

FIGURE 4. The final step in constructing $\mathrm{Hull}(\mathcal{C}_g)$ for $g = 5$. The boundary components d_1 and d_2 are both inessential in $\Sigma_{g,1}$ and so are filled in when constructing $\mathrm{Hull}(\mathcal{C}_g)$.

4. PROOF OF THEOREM 1.5

We prove Theorem 1.5 in Section 4.2. The argument is inductive: we construct a sequence $S_0 \subset S_1 \subset \cdots \subset S_k = \Sigma_{g,1}$ of subsurfaces and show that $\mathrm{Mod}(S_i) \leq \Gamma$ for $i = 1, \ldots, k$. The inductive step is fairly simple and relies on the notion of a "stabilization" of subsurfaces to be discussed in Section 4.1. We consider separate base cases for the regimes $g \geq 4, g = 3$, and $g = 2$; these arguments are deferred to Sections 4.3 – 4.5.

4.1. **Stabilizations.**

Definition 4.1 (Stabilization)**.** Let $S \subset \Sigma$ be a subsurface, and let $c \subset \Sigma$ be a simple closed curve such that $c \cap S$ is a single arc (the endpoints of c do not necessarily lie on distinct boundary components of S). The *stabilization of* S *along* c is the subsurface S^+ constructed as an ϵ-neighborhood of $S \cup c$ inside Σ.

Stabilizations are useful because they allow for simple inductive generating sets for the associated mapping class groups.

Lemma 4.2 (Stabilization)**.** *Let $S \subset \Sigma$ be a subsurface of genus at least 2, and let S^+ denote the stabilization of S along the simple closed curve c. Then*

$$\mathrm{Mod}(S^+) = \langle T_c, \mathrm{Mod}(S) \rangle.$$

Proof. There are two cases to consider: either c enters and exits S via the same boundary component, or else it enters along one component of ∂S and exits along a distinct component. In the former, S^+ has the same genus as S but gains an additional boundary component, and in the latter, S^+ has genus $g(S) + 1$ but one fewer boundary component. In either case, the change–of–coordinates principle for S implies that c can be extended to a configuration of curves $c_0 = c, \ldots, c_n$ such that $c_i \subset S$ for $i > 0$ and such that the associated twists generate $\mathrm{Mod}(S^+)$. For instance, one can take $c_0, \ldots, c_n$ to be the Humphries generating set (see, e.g., [FM12, Theorem 4.14, Figure 4.10]) for S^+ and $c_1, \ldots, c_n$ to be the Humphries generating set for S, so long as $g(S) \geq 2$. The result follows. □

4.2. **Proof of Theorem 1.5.**

The result for $g = 2$ will be established by separate methods in Section 4.5; we therefore assume $g \geq 3$. We will express $\Sigma_{g,1}$ as a sequence

$$S_0 \subset S_1 \subset \cdots \subset S_k = \Sigma_{g,1}$$

of stabilizations of S along curves in the set $\mathcal{C}_g$ of basic chordal curves of type p and r. At each stage we will see that $\mathrm{Mod}(S_i) \leq \Gamma$.

The base case. In Sections 4.3 and 4.4, we will establish the following lemma.

Lemma 4.3. *For each of the model surfaces shown in Figure 1 (excluding $g = 2$) as well as the model surface in genus 3 shown in Figure 6, there is a D-convex subsurface $S_0 \subset \Sigma_{g,1}$ of genus 2 such that* $\mathrm{Mod}(S_0) \leq \Gamma$.

The inductive step. Suppose that S_i is given as a D-convex subsurface with $\mathrm{Mod}(S_i) \leq \Gamma$. Suppose first that every curve $c_j \in \mathcal{C}_g$ is contained in S_i. Since S_i is D-convex and the D-convex hull of $\mathcal{C}_g$ is $\Sigma_{g,1}$ by Lemma 3.4, in this case $S_i = \Sigma_{g,1}$ and the theorem is proved.

