ON THE MONODROMY GROUP OF THE FAMILY OF SMOOTH QUINTIC PLANE CURVES

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ABSTRACT. We consider the space $\mathcal{P}_d$ of smooth complex projective plane curves of degree d. There is the tautological family of plane curves defined over $\mathcal{P}_d$, which has an associated monodromy representation $\rho_d : \pi_1(\mathcal{P}_d) \to \mathrm{Mod}(\Sigma_g)$ into the mapping class group of the fiber. For $d \leq 4$, classical algebraic geometry implies the surjectivity of ρ_d. For $d \geq 5$, the existence of a $(d-3)^{rd}$ root of the canonical bundle implies that ρ_d cannot be surjective. The main result of this paper is that for $d = 5$, the image of ρ_5 is as large as possible, subject to this constraint. This requires combining the algebro-geometric work of Lönne [Lön09] with Johnson's theory [Joh83] of the Torelli subgroup of $\mathrm{Mod}(\Sigma_g)$.

1. INTRODUCTION

Let[1] $\mathcal{P}_d$ denote the moduli space of smooth degree d plane curves. The tautological family of plane curves over $\mathcal{P}_d$ determines a monodromy representation

$$\rho_d : \pi_1(\mathcal{P}_d) \to \mathrm{Mod}(\Sigma_g),$$

where $g = \binom{d-1}{2}$ and $\mathrm{Mod}(\Sigma_g)$ is the mapping class group of the surface Σ_g of genus g. This note concerns the problem of computing the image of ρ_d.

The first step towards determining the image of ρ_d was carried out by A. Beauville in [Bea86], building off of earlier work of W. Janssen [Jan83] and S. Chmutov [Chm82]. Let $\Psi : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ denote the symplectic representation of $\mathrm{Mod}(\Sigma_g)$ on $H_1(\Sigma_g; \mathbb{Z})$. Beauville determined $\Psi \circ \rho_d$. He shows that for d even it is a surjection, while for d odd it is the (finite-index) stabilizer of a certain spin structure. Naively, it is therefore possible that ρ_d could surject onto $\mathrm{Mod}(\Sigma_g)$ or onto a spin mapping class group, depending on the parity of d.

It is a folklore result that in general, this does not happen. There is an invariant called an *r-spin structure* that provides an obstruction for $f \in \mathrm{Mod}(\Sigma_g)$ to be contained in $\mathrm{Im}(\rho_d)$. This r-spin structure is constructed in a natural way from a $(d-3)^{rd}$ root of the canonical bundle of a plane curve; see Section 2.2 for further discussion. It follows from work of Sipe [Sip82] that for $d \geq 6$, this obstruction is not detectable on the level of homology, i.e. that Beauville's upper bound is not

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[1]This paper originally appeared in 2016 and languished as a preprint for several years. Accordingly, the literature on this topic has advanced considerably. In the interest of not being ahistorical, we leave the discussion of the introduction in its original form. See the "Update" section in the introduction for the current state of the questions discussed here.

sharp. We formulate this as a theorem, in the interest of giving a self-contained proof of a result which does not seem to be easily accessible in the literature.

Theorem 1.1 (Folklore, Sipe [Sip82])**.** *For all $d \geq 4$, there is a finite-index subgroup $\mathrm{Mod}(\Sigma_g)[\phi_d] \leq \mathrm{Mod}(\Sigma_g)$ for which*

$$\mathrm{Im}(\rho_d) \subseteq \mathrm{Mod}(\Sigma_g)[\phi_d].$$

For $d \geq 6$, the containment

$$\mathrm{Mod}(\Sigma_g)[\phi_d] \subsetneq \Psi^{-1}(\Psi(\mathrm{Mod}(\Sigma_g)[\phi_d]))$$

is strict. *Consequently, for $d \geq 6$, the same is true for $\mathrm{Im}(\rho_d)$:*

$$\mathrm{Im}(\rho_d) \subsetneq \Psi^{-1}(\Psi(\mathrm{Im}(\rho_d))).$$

In the statement of Theorem 1.1, $\phi_d \in H^1(T^{*,1}\Sigma_g; \mathbb{Z}/(d-3)\mathbb{Z})$, where $T^{*,1}\Sigma_g$ denotes the unit cotangent bundle of Σ_g, and $\mathrm{Mod}(\Sigma_g)[\phi_d]$ denotes the stabilizer of ϕ_d in the natural action of $\mathrm{Mod}(\Sigma_g)$ on $H^1(\Sigma_g; \mathbb{Z}/(d-3)\mathbb{Z})$. The class ϕ_d is an instance of an r-spin structure for $r = d - 3$.

Our proof of Theorem 1.1 will be obtained by giving a construction of ϕ_d that makes the invariance of ϕ_d under $\mathrm{Im}(\rho_d)$ transparent. Using a topological interpretation of r-spin structures based on the work of S. Humphries - D. Johnson [HJ89], it will then be possible to see how the invariance of ϕ_d provides a strictly stronger constraint on $\mathrm{Im}(\rho_d)$ than that of Beauville.

The main original result of this paper concerns the problem of determining *sufficient* conditions for an element $f \in \mathrm{Mod}(\Sigma_g)$ to be contained in $\mathrm{Im}(\rho_d)$. For degrees $d = 3, 4$ (where $g = 1, 3$), it is known that ρ_d is a surjection. This is ultimately a consequence of the fact that generic curves of genus $g = 1, 3$ admit planar embeddings, unique up to an automorphism of $\mathbb{C}P^2$ (the only exceptions being hyperelliptic curves in genus 3). The case $d = 3$ also follows immediately from Beauville's computation, in light of the fact that Ψ is an isomorphism $\Psi : \mathrm{Mod}(\Sigma_1) \to SL_2(\mathbb{Z})$ for $g = 1$. This case is also closely related to the work of I. Dolgachev - A. Libgober [DL81]. The case $d = 4$ was treated by Y. Kuno [Kun08]. Theorem A thus treats the first case where planarity is an exceptional property for a curve to possess, and shows that despite this, the monodromy of the family of plane curves of degree 5 is still as large as possible.

Theorem A. *Let $d = 5$. There is a (classical) spin structure of odd parity $\phi_5 \in H^1(T^{*,1}\Sigma_6; \mathbb{Z}/2\mathbb{Z})$ with associated stabilizer $\mathrm{Mod}(\Sigma_6)[\phi_5] \leq \mathrm{Mod}(\Sigma_6)$ for which there is an equality*

$$\mathrm{Im}(\rho_5) = \mathrm{Mod}(\Sigma_6)[\phi_5].$$

The methods of the paper are special to the case of $d = 5$ only in that a finite generating set for $\mathrm{Mod}(\Sigma_g)[\phi_d] \cap \ker(\Psi)$ is known only for $d \leq 5$; we are able to produce large collections of Dehn twists in $\mathrm{Im}(\rho_d)$ for all d. On the basis of this, we offer the following conjecture.

Conjecture 1.2 (Monodromy of the universal plane curve)**.** *For all $d \geq 4$, there is an equality*

$$\mathrm{Im}(\rho_d) = \mathrm{Mod}(\Sigma_g)[\phi_d].$$

A version of Conjecture 1.2 was also formulated by R. Crétois - L. Lang [CL18]. The paper [CL18] is closely related to this one, treating the problem of monodromies of linear systems on toric surfaces.

Theorem A is obtained by a combination of techniques both from algebraic geometry and from the theory of the mapping class group. The starting point is Beauville's work, which allows one to restrict attention to $\mathrm{Im}(\rho_5) \cap \mathcal{I}_6$, where $\mathcal{I}_6$ is the *Torelli group* (see Section 4.3 for the definition of the Torelli group).

The bridge between algebraic geometry and mapping class groups arises from the work of M. Lönne [Lön09]. The main theorem of [Lön09] gives an explicit presentation for the fundamental group of the space $\mathcal{P}_{n,d}$ of smooth hypersurfaces in $\mathbb{C}P^n$ of degree d. Picard-Lefschetz theory allows one to recognize Lönne's generators as Dehn twists. Theorem A is then proved by carrying out a careful examination of the configuration of vanishing cycles as simple closed curves on a surface of genus 6. This analysis is used to exhibit the elements of Johnson's generating set for the Torelli group inside $\mathrm{Im}(\rho_5)$.

In genus 6, Johnson's generating set has 4470 elements. In order to make this computation tractable, we find a new relation in $\mathrm{Mod}(\Sigma_g)$ known as the "genus-g star relation". Using this, we reduce the problem to eight easily-verified cases. An implicit corollary of the proof is a determination of a simple finite set of Dehn twist generators for the spin mapping class group $\mathrm{Mod}(\Sigma_6)[\phi_5]$. An alternative set of generators was obtained by S. Hirose [Hir05, Theorem 6.1].

Update. As discussed in Footnote 1, this paper first appeared in 2016, and much has advanced in the intervening years. We provide here an up-to-date account of the state of affairs. Conjecture 1.2 has now been completely resolved, indeed in the broader setting of linear systems on smooth toric surfaces. The case when the r-spin structure has $r = 2$ was treated by Crétois–Lang in [CL19]. The general case was resolved by the author in [Sal19] (only up to finite index, in the case of r even), and subsequently the full conjecture was obtained by the author and Calderon in [CS21]. As first observed by Calderon [Cal20], r-spin mapping class groups (and the refinements to "framed mapping class groups") appear in a diverse array of other monodromy problems, including strata of Abelian differentials [Cal20, CS21, CS23] and versal deformations of isolated plane curve singularities [PCS21].

In [Har21], R. Harris investigates the *kernel* of the monodromy map, finding that for plane quartics, the kernel is isomorphic to the product of a free group of infinite rank with $\mathbb{Z}/3\mathbb{Z}$. And in [Ban24], Banerjee finds that the fundamental group of the ambient algebraic surface provides a further constraint on the monodromy of a linear system. This constraint is quite strong, forcing the monodromy to be of infinite index. It would be interesting to investigate Conjecture 1.2 for any linear system on a simply-connected smooth algebraic surface.

Outline. Section 2 is devoted to the construction of ϕ_d. In Section 3, we recall some work of S. Humphries and D. Johnson that relates $H^1(T^{*,1}\Sigma_g; V)$ for an abelian group V to the notion of a

"generalized winding number function". We will use this perspective to prove that the invariance of ϕ_d under $\operatorname{Im}(\rho_d)$ provides an obstruction to the surjectivity of ρ_d.