Otherwise, select $c_j \in \mathcal{C}_g$ a curve *not entirely contained* in S_i. We then define S_{i+1} to be the D-convex hull of $S_i \cup c_j$. Since S_i is D-convex and c_j is a basic chordal curve, necessarily c_j enters and exits S_i exactly once, and hence S_{i+1} is the stabilization of S_i along c_j. By Lemma 3.2, $T_{c_j} \in \Gamma$, and by hypothesis, $\mathrm{Mod}(S_i) \le \Gamma$. By the stabilization lemma (Lemma 4.2), therefore $\mathrm{Mod}(S_{i+1}) \le \Gamma$ as well. □

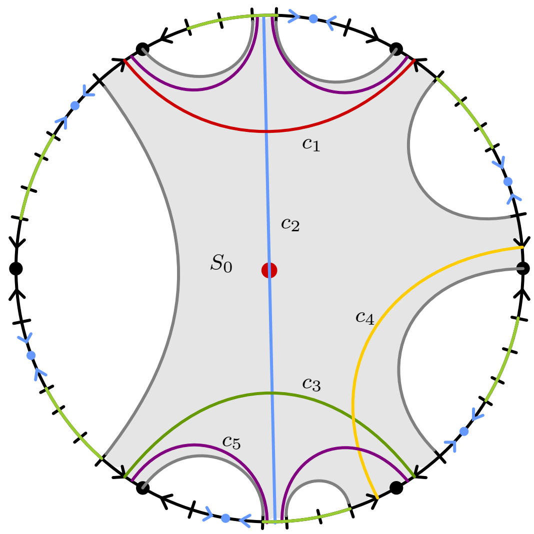

FIGURE 5. For $g \equiv 1 \pmod 3$, the surface S_0 is taken as the D-convex hull of the curves $c_1, \ldots, c_5$ as shown.

4.3. **Proof of Lemma 4.3 for $g \ge 4$.** We will describe a chain of five curves $c_1, \ldots c_5$ such that $c_1 \cup c_3 \cup c_5$ bounds a pair of pants. Such a configuration is supported on a surface S_0 of genus 2 with one boundary component, and the associated Dehn twists form the Humphries generating set (see, e.g., [FM12, Theorem 4.14, Figure 4.10]) for $\mathrm{Mod}(S_0)$. The curves we will describe will either be of type p (hence in Γ by Lemma 3.2) or else invariant under α_g^3 (and hence in $C(\alpha_g^3) \le \Gamma$). Such a configuration is illustrated in Figure 5 in the case of $g \equiv 1 \pmod 4$, but the construction we describe below works on all the model surfaces.

Let c_1 be the curve of type p connecting the edges of type p labeled 1 in Figure 1. We take $c_3 = \alpha_g^3(c_1)$ and $c_4 = \alpha_g^4(c_1)$. Let c_2 be the curve of type r intersecting c_1 and c_3. Finally, let c_5 be the curve obtained by connect-summing c_1 and c_3 along one of the segments of c_2. As shown in Figure 5, c_5 is invariant under α_g^3 as required.

We find that c_1, c_3, c_4 are curves of type p, and c_2, c_5 are invariant under α_g^3, so all associated twists are elements of Γ, and hence $\mathrm{Mod}(S_0) \le \Gamma$ as claimed. □

4.4. **Proof of Lemma 4.3 for $g = 3$.** Recall from (2) that the monodromy tuple for $g = 3$ is $1\ 5\ 3^2$. The model surface for this tuple is shown in Figure 6. To establish Lemma 4.3 in this case, we first consider the subsurface S_0' shown at left in Figure 6. By construction S_0' is an ϵ-neighborhood of $c_1, \ldots, c_5$. Observe that c_1 and c_5 are α_g^3-invariant, c_3 is α_g^2-invariant,