The proof of Theorem A is carried out in sections 4 through 7. Section 4 collects a number of results from the theory of mapping class groups. Section 5 recalls Lönne's presentation and establishes some first properties of $\operatorname{Im}(\rho_d)$. Section 6 continues the analysis of $\operatorname{Im}(\rho_d)$. Finally Section 7 collects these results together to prove Theorem A.

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2. r^{th} ROOTS OF THE CANONICAL BUNDLE AND GENERALIZED SPIN STRUCTURES

2.1. **Plane curves and $\mathcal{P}_d$.** A general reference for this paragraph is [DL81, Section 2].

By definition, a (projective) *plane curve* of degree d is the vanishing locus $V(f)$ in $\mathbb{C}P^2$ of a nonzero homogeneous polynomial $f(x, y, z)$ of degree d. The space of all plane curves is identified with $\mathbb{C}P^N$, where $N = \binom{d+2}{2} - 1$. A plane curve X of degree d is *smooth* if $X \cong \Sigma_g$ with $g = \binom{d-1}{2}$, and otherwise X is said to be *singular*.

We define the *discriminant* as the set

$$\mathcal{D}_d = \{f \in \mathbb{C}P^N \mid V(f) \text{ is singular.}\}.$$

The discriminant $\mathcal{D}_d$ is the vanishing locus of a polynomial p_d known as the *discriminant polynomial*, and is therefore a hypersurface in $\mathbb{C}P^N$. The space of smooth plane curves is then defined as

$$\mathcal{P}_d = \mathbb{C}P^N \setminus \mathcal{D}_d.$$

The *universal family of plane curves* is the space $\mathfrak{X}_d \subset \mathcal{P}_d \times \mathbb{C}\mathbb{P}^2$ defined via

$$\mathfrak{X}_d = \{(f, [x : y : z]) \in \mathcal{P}_d \times \mathbb{C}\mathbb{P}^2 \mid f(x, y, z) = 0\}.$$

The projection $\pi : \mathfrak{X}_d \to \mathcal{P}_d$ is the projection map for a C^∞ fiber bundle structure on $\mathfrak{X}_d$ with fibers diffeomorphic to Σ_g.

2.2. **r-spin structures.** Let X be a smooth projective algebraic curve over $\mathbb{C}$ and let $K \in \operatorname{Pic}(X)$ denote the canonical bundle.[2] Recall that a *spin structure* on X is an element $L \in \operatorname{Pic}(X)$ satisfying $L^{\otimes 2} = K$. This admits an obvious generalization.

Definition 2.1. An r-spin structure is a line bundle $L \in \operatorname{Pic}(X)$ satisfying $L^{\otimes r} = K$.

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[2]Recall that the *canonical bundle* is the line bundle whose underlying $\mathbb{R}^2$ bundle is T^*X, the cotangent bundle.

Let $T^{*,1}X$ denote the unit cotangent bundle of X, relative to an arbitrary Riemannian metric on X. Just as ordinary spin structures are closely related to $H^1(T^{*,1}X; \mathbb{Z}/2\mathbb{Z})$, there is an analogous picture of r-spin structures.

Proposition 2.2. *Let L be an r-spin structure on X. Associated to L are*

    (1) a regular r-sheeted covering space $\widetilde{T^{*,1}X} \to T^{*,1}X$ with deck group $\mathbb{Z}/n\mathbb{Z}$, and

    (2) a cohomology class $\phi_L \in H^1(T^{*,1}X; \mathbb{Z}/r\mathbb{Z})$ restricting to a generator of the cohomology $H^1(S^1; \mathbb{Z}/r\mathbb{Z})$ of the fiber of $T^{*,1}X \to X$.

Proof. In view of the equality $L^{\otimes r} = K$ in $\mathrm{Pic}(X)$, taking r^{th} powers in the fiber induces a map $\mu : L \to K$. Let L° denote the complement of the zero section in L, and define K° similarly. Then $\mu : L^\circ \to K^\circ$ is an r-sheeted covering space with deck group $\mathbb{Z}/r\mathbb{Z}$ induced from the multiplicative action of the rth roots of unity. The covering space $\widetilde{T^{*,1}X} \to T^{*,1}X$ is obtained from $L^\circ \to K^\circ$ by restriction.

As $\widetilde{T^{*,1}X} \to T^{*,1}X$ is a regular cover with deck group $\mathbb{Z}/r\mathbb{Z}$, the Galois correspondence for covering spaces asserts that $\widetilde{T^{*,1}X}$ is associated to some homomorphism $\phi_L : \pi_1(T^{*,1}X) \to \mathbb{Z}/r\mathbb{Z}$. This gives rise to a class, also denoted ϕ_L, in $H^1(T^{1,*}X; \mathbb{Z}/r\mathbb{Z})$. On a given fiber of $T^{*,1}X \to X$, the covering $\widetilde{T^{*,1}X} \to T^{*,1}X$ restricts to an r-sheeted cover $S^1 \to S^1$; this proves the assertion concerning the restriction of ϕ_L to $H^1(S^1; \mathbb{Z}/r\mathbb{Z})$. $\qquad\square$

Our interest in r-spin structures arises from the fact that degree-d plane curves are equipped with a canonical $(d-3)$-spin structure.

Fact 2.3. *Let X be a smooth degree-d plane curve, $d \geq 3$. The canonical bundle $K \in \mathrm{Pic}(X)$ is induced from the restriction of $\mathcal{O}(d-3) \in \mathrm{Pic}(\mathbb{CP}^2)$. Consequently, $\mathcal{O}(1)$ determines a $(d-3)$-spin structure on X for $d \geq 4$.*

Let $\varpi : \mathfrak{X}_d \to \mathbb{CP}^2$ denote the projection onto the second factor. Then $\varpi^*(\mathcal{O}(d-3)) \in \mathrm{Pic}(\mathfrak{X}_d)$ restricts to the canonical bundle on each fiber, and $\varpi^*(\mathcal{O}(1))$ determines a $(d-3)$-spin structure. Let $T^{*,1}\mathfrak{X}_d$ denote the S^1-bundle over $\mathfrak{X}_d$ for which the fiber over (X, x) consists of the unit cotangent vectors $T_x^{*,1}X$.

Definition 2.4. The cohomology class

$$\phi_d \in H^1(T^{*,1}\mathfrak{X}_d; \mathbb{Z}/(d-3)\mathbb{Z})$$

is obtained by applying the construction of Proposition 2.2 to the pair of line bundles $\varpi^*(\mathcal{O}(1))$, $\varpi^*(\mathcal{O}(d-3)) \in \mathrm{Pic}(\mathfrak{X}_d)$.

3. Generalized winding numbers and obstructions to surjectivity

In this section, we prove that the existence of ϕ_d gives rise to an obstruction for a mapping class $f \in \mathrm{Mod}(\Sigma_g)$ to be contained in $\mathrm{Im}(\rho_d)$. For any system of coefficients V, there is a natural action of $\mathrm{Mod}(\Sigma_g)$ on $H^1(T^{*,1}\Sigma_g; V)$ which extends the action of $\mathrm{Mod}(\Sigma_g)$ on $H^1(\Sigma_g; V)$ via Ψ. To prove

Theorem 1.1, it therefore suffices to prove that the stabilizer $\mathrm{Mod}(\Sigma_g)[\phi_d]$ of each nonzero element of $H^1(T^{*,1}\Sigma_g; \mathbb{Z}/(d-3)\mathbb{Z})$ is not the full group $\Psi^{-1}(\Psi(\mathrm{Im}(\rho_d)))$.

The natural setting for what follows is in the unit *tangent* bundle of a surface, which we write $T^1\Sigma$. Of course, a choice of Riemannian metric on Σ identifies $T^1\Sigma$ with $T^{*,1}\Sigma$, and a choice of metric in each fiber identifies $T^{*,1}\mathfrak{X}_d$ with the "vertical unit tangent bundle" $T^1\mathfrak{X}_d$; we will make no further comment on these matters.

The basis for our approach is the work of Humphries-Johnson [HJ89], which gives an interpretation of $H^1(T^1\Sigma_g; V)$ as the space of "V-valued generalized winding number functions". A basic notion here is that of a *Johnson lift*. For our purposes, a *simple closed curve* is a C^1-embedded S^1-submanifold.

Definition 3.1. Let a be a simple closed curve on the surface Σ given by a unit-speed C^1 embedding $a : S^1 \to \Sigma$. A choice of orientation on S^1 induces an orientation on a, as well as providing a coherent identification $T^1_x S^1 = \{-1, 1\}$ for each $x \in S^1$. The *Johnson lift* of a, written $\vec{a}$, is the map $\vec{a} : S^1 \to T^1\Sigma$ given by

$$\vec{a}(t) = (a(t), D_t a(1)).$$

That is, the Johnson lift of a is simply the curve in $T^1\Sigma$ induced from a by tracking the tangent vector.

The Johnson lift allows for the evaluation of elements of $H^1(T^1\Sigma; V)$ on simple closed curves. Let Σ be a surface, V an abelian group, and $\alpha \in H^1(T^1\Sigma; V)$ a cohomology class. Let a be a simple closed curve. By an abuse of notation, we write $\alpha(a)$ for the evaluation of α on the 1-cycle determined by the Johnson lift $\vec{a}$. In this context we call α a "generalized winding number function".[3] In [HJ89], it is shown that this pairing satisfies the following properties:

Theorem 3.2 (Humphries-Johnson)**.**

(i) *The evaluation $\alpha(a) \in V$ is well-defined on the isotopy class of a.*

(ii) *(Twist-linearity) If b is another simple closed curve and T_b denotes the Dehn twist about b, then α is "twist-linear" in the following sense:*

$$\alpha(T_b(a)) = \alpha(a) + \langle a, b \rangle \alpha(b), \tag{1}$$

where $\langle a, b \rangle$ denotes the algebraic intersection pairing.

(iii) *Let ζ be a curve enclosing a small null-homotopic disk on Σ, and let $S \subset \Sigma$ be a subsurface with boundary components $b_1, \ldots, b_k$. If each b_i is oriented so that S is on the left and ζ is oriented similarly, then*

$$\alpha(b_1) + \cdots + \alpha(b_k) = \chi(S)\alpha(\zeta), \tag{2}$$

where $\chi(S)$ is the Euler characteristic of S.

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[3]The terminology "generalized winding number" is inspired by the fact that the twist-linearity property was first encountered in the context of computing winding numbers of curves on surfaces relative to a vector field.

Remark 3.3. Humprhies-Johnson in fact establish much more: they prove that *every* V-valued twist-linear function arises as a class $\alpha \in H^1(T^1\Sigma; V)$. For what follows we only need the results of Theorem 3.2.

Proof of Theorem 1.1. Consider the class $\phi_d \in H^1(T^{*,1}\mathfrak{X}_d; \mathbb{Z}/(d-3)\mathbb{Z})$. The above discussion implies that on a given fiber X of $\mathfrak{X}_d \to \mathcal{P}_d$, the restriction of ϕ_d determines a generalized winding number function; we write $\alpha_d \in H^1(T^1X; \mathbb{Z}/(d-3)\mathbb{Z})$ for this class. Since α_d is induced from the globally-defined form ϕ_d, it follows that α_d is monodromy-invariant: if $f \in \mathrm{Im}(\rho_d)$, then for any simple closed curve a on X, the equation

$$\alpha_d(f(a)) = \alpha_d(a) \tag{3}$$

must hold. Consequently,

$$\mathrm{Im}(\rho_d) \subseteq \mathrm{Mod}(\Sigma_g)[\phi_d]$$

as claimed.

We wish to exhibit a nonseparating simple closed curve b for which $\alpha_d(b) \neq 0$. Given such a b, there is another simple closed curve a satisfying $\langle a, b \rangle = 1$. Then the twist-linearity condition (1) will show that

$$\alpha_d(T_b(a)) = \alpha_d(a) + \alpha_d(b) \neq \alpha_d(a);$$

this contradicts (3). It follows that the Dehn twist T_b for such a curve cannot be contained in $\mathrm{Mod}(\Sigma_g)[\phi_d]$.

In the case when d is even, when $\Psi^{-1}(\Psi(\mathrm{Im}(\rho_d))) = \mathrm{Mod}(\Sigma_g)$, this will prove Theorem 1.1. For d odd, there is an additional complication. Here, the class $\frac{d-3}{2}\phi_d \in H^1(T^{*,1}\mathfrak{X}_d; \mathbb{Z}/2\mathbb{Z})$ determines an ordinary spin structure, and according to Beauville, the group $\Psi(\mathrm{Mod}(\Sigma_g)[\phi_d])$ is the stabilizer of $\frac{d-3}{2}\phi_d$ in $\mathrm{Sp}(2g, \mathbb{Z})$. We must therefore exhibit a curve b for which $\alpha_d(b)$ is nonzero and $\frac{d-3}{2}$-torsion. Equation (1) shows that such a curve *does* stabilize the spin structure $\frac{d-3}{2}\phi_d$, but not the refinement to a $(d-3)$-spin structure ϕ_d.

It remains to exhibit a suitable curve b. It follows easily from the twist-linearity condition (1) that given any subsurface $S \subset X$ of genus 1 with one boundary component, there is some (necessarily nonseparating) curve c contained in S with $\alpha_d(c) = 0$. Let S_1, S_2, S_3 be a collection of mutually-disjoint subsurfaces of genus 1 with one boundary component, and let c_1, c_2, c_3 be curves satisfying $\alpha_d(c_i) = 0$, and for which c_i is contained in S_i (recall that $d \geq 6$ and so the genus of X is $g \geq 10$). Choose b disjoint from all c_i so that the collection of curves b, c_1, c_2, c_3 encloses a subsurface Σ homeomorphic to a sphere with 4 boundary components. From (2) and the construction of the c_i, it follows that when b is suitably oriented, it satisfies

$$\alpha_d(b) = \chi(\Sigma)\alpha_d(\zeta) = -2\alpha_d(\zeta).$$

Recall that by Proposition 2.2.2, the element $\alpha_d(\zeta) \in \mathbb{Z}/(d-3)\mathbb{Z}$ is primitive. Thus $\alpha_d(b) \neq 0$ for any d, but is $\frac{d-3}{2}$-torsion when d is odd, as required. $\qquad\square$

4. Results from the theory of the mapping class group

We turn now to the proof of Theorem A. From this section onwards, we adopt the conventions and notations of the reference [FM12]. In particular, the *left-handed* Dehn twist about a curve c is written T_c, and the geometric intersection number between curves a, b is written $i(a, b)$. We pause briefly to establish some further conventions. We will often refer to a simple closed curve as simply a "curve", and will often confuse the distinction between a curve and its isotopy class. Unless otherwise specified, we will assume that all intersections between curves are essential.

4.1. **The change-of-coordinates principle.** The *change-of-coordinates principle* roughly asserts that if two configurations of simple closed curves and arcs on a surface have the same intersection pattern, then there is a homeomorphism taking one configuration to the other. There are many variants of the change-of-coordinates principle, all based on the classification of surfaces. See the discussion in [FM12, Section 1.3.2].

Basic principle. Suppose $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ are configurations of curves on a surface S all meeting transversely. The surface $\overline{S \setminus \{c_i\}}$ has a labeling on segments of its boundary, corresponding to the segments of the curves c_i from which the boundary component arises. Suppose there is a homeomorphism

$$f : \overline{S \setminus \{c_i\}} \to \overline{S \setminus \{d_i\}}$$

taking every boundary segment labeled by c_i to the corresponding d_i segment. Then f can be extended to a homeomorphsim $f : S \to S$ taking the configuration c_i to d_i.

We illustrate this in the case of *chains*.

Definition 4.1. Let S be a surface. A *chain* on S of length k is a collection of curves $(c_1, \ldots, c_k)$ for which the geometric intersection number $i(c_i, c_j)$ is 1 if $|i - j| = 1$ and 0 otherwise. If $C = (c_1, \ldots, c_k)$ is a chain, the *boundary* of C, written ∂C, is defined to be the boundary of a small regular neighborhood of $c_1 \cup \cdots \cup c_k$. When k is even, ∂C is a single (necessarily separating) curve, and when k is odd, $\partial C = d_1 \cup d_2$ consists of two curves d_1, d_2 whose union separates S.

Lemma 4.2 (Change-of-coordinates for chains). *Let $(c_1, \ldots, c_k)$ and $(d_1, \ldots, d_k)$ be chains of even length k on a surface S. Then there is a homeomorphism $f : S \to S$ for which $f(c_i) = d_i, 1 \le i \le k$.*

Proof. See [FM12, Section 1.3.2]. $\square$
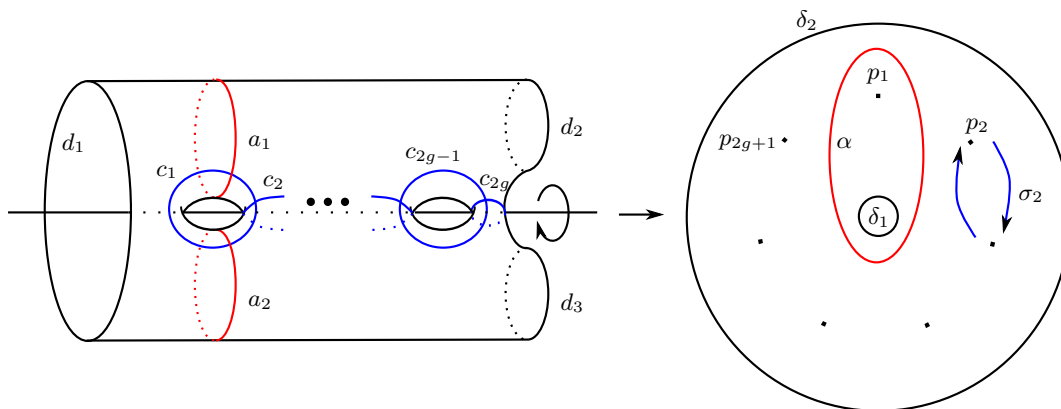
4.2. **Some relations in the mapping class group.**

Proposition 4.3 (Braid relation). *Let S be a surface, and a, b curves on S satisfying $i(a, b) = 1$. Then*

$$T_a T_b T_a = T_b T_a T_b. \tag{4}$$

On the level of curves,

$$T_a T_b(a) = b.$$

FIGURE 1. The genus-g star relation.

Any such a, b are necessarily non-separating.

Conversely, if a, b are curves on S in distinct isotopy classes that satisfy the braid relation (4), then $i(a, b) = 1$.

Proof. See [FM12, Proposition 3.11] for the proof of the first assertion, and [FM12, Proposition 3.13] for the second.   □

The chain relation. The *chain relation* relates Dehn twists about curves in a chain to Dehn twists around the boundary. We will require a slightly less well-known form of the chain relation for chains of odd length; see [FM12, Section 4.4.1] for details.

Proposition 4.4 (Chain relation). *Let $C = (c_1, \ldots, c_k)$ be a chain with k odd. Let d_1, d_2 denote the components of ∂C. Then the following relation holds:*

$$(T_{c_1}^2 T_{c_2} \ldots T_{c_k})^k = T_{d_1} T_{d_2}.$$

The genus-g star relation. We will also need to make use of a novel[4] relation generalizing the star relation (setting $g = 1$ below recovers the classical star relation).

Proposition 4.5 (Genus-g star relation). *With reference to the curves $a_1, a_2, c_1, \ldots, c_{2g}, d_1, d_2, d_3$ on the surface $\Sigma_{g,3}$ of Figure 1, the following relation holds in $\mathrm{Mod}(\Sigma_{g,3})$:*

$$(T_{a_1} T_{a_2} T_{c_1} \ldots T_{c_{2g}})^{2g+1} = T_{d_1}^g T_{d_2} T_{d_3}. \tag{5}$$

Proof. We will derive the genus-g star relation from a more transparent relation in a braid group, making use of the theory of the *liftable mapping class group* as developed by Birman–Hilden [BH73]. Figure 1 depicts a $2 : 1$ covering $\Sigma_{g,3} \to \Sigma_{0,2}$ ramified at $2g + 1$ points. Number the ramification points clockwise $p_1, \ldots, p_{2g+1}$, and consider the mapping class group $\mathrm{Mod}(\Sigma_{0,2,2g+1})$

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[4]Added in 2024 revisions: In later work I realized this is actually a relation induced from a homomorphism of an Artin group of type D into the mapping class group, and that this was known to prior authors, e.g. [Mat00].

relative to these points. The theory of Birman–Hilden implies that there is a finite-index subgroup of $\mathrm{Mod}(\Sigma_{0,2,2g+1})$ (the "liftable subgroup") whose elements lift to mapping classes in $\mathrm{Mod}(\Sigma_{g,3})$. Such lifts are unique up to the action of the deck group, and so there is a *homomorphsim* from the liftable subgroup into $\mathrm{Mod}(\Sigma_{g,3})$. Consequently, any relation in the liftable subgroup induces a relation upstairs in $\mathrm{Mod}(\Sigma_{g,3})$; this is the underlying principle of our argument.