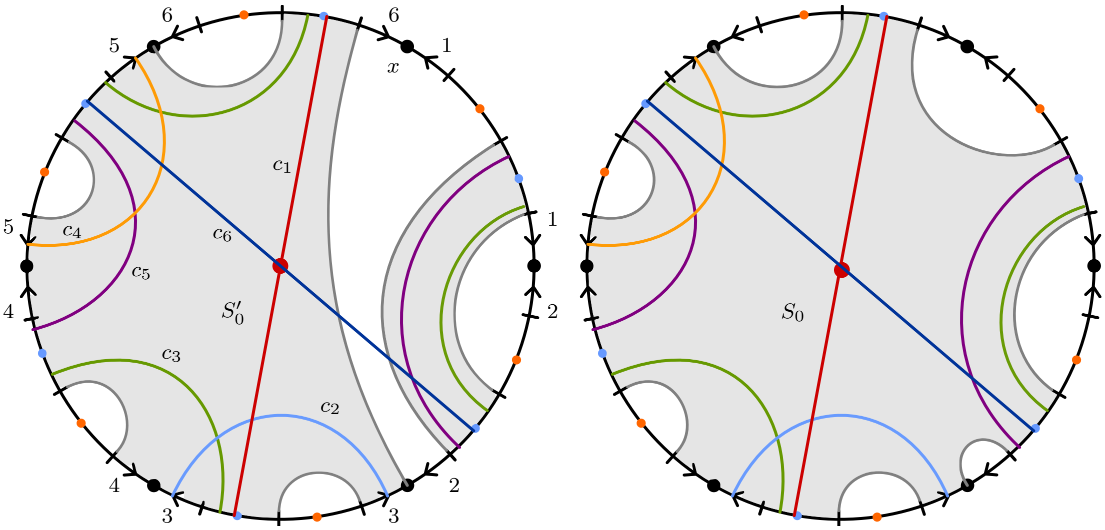

FIGURE 6. The model surface for $g = 3$, built from the monodromy tuple $1\ 5\ 3^2$. The surface S_0' is taken as an ϵ-neighborhood of the curves $c_1, \ldots, c_5$ as shown, and then S_0 is the D-convex hull of $c_1, \ldots, c_6$.

and c_2 and c_4 are basic chordal curves of type p. Thus each associated Dehn twist is an element of Γ. As above, $T_{c_1}, \ldots, T_{c_5}$ determines the Humphries generating set for S_0', and we conclude that $\mathrm{Mod}(S_0') \leq \Gamma$.

We next consider S_0. By construction, S_0 is the D-convex hull of $c_1, \ldots, c_6$, and it is also clear that S_0 is the stabilization of S_0' along c_6. Since c_6 is a basic chordal curve of type r, we have $T_{c_6} \in \Gamma$ by Lemma 3.2. By the stabilization lemma (Lemma 4.2), it follows that $\mathrm{Mod}(S_0) \leq \Gamma$ as required. □

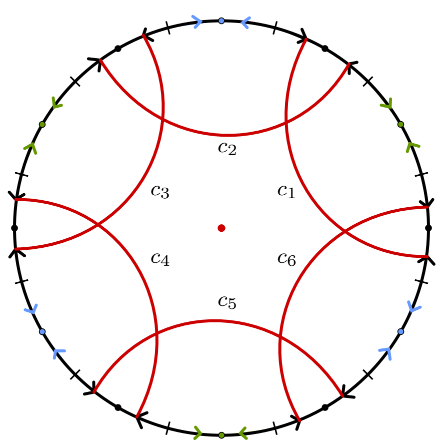

FIGURE 7. The model surface for $g = 2$ and the curves $c_1, \ldots, c_6$.

4.5. **Proof of Theorem 1.5 for $g = 2$.** For $g = 2$, we take a different approach based around an explicit factorization of α_2 into Dehn twists. The model for $g = 2$ is shown in

Figure 7. For ease of notation, we write T_i in place of T_{c_i} throughout the argument. The mapping class group is generated by the twists T_i for $i = 1, \dots, 5$; we will show that all $T_i \in \Gamma$. The fundamental observation is that

$$\alpha_2 = T_1T_2T_3T_4T_5.$$

We also observe that

$$T_1T_4,\ T_2T_5,\ T_3T_6 \in \Gamma$$

since these pairs of curves are invariant under α_2^3, and also

$$T_1T_3T_5,\ T_2T_4T_6 \in \Gamma$$

since these triples are invariant under α_2^2.

We consider the expression of elements of Γ

$$\begin{aligned} \alpha_2(T_2T_5)^{-1}(T_1T_4)^{-1} &= T_1T_2T_3T_4T_5T_5^{-1}T_2^{-1}T_4^{-1}T_1^{-1} \\ &= T_1T_2T_3T_2^{-1}T_1^{-1}, \end{aligned}$$

with the second equality holding by the commutativity of T_i and T_j whenever $i \neq j \pm 1$. Conjugating $T_1T_2T_3T_2^{-1}T_1^{-1}$ by $(T_1T_3T_5)^{-1}$ shows that the element

$$(T_1T_3T_5)^{-1}T_1T_2T_3T_2^{-1}T_1^{-1}(T_1T_3T_5) = T_3^{-1}T_2T_3T_2^{-1}T_3$$

is also in Γ. Conjugating this by (T_3T_6) shows that

$$T_2T_3T_2^{-1} \in \Gamma;$$

a final conjugation by $(T_2T_5)^{-1}$ reveals that $T_3 \in \Gamma$. Conjugation by α_2 now exhibits all T_i in Γ. ☐

5. Proofs of the main theorems

We first explain how Theorem 1.5 implies Theorem 1.2.