Under the covering, the double-twist $T_{\delta_1}^2$ lifts to $T_{d_1} \in \mathrm{Mod}(\Sigma_{g,3})$, and the twist T_{δ_2} lifts to $T_{d_2}T_{d_3}$. The twist T_α lifts to $T_{a_1}T_{a_2}$, and the half-twist σ_i lifts to T_{c_i}. Let $f \in \mathrm{Mod}(\Sigma_{0,2,2g+1})$ be the push map moving each p_i clockwise to p_{i+1}, with subscripts interpreted mod $2g+1$. One verifies the equality

$$f = T_\alpha \sigma_1 \ldots \sigma_{2g} T_{\delta_1}^{-1}.$$

It follows that

$$f^{2g+1} = (T_\alpha \sigma_1 \ldots \sigma_{2g})^{2g+1} T_{\delta_1}^{-(2g+1)},$$

since T_{δ_1} is central. As f^{2g+1} is the push map around the core of the annulus, there is an equality

$$f^{2g+1} = T_{\delta_1}^{-1} T_{\delta_2}.$$

Combining these results,

$$T_{\delta_2} T_{\delta_1}^{2g} = (T_\alpha \sigma_1 \ldots \sigma_{2g})^{2g+1}. \tag{6}$$

Under the lifting described above, the relation (6) in $\mathrm{Mod}(\Sigma_{0,2,2g+1})$ lifts to the relation (5) in $\mathrm{Mod}(\Sigma_{g,3})$. □

4.3. **The Johnson generating set for $\mathcal{I}_g$.** There is a natural map

$$\Psi : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$$

taking a mapping class f to the induced automorphism f_* of $H_1(\Sigma_g; \mathbb{Z})$. The *Torelli group* $\mathcal{I}_g$ is defined to be the kernel of this map:

$$\mathcal{I}_g = \ker(\Psi).$$

In [Joh83], Johnson produced a finite set of generators for $\mathcal{I}_g$, for all $g \geq 3$. Elements of this generating set are known as *chain maps*. Let $C = (c_1, \ldots, c_k)$ be a chain of *odd* length with boundary $\partial C = d_1 \cup d_2$. There are exactly two ways to orient the collection of curves $c_1, \ldots, c_k$ in such a way that the algebraic intersection number $c_i \cdot c_{i+1} = +1$. Relative to such a choice, the *chain map* associated to C is then the mapping class $T_{d_1} T_{d_2}^{-1}$, where d_1 is distinguished as the boundary component to the *left* of the curves $c_1, c_3, \ldots, c_k$. The mapping class $T_{d_1} T_{d-2}^{-1}$ is also called the *bounding pair map* for d_1, d_2.

While a complete description of Johnson's generating set is quite tidy and elegant, it has the disadvantage of requiring several preliminary notions before it can be stated. We therefore content ourselves with a distillation of his work that is more immediately applicable to our situation.

Theorem 4.6 (Johnson). *For $g \geq 3$, let $\Gamma \leq \mathrm{Mod}(\Sigma_g)$ be a subgroup that contains the Dehn twists about the curves $c_1, \ldots, c_{2g}$ shown in Figure 2. Suppose that Γ contains all chain maps for the odd-length chains of the form $(c_1, \ldots, c_k)$ and $(\beta, c_5, \ldots, c_k)$. Then $\mathcal{I}_g \leq \Gamma$.*
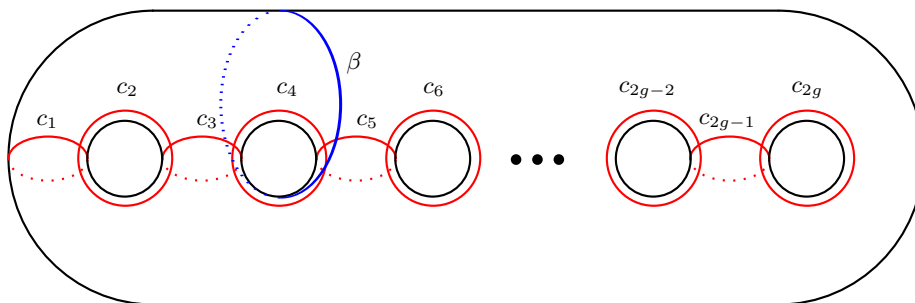
FIGURE 2. Curves involved in the Johnson generating set.

Proof. The interested reader should have no trouble deducing Theorem 4.6 from the Main Theorem and Lemma 1(f) of [Joh83].   □

5. THE LÖNNE PRESENTATION

In this section, we recall Lönne's work [Lön09] computing $\pi_1(\mathcal{P}_d)$, and apply this to derive some first properties of the monodromy map $\rho_d : \pi_1(\mathcal{P}_d) \to \mathrm{Mod}(\Sigma_g)$.

5.1. **Picard-Lefschetz theory.** Picard-Lefschetz theory concerns the problem of computing monodromies attached to singular points of holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$. This then serves as the local theory underpinning more global monodromy computations. Our reference is [AGZV12].

Let $U \subset \mathbb{C}^2$ and $V \subset \mathbb{C}$ be open sets for which $0 \in V$. Let $f(u,v) : U \to V$ be a holomorphic function. Suppose f has an isolated critical value at $z = 0$, and that there is a single corresponding critical point $p \in \mathbb{C}^2$. Suppose that p is *of Morse type* in the sense that the Hessian

$$\begin{pmatrix} \frac{\partial^2 f}{\partial^2 x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} \end{pmatrix}$$

is non-singular at p. Equivalently, f is analytically equivalent to $f(u,v) = uv$, the union of two lines crossing transversely.

In such a situation, the fiber $f^{-1}(z)$ for $z \neq 0$ is diffeomorphic to an open annulus. The core curve of such an annulus is called a *vanishing cycle*. Let γ be a small circle in $\mathbb{C}$ enclosing only the critical value at $z = 0$. Let $z_1 \in \gamma$ be a basepoint with corresponding core curve $c \subset f^{-1}(z_1)$. The Picard-Lefschetz theorem describes the monodromy obtained by traversing γ.

More generally, an algebraic curve C is said to have a *nodal singularity* at $p \in C$ if there is an analytic local equation for C near p of the form $f(u,v) = uv$. Perturbing C slightly to a smooth C', the intersection of C' with a small ball near p is again an annulus, and the core curve is again called a vanishing cycle.

Theorem 5.1 (Picard-Lefschetz for $n = 2$). *With reference to the preceding discussion, the monodromy $\mu \in \mathrm{Mod}(f^{-1}(z_1))$ attached to traversing γ counter-clockwise is given by a right-handed*

Dehn twist about the vanishing cycle:

$$\mu = T_c^{-1}.$$

More generally, let D^* denote the punctured unit disk

$$D^* = \{w \in \mathbb{C} \mid 0 < |w| \leq 1\},$$

and write $D = \{w \in \mathbb{C} \mid |w| \leq 1\}$ for the closed unit disk.

Let $f(x, y, z)$ be a homogeneous polynomial of degree d with the following properties:

(1) For $c \in D$, the plane curve $cz^d - f(x, y, z)$ is singular only for $c = 0$.
(2) The only critical point for f of the form $(x, y, 0)$ is the point $(0, 0, 0)$.
(3) The function $f(x, y, 1)$ has a single critical point of Morse type at $(x, y) = (0, 0)$.

In this setting, the local theory of Theorem 5.1 can be used to analyze the monodromy of the family

$$E = \{(c, [x : y : z]) \mid cz^d = f(x, y, z)\} \subset D^* \times \mathbb{C}P^2$$

around the boundary ∂D^*.

Theorem 5.2 (Picard-Lefschetz for plane curve families)**.** *Let $f \in \mathbb{C}P^N$ satisfy the properties (1), (2), (3) listed above. Let $X = V(z^d - f(x, y, z))$ denote the fiber above $1 \in D^*$. Then there is a vanishing cycle $c \subset X$ so that the monodromy $\mu \in \mathrm{Mod}(X)$ obtained by traversing ∂D^* counter-clockwise is given by a right-handed Dehn twist about the vanishing cycle:*

$$\mu = T_c^{-1}.$$

Proof. Condition (2) asserts that the monodromy can be computed by restricting attention to the affine subfamily obtained by setting $z = 1$. Define

$$E^\circ = \{(c, x, y) \mid c = f(x, y, 1)\} \subset D^* \times \mathbb{C}^2.$$

Define $U = \{(x, y) \in \mathbb{C}^2 \mid |f(x, y, 1)| \leq 1\}$, and consider $f(x, y, 1)$ as a holomorphic function $f : U \to D$. The monodromy of this family then corresponds to the monodromy of the original family $E \to D^*$. The result now follows from Condition (3) in combination with Theorem 5.1 as applied to $f(x, y, 1)$. $\square$

5.2. **Lönne's theorem.** There are some preliminary notions to establish before Lönne's theorem can be stated. We begin by introducing the *Lönne graphs* Γ_d. Lönne obtains his presentation of $\pi_1(\mathcal{P}_d)$ as a quotient a certain group constructed from Γ_d.

Definition 5.3. [Lönne graph] Let $d \geq 3$ be given. The *Lönne graph* Γ_d has vertex set

$$I_d = \{(a, b) \mid 1 \leq a, b \leq d - 1\}.$$

Vertices (a_1, b_1) and (a_2, b_2) are connected by an edge if and only if both of the following conditions are met:

(1) $|a_1 - a_2| \leq 1$ and $|b_1 - b_2| \leq 1$.
(2) $(a_1 - a_2)(b_1 - b_2) \leq 0$.

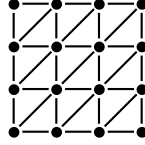The set of edges of Γ_d is denoted E_d.

FIGURE 3. The Lönne graph Γ_5.

Vertices $i, j, k \in \Gamma_d$ are said to form a *triangle* when i, j, k are mutually adjacent. The triangles in the Lönne graph are crucial to what follows. It will be necessary to endow them with orientations.

Definition 5.4 (Orientation of triangles). Let i, j, k determine a triangle in Γ_d.

(1) If
$$i = (a, b), \quad j = (a, b+1), \quad k = (a+1, b),$$
then the triangle i, j, k is positively-oriented by traversing the boundary clockwise.

(2) If
$$i = (a, b), \quad j = (a, b+1), \quad k = (a-1, b+1),$$
then the triangle i, j, k is positively-oriented by traversing the boundary counterclockwise.

We say that the ordered triple (i, j, k) of vertices determining a triangle is positively-oriented if traversing the boundary from i to j to k agrees with the orientation specified above.

Definition 5.5 (Artin group). Let Γ be a graph with vertex set V and edge set E. The *Artin group* $A(\Gamma)$ is defined to be the group with generators

$$\sigma_i, i \in V,$$

subject to the following relations:

(1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for all $(i, j) \notin E$.
(2) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for all $(i, j) \in E$.

Theorem 5.6 (Lönne). *For $d \geq 3$, the group $\pi_1(\mathcal{P}_d)$ is isomorphic to a quotient of the Artin group $A(\Gamma_d)$, subject to the following additional relations:*

(3) $\sigma_i \sigma_j \sigma_k \sigma_i = \sigma_j \sigma_k \sigma_i \sigma_j$ *if (i, j, k) forms a positively-oriented triangle in Γ_d.*
(4) *An additional family of relations $R_i, i \in I_d$.*
(5) *An additional relation $\widetilde{R}$.*

Remark 5.7. Define the group $B(\Gamma_d)$ as the quotient of the Artin group $A(\Gamma_d)$ by the family of relations (3) in Theorem 5.6. As our statement of Lönne's theorem indicates, the additional relations will be of no use to us, and our theorem really concerns the lift of the monodromy representation $\tilde{\rho}_d : B(\Gamma_d) \to \text{Mod}(\Sigma_g)$.

For the analysis to follow, it is essential to understand the mapping classes $\rho_d(\sigma_i), i \in I_d$.

Proposition 5.8. *For each generator σ_i of Theorem 5.6, the image*

$$\rho_d(\sigma_i) = T_{c_i}^{-1}$$

is a right-handed Dehn twist about some vanishing cycle c_i on a fiber $X \in \mathcal{P}_d$.

Proof. The result will follow from Theorem 5.2, once certain aspects of Lönne's proof are recalled. The generators σ_i of Theorem 5.6 correspond to specific loops in $\mathcal{P}_d$ known as *geometric elements*.

Definition 5.9 (Geometric element)**.** Let $D = V(p)$ be a hypersurface in $\mathbb{C}^n$ defined by some polynomial p. An element $x \in \pi_1(\mathbb{C}^n \setminus D)$ that can be represented by a path isotopic to the boundary of a small disk transversal to D is called a *geometric element*. If $\widetilde{D}$ is a projective hypersurface, an element $x \in \pi_1(\mathbb{C}P^n \setminus \widetilde{D})$ is said to be a geometric element if it can be represented by a geometric element in some affine chart.

In Lönne's terminology, the generators $\sigma_i, i \in I_d$ arise as a "Hefez-Lazzeri basis" - this will require some explanation. Consider the linearly-perturbed Fermat polynomial

$$f(x, y, z) = x^d + y^d + \nu_x x z^{d-1} + \nu_y y z^{d-1}$$

for well-chosen constants ν_x, ν_y. Such an f satisfies the conditions (1)-(3) of Theorem 5.2 near each critical point. Moreover, there is a bijection between the critical points of $f(x, y, 1)$ and the set I_d of Definition 5.3. If ν_x, ν_y are chosen carefully, each critical point lies above a distinct critical value - in this way we embed $I_d \subset \mathbb{C}$.

Each $c \in \mathbb{C}$ determines a plane curve $V(cz^d - f)$. The values of c for which $V(cz^d - f)$ is not smooth are exactly the critical values of $f(x, y, 1)$. The family

$$H = \{V(cz^d - f) \mid V(cz^d - f) \text{ is smooth}\}$$

is a subfamily of $\mathcal{P}_d$ defined over $\mathbb{C} \setminus I_d$. The Hefez-Lazzeri basis $\{\sigma_i \mid i \in I_d\}$ is a carefully-chosen set of paths in $\mathbb{C} \setminus I_d$ with each σ_i encircling an individual $i \in I_d$. Lönne shows that the inclusions of these paths into $\mathcal{P}_d$ via the family H generate $\pi_1(\mathcal{P}_d)$. The result now follows from an application of Theorem 5.2.   □

5.3. **First properties of ρ_d.** Proposition 5.8 establishes the existence of a collection $c_i, i \in I_d$ of vanishing cycles on X. In this section, we derive some basic topological properties of this configuration arising from the fact that the Dehn twists $T_{c_i}^{-1}$ must satisfy the relations (1)-(3) of Lönne's presentation.

Lemma 5.10.

> (1) If the vertices v_i, v_j are adjacent, then the curves c_i, c_j satisfy $i(c_i, c_j) = 1$.
> (2) For $d \geq 4$, the curves $c_i, i \in I_d$ are pairwise distinct, and all c_i are non-separating.
> (3) If the vertices v_i, v_j in Γ_d are non-adjacent, then the curves c_i and c_j are disjoint.

(4) *For $d \geq 4$, if the vertices v_i, v_j, v_k form a triangle in Γ_d, then the curves c_i, c_j, c_k are supported on an essential subsurface[5] $S_{ijk} \subset X$ homeomorphic to $\Sigma_{1,2}$. Moreover, if the triangle determined by v_i, v_j, v_k is positively oriented, then $i(c_i, T_{c_j}^{-1}(c_k)) = 0$.*

Proof. (1): If v_i and v_j are adjacent, then the Dehn twists $T_{c_i}^{-1}$ and $T_{c_j}^{-1}$ satisfy the braid relation. It follows from Proposition 4.3 that $i(c_i, c_j) = 1$.

(2): Suppose v_i and v_j are distinct vertices. For $d \geq 4$, no two vertices have the same set of adjacent vertices, so that there is some v_k adjacent to v_i and not v_j. By (1) above, it follows that $T_{c_i}^{-1}$ and $T_{c_k}^{-1}$ satisfy the braid relation, while $T_{c_j}^{-1}$ and $T_{c_k}^{-1}$ do not, showing that the isotopy classes of c_i and c_j are distinct. Since each c_i satisfies a braid relation with some other c_j, Proposition 4.3 shows that c_i is non-separating.

(3): If v_i and v_j are non-adjacent, then the Dehn twists $T_{c_i}^{-1}$ and $T_{c_j}^{-1}$ commute. According to [FM12, Section 3.5.2], this implies that either $c_i = c_j$ or else c_i and c_j are disjoint. By (2), the former possibility cannot hold.

(4): Via the change-of-coordinates principle, it can be checked that if c_i, c_j, c_k are curves that pairwise intersect once, then $c_i \cup c_j \cup c_k$ is supported on an essential subsurface of the form $\Sigma_{1,b}$ for $1 \leq b \leq 3$. In the case $b = 1$, the curve c_k must be of the form $c_k = T_{c_i}^{\pm 1}(c_j)$. It follows that if d is a curve such that $i(d, c_k) \neq 0$, then at least one of $i(d, c_i)$ and $i(d, c_j)$ must also be nonzero. However, as $d \geq 4$, there is always some vertex v_l adjacent to exactly one of c_i, c_j, c_k. The corresponding curve c_l would violate the condition required of d above (possibly after permuting the indices i, j, k).

It remains to eliminate the possibility $b = 3$. In this case, the change-of-coordinates principle implies that up to homeomorphism, the curves c_i, c_j, c_k must be arranged as in Figure 4. It can be checked directly (e.g. by examining the action on $H_1(\Sigma_{1,3})$) that for this configuration, the relation

$$T_{c_i}^{-1} T_{c_j}^{-1} T_{c_k}^{-1} T_{c_i}^{-1} = T_{c_j}^{-1} T_{c_k}^{-1} T_{c_i}^{-1} T_{c_j}^{-1}$$

does not hold. This violates relation (3) in Lönne's presentation of $\pi_1(\mathcal{P}_d)$. We conclude that necessarily $b = 2$.

Having shown that $b = 2$, it remains to show the condition $i(c_i, T_{c_j}^{-1}(c_k)) = 0$ for a positively-oriented triangle. Let (x, y, z) denote a 3-chain on $\Sigma_{1,2}$. The change-of-coordinates principle implies that without loss of generality, $c_i = x, c_j = y$, and $c_k = T_y^{\pm 1}(z)$. We wish to show that necessarily $c_k = T_y(z)$. It can be checked directly (e.g. by considering the action on $H_1(\Sigma_{1,2})$) that only in the case $c_k = T_y(z)$ does relation (3) in the Lönne presentation hold. $\square$

6. Configurations of vanishing cycles

The goal of this section is to derive an explicit picture of the configuration of vanishing cycles on a plane curve of degree 5. The main result of the section is Lemma 6.1.

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[5]A subsurface $S' \subset S$ is *essential* if every component of $\partial S'$ is not null-homotopic.
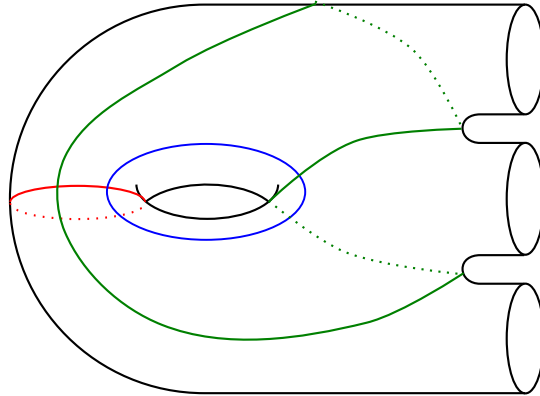
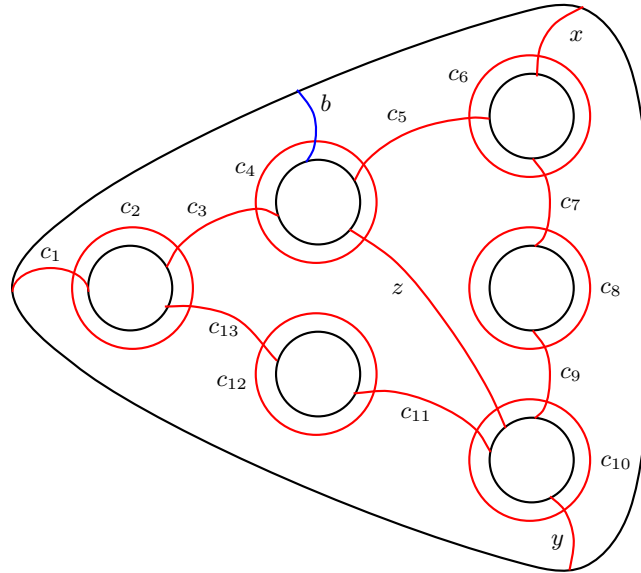FIGURE 4. Lemma 5.10.4: the configuration of c_i, c_j, c_k in the $b = 3$ case.