Proof of Theorem 1.2. Since $\mathrm{Mod}(\Sigma_{g,1})$ is perfect, it has no nontrivial maps to $\mathbb{Z}/2$. Thus any action on $\mathbb{R}^n$ is orientation-preserving. Let α_g be the symmetry of Theorem 1.5. Any continuous action of the finite-order element α_g on $\mathbb{R}^2$ has a unique fixed point O (see [CK94] and [vK19]). Since this fixed point is unique, both $C(\alpha_g^2)$ and $C(\alpha_g^3)$ fix O. By Theorem 1.5, $\mathrm{Mod}(\Sigma_{g,1})$ fixes O, showing Theorem 1.2. ☐

We now explain how Theorem 1.2 implies Theorem 1.1.

Proof of Theorem 1.1. Let $\mathrm{Homeo}_+(\mathbb{D}^2)^{\pi_1(\Sigma_g)}$ denote the group of $\pi_1(\Sigma_g)$-equivariant orientation-preserving homeomorphisms of the universal cover $U : \mathbb{D}^2 \to \Sigma_g$. Choose a basepoint $x \in \mathbb{D}^2$. Then there is a map $p_g' : \mathrm{Homeo}_+(\mathbb{D}^2)^{\pi_1(\Sigma_g)} \to \mathrm{Mod}(\Sigma_{g,1})$: Given $f \in \mathrm{Homeo}_+(\mathbb{D}^2)^{\pi_1(\Sigma_g)}$, let γ be a path in $\mathbb{D}^2$ connecting $f(x)$ to x, and let f' be the composition of f with a $\pi_1(\Sigma_g)$-equivariant map that pushes the $\pi_1(\Sigma_g)$- orbit of $f(x)$ back to the orbit of x along the orbit of γ. This equivariant pushing P_γ can be obtained by first defining the point-pushing map on

Σ_g from $U(x)$ to $U(f(x))$ along the path $U(\gamma)$, then lifting to $\mathbb{D}^2$. Since $\mathbb{D}^2$ is contractible, two different choices of path γ_1, γ_2 will give isotopic push-maps P_{γ_1} and P_{γ_2}. Therefore $p'_g(f) = f'$ as the induced mapping class on $\Sigma_{g,1}$ is well-defined.

The map p'_g is compatible with the Birman exact sequence for the mapping class group, realizing $\mathrm{Homeo}_+(\mathbb{D}^2)^{\pi_1(\Sigma_g)}$ as the pullback of $\mathrm{Homeo}_+(\Sigma_g)$ and $\mathrm{Mod}(\Sigma_{g,1})$ along $\mathrm{Mod}(\Sigma_g)$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\Sigma_g) & \longrightarrow & \mathrm{Homeo}_+(\mathbb{D}^2)^{\pi_1(\Sigma_g)} & \longrightarrow & \mathrm{Homeo}_+(\Sigma_g) & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle p'_g} & & \downarrow{\scriptstyle p_g} & & \\ 1 & \longrightarrow & \pi_1(\Sigma_g) & \longrightarrow & \mathrm{Mod}(\Sigma_{g,1}) & \longrightarrow & \mathrm{Mod}(\Sigma_g) & \longrightarrow & 1 \end{array}$$

By the universal property of pullbacks, a section of p_g gives rise to a section of p'_g; any such section p'_g must realize $\pi_1(\Sigma_g)$ as the group of deck transformations of Σ_g.

By Theorem 1.2, the action of $\mathrm{Mod}(\Sigma_{g,1})$ on $\mathbb{D}^2 \cong \mathbb{R}^2$ via a section of p'_g has a global fixed point, which contradicts the fact that deck transformations act freely. □

6. Mapping class group actions on $\mathbb{R}^3$

In this final section, we show how Theorem 1.5 implies Theorem 1.3 and Corollary 1.4. The objective is to show that for $g \geq 4$, any action of $\mathrm{Mod}(\Sigma_{g,1})$ on $\mathbb{R}^3$ has a globally-invariant line; from this we will deduce that $\mathrm{Mod}(\Sigma_{g,1})$ does not act by C^1 diffeomorphisms on $\mathbb{R}^2$ or $\mathbb{R}^3$.

We consider an action ρ of $\mathrm{Mod}(\Sigma_{g,1})$ on $\mathbb{R}^3$. Recall that any such action must necessarily preserve orientation. We will appeal to the following result of Lanier–Margalit [LM18, Theorem 1.1].