FIGURE 5. The curves of Lemma 6.1. The bottom halves of curves b, x, y, z, and c_i for i odd have been omitted for clarity; on the bottom half, each curve follows its mirror image on the top.

Lemma 6.1. *There is a homeomorphism $f : X \to \Sigma_6$ such that with reference to Figure 5,*

(1) *The curves $c_1, \ldots, c_{12}$ are vanishing cycles; that is, $T_{c_i} \in \text{Im}(\rho_5)$ for $1 \leq i \leq 12$. The curves x, y, z are also vanishing cycles.*

(2) *The curve b satisfies $T_b^2 \in \text{Im}(\rho_5)$.*

Proof. Lemma 6.1 will be proved in three steps.

Step 1: Uniqueness of Lönne configurations.

Lemma 6.2. *Suppose $d \geq 5$ is odd. Up to homeomorphism, there is a unique configuration of curves $c_i, i \in I_d$ on Σ_g whose intersection pattern is prescribed by Γ_d and such that the twists $T_{c_i}^{-1}$ satisfy the relations (1),(2),(3) given by Lönne's presentation.*

A configuration of curves $c_i, i \in I_d$ as in Lemma 6.2 will be referred to as a *Lönne configuration.*

Proof. Let $a_{1,1}, \ldots, a_{d-1,d-1}$ determine a Lönne configuration on Σ_g. We will exhibit a homeomorphism of Σ_g taking each $a_{i,j}$ to a corresponding $b_{i,j}$ in a "reference" configuration $\{b_{i,j}\}$ to be constructed in the course of the proof. This will require three steps.

Step 1: A collection of disjoint chains. Each row in the Lönne graph determines a chain of length $d-1$. The change of coordinates principle for chains of even length (Lemma 4.2) asserts that any two chains of length $d-1$ are equivalent up to homeomorphism. Considering the odd-numbered rows of Γ_d, it follows that there is a homeomorphism f_1 of Σ_g that takes each $a_{2i-1,j}$ for $1 \leq i \leq d-1$ to a curve $b_{2i-1,j}$ in a standard picture of a chain. We denote the subsurface of Σ_g determined by the chain $a_{2i-1,1}, \ldots, a_{2i-1,d-1}$ as A_i, and similarly we define the subsurfaces B_i of the reference configuration. Each A_i, B_i is homeomorphic to $\Sigma_{(d-1)/2,1}$.

Step 2: Arcs on A_i. The next step is to show that up to homeomorphism, there is a unique picture of what the intersection of the remaining curves $a_{2i,j}$ with $\bigcup A_i$ looks like. Consider a curve $a_{2i,j}$. Up to isotopy, $a_{2i,j}$ intersects only the subsurfaces A_i and A_{i+1}. We claim that $a_{2i,j}$ can be isotoped so that its intersection with A_i is a single arc, and similarly for A_{i+1}. If $j = d-1$, then $a_{2i,d-1}$ intersects only the curve $a_{2i-1,d-1}$, and $i(a_{2i,d-1}, a_{2i-1,d-1}) = 1$. It follows that if $a_{2i,d-1} \cap A_i$ has multiple components, exactly one is essential, and the remaining components can be isotoped off of A_i.

In the general case where $a_{2i,j}$ intersects both $a_{2i-1,j}$ and $a_{2i-1,j+1}$, an analogous argument shows that $a_{2i,j} \cap A_i$ consists of one or two essential arcs. Consider the triangle in the Lönne graph determined by $a_{2i,j}, a_{2i-1,j}, a_{2i-1,j+1}$. According to Lemma 5.10.4, the union $a_{2i,j} \cup a_{2i-1,j} \cup a_{2i-1,j+1}$ is supported on an essential subsurface of the form $\Sigma_{1,2}$. Figure 6 shows that if $a_{2i,j} \cap A_i$ consists of two essential arcs, then $a_{2i,j} \cup a_{2i-1,j} \cup a_{2i-1,j+1}$ is supported on an essential subsurface of the form $\Sigma_{1,3}$, in contradiction with Lemma 5.10.4. Similar arguments establish that $a_{2i-2,j} \cap A_i$ is a single essential arc as well.

We next show that all points of intersection $a_{2i,j} \cap a_{2i,j+1}$ can be isotoped to occur on both A_i and A_{i+1}. This also follows from Lemma 5.10.4. If some point of intersection $a_{2i,j} \cap a_{2i,j+1}$ could not be isotoped onto A_i, then the union $a_{2i,j} \cup a_{2i,j+1} \cup a_{2i-1,j+1}$ could not be supported on a subsurface homeomorphic to $\Sigma_{1,2}$. An analogous argument applies with A_{i+1} in place of A_i. This is explained in Figure 7.

It follows from this analysis that all crossings between curves in row $2i$ can be isotoped to occur in a collar neighborhood of ∂A_i. We define A_i^+ to be a slight enlargement of A_i along such a neighborhood, so that all crossings between curves in row $2i$ occur in $A_i^+ \setminus A_i$.
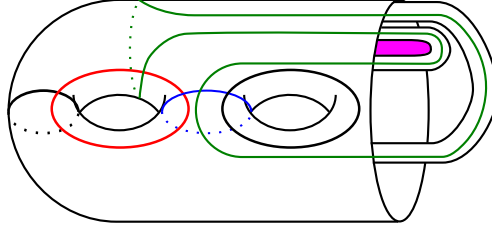
FIGURE 6. If $a_{2i,j}$ cannot be isotoped onto a single arc inside A_i, then the curve enclosed by the inner strip (shaded) is essential in Σ_g, causing $a_{2i,j} \cup a_{2i-1,j} \cup a_{2i-1,j+1}$ to be supported on a surface $\Sigma_{1,3}$.
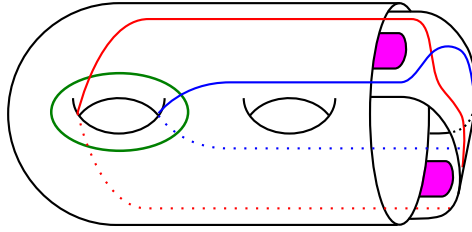

FIGURE 7. If the intersection $a_{2i,j} \cap a_{2i,j+1}$ cannot be isotoped to occur on A_i, then both curves indicated by the shaded regions are essential in Σ_g, causing $a_{2i,j} \cup a_{2i,j+1} \cup a_{2i-1,j+1}$ to be supported on a surface $\Sigma_{1,3}$.

We can now understand what the collection of arcs $a_{2i,1} \cap A_i^+, \ldots, a_{2i,d-1} \cap A_i^+$ looks like. To begin with, the change-of-coordinates principle asserts that up to a homeomorphism of A_i fixing the curves $\{a_{2i-1,j}\}$, the arc $a_{2i,1} \cap A_i$ can be drawn in one of two ways. The first possibility is shown in Figure 8(a), and the second is its mirror-image obtained by reflection through the plane of the page (i.e. the curve with the dotted and solid portions exchanged). In fact, $a_{2i,1} \cap A_i$ must look as shown. This follows from Lemma 5.10.4. The vertices $(a_{2i-1,1}, a_{2i-1,2}, a_{2i,1})$ form a positively-oriented triangle, and so $i(a_{2i-1,1}, T_{a_{2i-1,2}}^{-1}(a_{2i,1})) = 0$. This condition precludes the other possibility.

The pictures for $a_{2i,2}, \ldots, a_{2i,d-1}$ are obtained by proceeding inductively. In each case, there are exactly two ways to draw an arc satisfying the requisite intersection properties, and Lemma 5.10.4 precludes one of these possibilities. The result is shown in Figure 8(b).

It remains to understand how the crossings between curves in row $2i$ are organized on A_i^+. As shown, the arcs $a_{2i,j} \cap A_i$ and $a_{2i,j+1} \cap A_i$ intersect ∂A_i twice each, and in both instances the intersections are adjacent relative to the other arcs. There are thus apparently two possibilities for where the crossing can occur. However, one can see from Figure 8(c) that once a choice is made for one crossing, this enforces choices for the remaining crossings. Moreover, the two apparently distinct configurations are in fact equivalent: the cyclic ordering of the arcs along ∂A_i^+ is the same

in either case, and the combinatorial type of the cut-up surface

$$A_i^\circ := A_i^+ \setminus \bigcup \{a_{k,j} \mid 2i - 1 \le k \le 2i, 1 \le j \le d - 1\}$$

is the same in either situation. The change-of-coordinates principle then asserts the existence of a homeomorphism of A_i^+ sending each $a_{2i-1,j}$ to itself, and taking one configuration of arcs to the other.

Having seen that the arcs $a_{2i,j} \cap A_i^+$ can be put into standard form, it remains to examine the other collection of arcs on A_i^+, namely those of the form $a_{2i-2,j}$. It is easy to see by induction on d that the cut-up surface A_i° is a union of polyhedral disks for which the edges correspond to portions of the curves $a_{2i-1,j}$, the arcs $a_{2i,j} \cap A_i^+$, or else the boundary ∂A_i^+. It follows that the isotopy class of an arc $a_{2i-2,j} \cap A_I^+$ is uniquely determined by its intersection data with the curves $a_{2i-1,j}$ and $a_{2i,j}$.

For $j \ge 2$, the curve $a_{2i-2,j}$ intersects $a_{2i-1,j-1}$ and $a_{2i-1,j}$, and is disjoint from all curves $a_{2i,k}$. As $a_{2i,j-1}$ has the same set of intersections as $a_{2i-2,j}$, it follows that $a_{2i-2,j} \cap A_i^+$ must run parallel to $a_{2i,j-1}$. The curve $a_{2i-2,1}$ intersects only $a_{2i,1}$; consequently $a_{2i-2,1} \cap A_i^+$ is uniquely determined. As can be seen from Figure 8(c), this forces each subsequent $a_{2i-2,j}$ onto a particular side of $a_{2i,j-1}$.
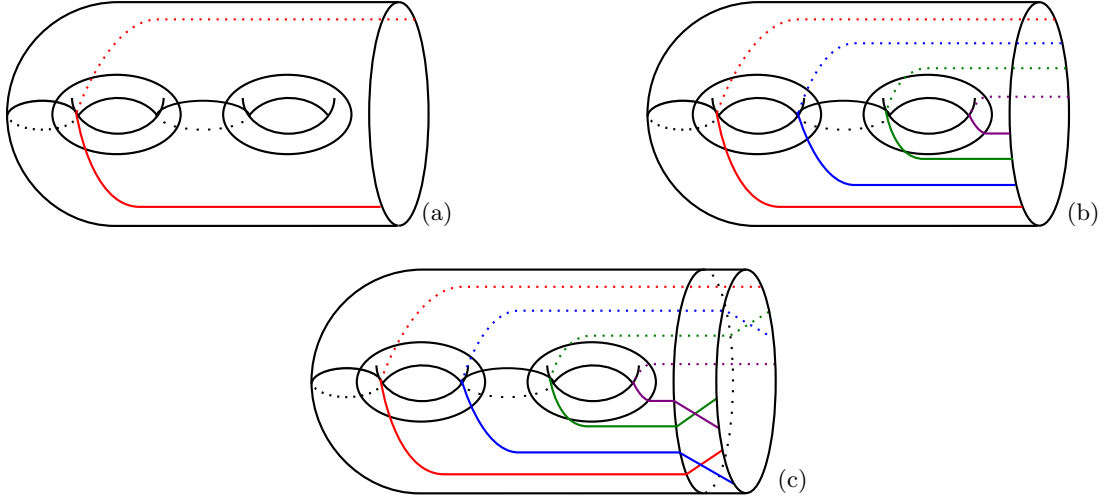

FIGURE 8. The surface A_i^+. (a): The correct choice for $a_{2i,1} \cap A_i$. (b): The configuration $a_{2i,j} \cap A_i$. (c) The configuration $a_{2i,j} \cap A_i^+$.

Step 3: Arcs on the remainder of Σ_g. Consider now the subsurface

$$\Sigma_g^\circ := \Sigma_g \setminus \bigcup A_i.$$

This has $(d-1)/2$ boundary components ∂_k, indexed by the corresponding A_k. The intersection $a_{2i,j} \cap (\Sigma_g \setminus \bigcup A_i)$ consists of two arcs, each connecting ∂_i with ∂_{i+1}. The strategy for the remainder

of the proof is to argue that when all these arcs are deleted from Σ_g°, the result is a union of disks. The change-of-coordinates principle will then assert the uniqueness of such a configuration of arcs, completing the proof.

For what follows, it will be convenient to refer to a product neighborhood $[0,1] \times [0,1] \subset \Sigma_g^\circ$ of some arc $a_{2i,j} \cap \Sigma_g^\circ$ as a *strip*. Our first objective is to compute the Euler characteristic χ of the surface $\Sigma_g^{\circ\circ}$ obtained by deleting strips for all arcs from Σ_g°. Then an analysis of the pattern by which strips are attached will determine the number of components of this surface.

To begin, we return to the setting of Figure 7. Above, it was argued that for $2i < (d-1)/2$, the intersection $a_{2i,j} \cap a_{2i,j+1}$ can be isotoped onto either A_i or A_{i+1}. This means that there is a strip that contains both $a_{2i,j} \cap \Sigma_g^\circ$ and $a_{2i,j+1} \cap \Sigma_g^\circ$. Grouping such strips together, it can be seen that for $1 \leq i \leq (d-3)/2$, the $2i^{th}$ row of the Lönne graph gives rise to d strips. In the last row, there are $d-1$ strips. So in total there are $1/2(d+1)(d-2)$ strips, and each strip contributes -1 to the Euler characteristic.

Recall the relation $g = (d-1)(d-2)/2$: this means that

$$\chi(\Sigma_g) = 2 - (d-1)(d-2).$$

Each A_i has Euler characteristic $\chi(A_i) = 2 - d$. It follows that

$$\chi(\Sigma_g^\circ) = \chi(\Sigma_g) - \sum_{i=1}^{(d-1)/2} \chi(A_i) = \quad 2 - (d-1)(d-2) + (d-1)(d-2)/2$$
$$= \quad\quad\quad\quad 2 - (d-1)(d-2)/2.$$

Therefore,

$$\chi(\Sigma_g^{\circ\circ}) = \quad \chi(\Sigma_g^\circ) + 1/2(d+1)(d-2)$$
$$= \quad\quad\quad\quad d.$$

We claim that $\Sigma_g^{\circ\circ}$ has d boundary components. This will finish the proof, as a surface of Euler characteristic d and $b = d$ boundary components must be a union of d disks. The claim can easily be checked directly in the case $d = 5$ of immediate relevance. For general d, this follows from a straightforward, if notationally tedious, verification, proceeding by an analysis of the cyclic ordering of the arcs $a_{i,j}$ around the boundary components ∂A_k^+. $\qquad\square$

Step 2: A convenient configuration.

Figure 9 presents a picture of a Lönne configuration in the case of interest $d = 5$. This was obtained by "building the surface" curve by curve, attaching one-handles in the sequence indicated by the numbering of the curves $a_1, \ldots, a_{16}$. There are other, more uniform depictions of Lönne configurations which arise from Akbulut-Kirby's picture of a plane curve of degree d derived from a Seifert surface of the (d, d) torus link (see [AK80] or [GS99, Section 6.2.7]), but the analysis to follow is easier to carry out using the model of Figure 9.
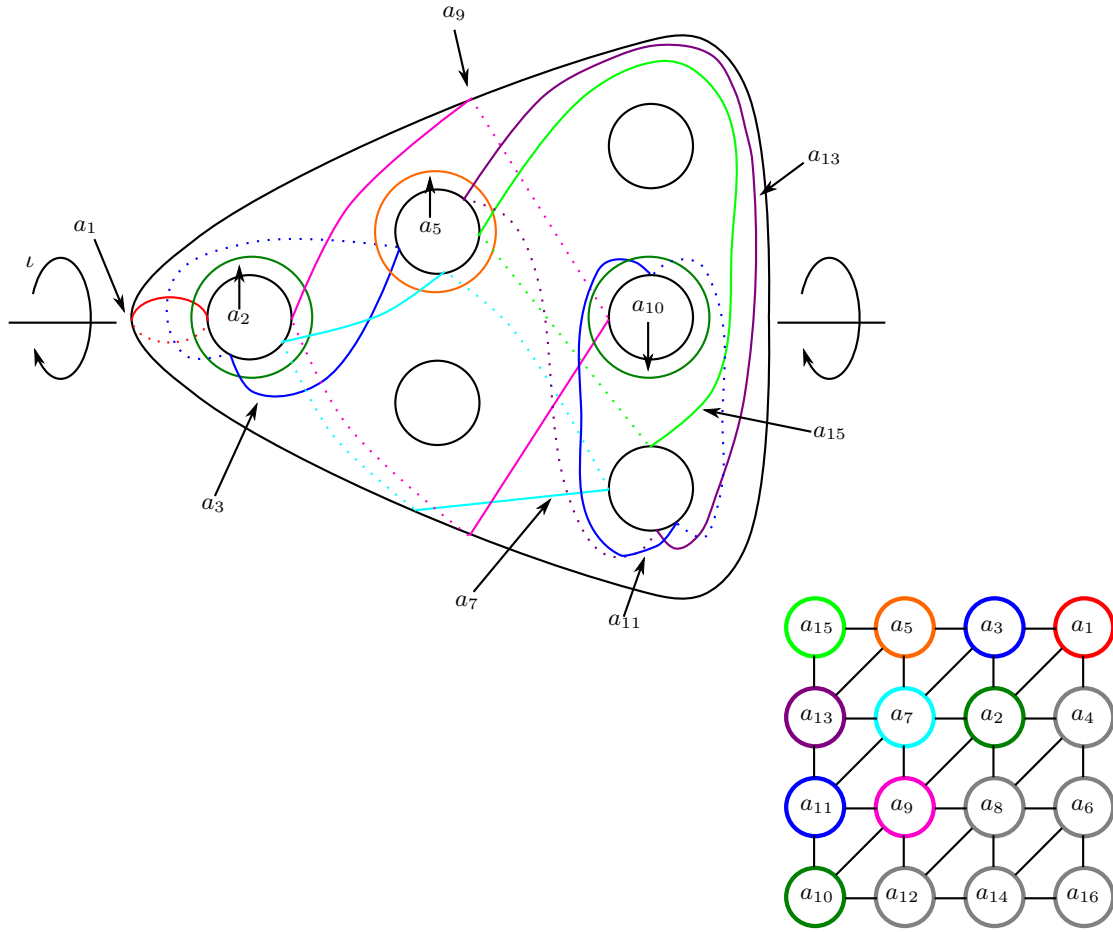
FIGURE 9. A Lönne configuration on Σ_6. Only a portion of the figure has been drawn: the omitted curves are obtained by applying the involution ι to the depicted curves.

Step 3: Producing vanishing cycles. The bulk of this step will establish claim (1); claim (2) follows as an immediate porism. The set of vanishing cycles is invariant under the action of the monodromy group, since acting by a monodromy element on a nodal degeneration amounts to changing the path along which the nodal degeneration is performed. In particular, if a and b are vanishing cycles, then so is $T_a(b)$. To begin with, curves $c_1, c_2, c_4, c_8,$ and c_{12} are elements of the Lönne configuration and so are already vanishing cycles. The curve c_3 is obtained as

$$c_3 = T_{a_2}^{-1}(a_3);$$

similarly,

$$c_{13} = T_{a_2}^{-1}(a_4).$$

Curve c_{10} is obtained as

$$c_{10} = T_{a_{15}}(a_{13});$$

c_6 is obtained from a_{14} and a_{16} analogously.

The curve c_9 is obtained as

$$c_9 = T_{c_{10}} T_{a_{10}}^{-1}(a_{11});$$

c_7 is obtained from a_{10}, a_{12}, and c_6 analogously.

To obtain c_5, twist a_{13} along the chain $c_6, \ldots, c_{10}$:

$$c_5 = T_{c_6}^{-1} T_{c_7}^{-1} T_{c_8}^{-1} T_{c_9}^{-1} T_{c_{10}}^{-1}(a_{13}).$$

c_{11} is obtained by an analogous procedure on a_{14}.

The sequence of twists used to exhibit x as a vanishing cycle is illustrated in Figure 10. Symbolically,

$$x = T_{c_6}^{-1} T_{c_7}^{-1} T_{c_8}^{-1} T_{c_9}^{-1} T_{c_5} T_{c_4} T_{c_6}^{-1} T_{c_7}^{-1} T_{c_8}^{-1} T_{a_9}^{-1}(a_7).$$

y is produced in an analogous fashion, starting with a_8 in place of a_6.