Theorem 6.1 (Lanier–Margalit)**.** *For $g \geq 3$, every nontrivial periodic mapping class that is not hyperelliptic normally generates* $\mathrm{Mod}(\Sigma_{g,1})$ *or* $\mathrm{Mod}(\Sigma_g)$*.*

We remark that [LM18, Theorem 1.1] only discusses the case $\mathrm{Mod}(\Sigma_g)$, however the same method applies to $\mathrm{Mod}(\Sigma_{g,1})$.

By Theorem 6.1, α_g^2 normally generates $\mathrm{Mod}(\Sigma_{g,1})$. Therefore if ρ is not trivial, then $\rho(\alpha_g^2)$ is not trivial. By local Smith theory ([Bre97, Theorem 20.1]), the fixed point set of α_g^2 is a $\mathbb{Z}/3\mathbb{Z}$–homology manifold of dimension less than 2. By [Bre97, Theorem 16.32], when the dimension of a homology manifold is less than 2, it is a topological manifold. We claim that the fixed set $F(\alpha_g^2)$ of $\rho(\alpha_g^2)$ is a single topological line in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is acyclic, also $F(\alpha_g^2)$ is also acyclic (c.f. [Bre97, Corollary 19.8]). Hence $F(\alpha_g^2)$ must have exactly one component. By [Bre97, Corollary 19.11], $F(\alpha_g^2)$ is a line, since we can consider the action on the one-point compactification of $\mathbb{R}^3$. This can be compared with the fact that the fixed set of a torsion element in $SO(3)$ is a single line in $\mathbb{R}^3$.

Since α_g^3 is not hyperelliptic, the same argument shows that the fixed set of $\rho(\alpha^3)$ is also a line $F(\alpha_g^3)$. We claim that $F(\alpha_g^2) = F(\alpha_g^3)$, which will be denoted by F. If these lines are distinct, then the action of $\rho(\alpha_g)$ on $F(\alpha_g^3)$ must be nontrivial (otherwise $\rho(\alpha_g^2)$ would act trivially as well, implying $F(\alpha_g^2) = F(\alpha_g^3)$). As $\rho(\alpha_g^3)$ acts trivially on $F(\alpha_g^3)$ by construction, it follows that $\rho(\alpha_g)$ acts as an element of order 3. This is a contradiction: there is no nontrivial action of $\mathbb{Z}/3\mathbb{Z}$ on a line. Thus by Theorem 1.5, $\mathrm{Mod}(\Sigma_{g,1})$ must preserve F, establishing Theorem 1.3.

Now suppose ρ acts by C^1 diffeomorphisms, and let $x \in F$ be any fixed point. Taking derivatives at x, we obtain a representation $R : \mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{GL}(3, \mathbb{R})$. According to [FH13, Theorem 1.1], any such homomorphism is trivial. The Thurston stability theorem [Thu74] then implies that the image of ρ must be locally-indicable, i.e. every finitely-generated subgroup admits a surjection onto $\mathbb{Z}$. In particular, $\mathrm{im}(\rho)$ must be torsion-free, and so $\rho(\alpha_g)$ is the identity map. By Theorem 6.1, it follows that the entire representation ρ is trivial. □

Remark. In fact, the conclusions of Theorem 1.3 and Corollary 1.4 hold for $g = 3$ as well, using slightly different arguments. We briefly discuss this. From the discussion above, if $\rho(\alpha_g^3)$ is not trivial, the same arguments apply. Otherwise, denote by $H : \mathrm{Mod}(S_{g,1}) \to \mathrm{Sp}(2g, \mathbb{Z})$ the induced action on $H_1(S_g; \mathbb{Z})$. If α_g^3 is hyperelliptic and $\rho(\alpha_g^3)$ is the identity, we claim that ρ factors through H. This is because the hyperelliptic involution α_g^3 normally generates the group $H^{-1}(\pm I)$ by [LM18, Proposition 3.3], whose proof also works for the punctured case.

To conclude, we claim that there is no action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{R}^3$, even by homeomorphisms. According to [Zim18, Corollary 1], $(\mathbb{Z}/3\mathbb{Z})^3$ is not a subgroup of $\mathrm{Homeo}(\mathbb{R}^3)$. The claim then follows from [CL20, Lemma 10].

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LC: Department of Mathematics, University of Maryland, College Park, MD 20742

NS: Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
Email address: chenlei@umd.edu
Email address: nsalter@nd.edu