To produce z, we appeal to the genus-2 star relation. Applied to the surface bounded by b, y, z, it shows that $T_b^2 T_y T_z \in \text{Im}(\rho_5)$, and hence $T_b^2 T_z \in \text{Im}(\rho_5)$ since $T_y \in \text{Im}(\rho_5)$ by above. Observe that $i(c_{10}, z) = 1$, and that $T_{c_{10}} \in \text{Im}(\rho_5)$. Making use of the fact that b is disjoint from both z and c_{10}, the braid relation gives

$$T_{c_{10}} T_b^2 T_z(c_{10}) = T_{c_{10}} T_z(c_{10}) = z.$$

This exhibits z as a vanishing cycle, establishing claim (1) of Lemma 6.1. As $T_b^2 T_z$ and T_z are now both known to be elements of $\text{Im}(\rho_5)$, it follows that $T_b^2 \in \text{Im}(\rho_5)$ as well, completing claim (2). $\qquad \square$

7. Proof of Theorem A

In this final section we assemble the work we have done so far in order to prove Theorem A.

Step 1: Reduction to the Torelli group. The first step is to reduce the problem of determining $\text{Im}(\rho_5)$ to the determination of $\text{Im}(\rho_5) \cap \mathcal{I}_6$. This will follow from [Bea86]. Recall that Beauville establishes that $\text{Im}(\Psi \circ \rho_5)$ is the entire stabilizer of an odd-parity spin structure on $H_1(\Sigma_6; \mathbb{Z})$. This spin structure was identified as ϕ_5 in Section 2. Therefore $\text{Im}(\Psi \circ \rho_5) = \text{Im}(\Psi \circ \text{Mod}(\Sigma_6)[\phi_5])$. It therefore suffices to show that

$$\text{Im}(\rho_5) \cap \ker \Psi = \text{Mod}(\Sigma_6)[\phi_5] \cap \ker \Psi = \mathcal{I}_6. \tag{7}$$

Step 2: Enumeration of cases. Equation (7) will be derived as a consequence of Theorem 4.6. Lemma 6.1.1 asserts that the curves $c_1, \ldots, c_{12}$ in the Johnson generating set are contained in $\text{Im}(\rho_5)$, so that the first hypothesis of Theorem 4.6 is satisfied. There are then eight cases to check: the four *straight chain maps* of the form $(c_1, \ldots, c_k)$ for $k = 3, 5, 7, 9$, and the four *β-chain maps* of the form $(\beta, c_5, \ldots, c_k)$ for $k = 6, 8, 10, 12$. See Figure 11.
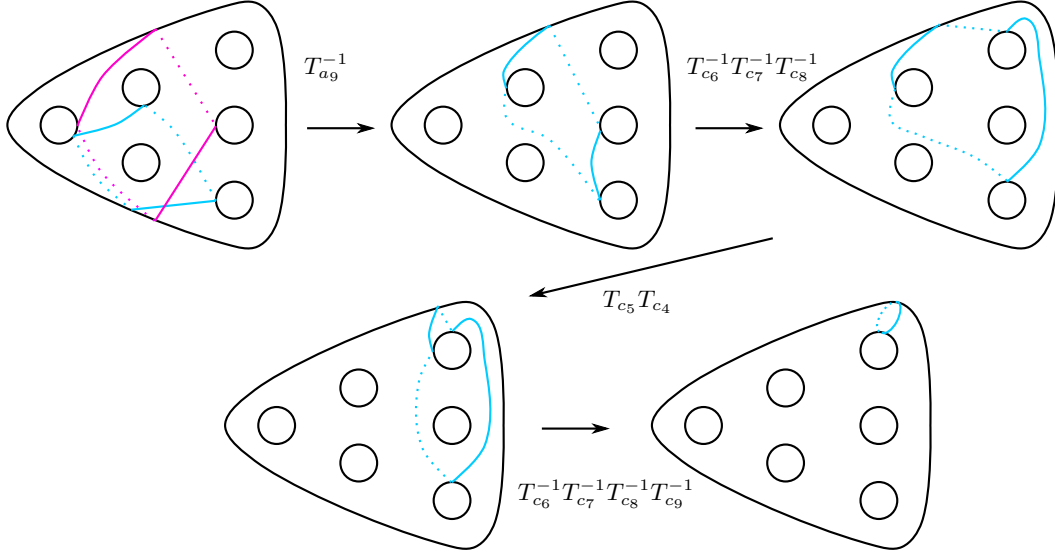
FIGURE 10. The sequence of twists used to obtain x.

The verification of the β-chain cases will be easier to accomplish after conjugating by the class $g = T_x T_{c_5}^{-1} T_{c_4}^{-1} \in \mathrm{Im}(\rho_5)$. This has the following effect on the curves in the β-chains (the curve γ is indicated in Figure 11 in the picture for $k = 6$):

$$g(\beta) = b, \qquad g(c_5) = c_4, \qquad g(c_6) = \gamma, \qquad g(c_k) = c_k \text{ for } k \geq 7.$$

Step 3: Producing bounding-pair maps. In this step, we explain the method by which we will obtain the necessary bounding-pair maps. This is an easy consequence of the chain relation.

Lemma 7.1. *Let $C = (c_1, \ldots, c_k)$ be a chain of odd length k and boundary $\partial C = d_1 \cup d_2$. Suppose that the mapping classes*

$$T_{c_1}^2, T_{c_2}^2, \ldots, T_{c_k}^2, T_{d_1}^2$$

are all contained in some subgroup $\Gamma \leq \mathrm{Mod}(\Sigma_g)$. Then the chain map associated to C (i.e. the bounding pair map $T_{d_1} T_{d_2}^{-1}$) is also contained in Γ.

Proof. The chain relation (Proposition 4.4) implies that $T_{d_1} T_{d_2} \in \Gamma$. By hypothesis, $T_{d_1}^2 \in \Gamma$, so the bounding pair map $T_{d_1} T_{d_2}^{-1} \in \Gamma$ as well. $\qquad\square$

Step 4: Verification of cases. Lemma 6.1 asserts that the classes $T_{c_i}, 1 \leq i \leq 12$, as well as T_b^2 are all contained in $\mathrm{Im}(\rho_5)$. The class γ is obtained from c_6 by the element $g \in \mathrm{Im}(\rho_5)$, so γ is a vanishing cycle as well. Via Lemma 7.1, it remains only to show that in each of the cases in Step 2, one of the boundary components d_1 satisfies $T_{d_1}^2 \in \mathrm{Im}(\rho_5)$.
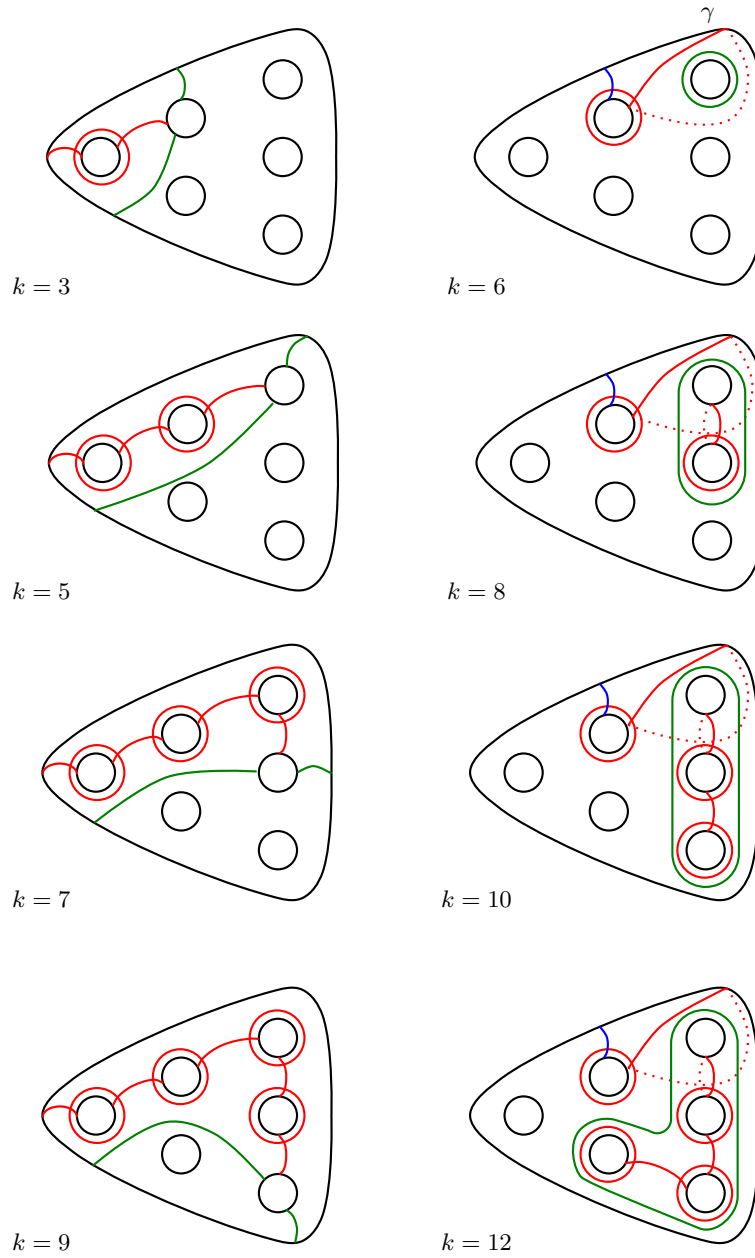
FIGURE 11. The cases of Step 2

The straight chain maps are depicted in the left-hand column of Figure 11. For $k = 3$, one boundary component is b; we have already remarked how $T_b^2 \in \mathrm{Im}(\rho_5)$. For $k = 5$, one of the boundary components is x. For $k = 7$, one uses the methods of Lemma 6.1 to show that the

right-hand boundary component c satisfies $T_c^2 \in \mathrm{Im}(\rho_5)$ (the proof is identical to that for b). Finally, for $k = 9$, one of the boundary components is y.

We turn to the β-chains. The images of the β-chains under the map g are depicted in the right-hand column of Figure 11. For $k = 6, 8, 10, 12$, let d_k denote the boundary component depicted there for the chain $(b, c_4, \gamma, c_7, \ldots, c_k)$. Observe that d_k is also a boundary component of the chain map for $(c_6, \ldots, c_k)$ (in the case $k = 6$, the boundary component d_6 is just c_6). Moreover, the chain map for $(c_6, \ldots, c_k)$ is conjugate to the chain map for $(c_1, \ldots, c_{k-5})$ by an element of $\mathrm{Im}(\rho_5)$ (this is easy to see using the isomorphism between the group generated by $c_1, \ldots, c_{12}$ and the braid group B_{13} on 13 strands). Via the verification of the straight-chain cases, it follows that $T_{d_k}^2 \in \mathrm{Im}(\rho_5)$, and so by Lemma 7.1 the β-chain maps are also contained in $\mathrm{Im}(\rho_5)$. $\qquad\square$

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