THE BIRMAN EXACT SEQUENCE DOES NOT VIRTUALLY SPLIT

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ABSTRACT. This paper answers a basic question about the Birman exact sequence in the theory of mapping class groups. We prove that the Birman exact sequence does not admit a section over any subgroup Γ contained in the Torelli group with finite index. *A fortiori* this implies that there is no multi-section for the universal surface bundle with Torelli monodromy. This theorem was announced in a 1990 preprint of G. Mess, but an error was uncovered and described in a recent paper of the first author.

1. INTRODUCTION

Let S be a surface of finite type. A fundamental tool in the study of the mapping class group $\mathrm{Mod}(S)$ of S is the *Birman exact sequence*, which describes the relationship between $\mathrm{Mod}(S)$ and $\mathrm{Mod}(S')$ of a surface S' obtained from S by filling in boundary components and/or punctures on S. In its most basic form, $S = \Sigma_{g,*}$ is a surface of genus $g \geq 2$ with a single puncture $* \in \Sigma_g$, and $S' = \Sigma_g$ is the closed surface obtained by filling in $*$. In this case, the Birman exact sequence takes the form

$$1 \to \pi_1(\Sigma_g, *) \to \mathrm{Mod}(\Sigma_{g,*}) \to \mathrm{Mod}(\Sigma_g) \to 1. \tag{1}$$

Given any subgroup $\Gamma \leqslant \mathrm{Mod}(\Sigma_g)$, we can form the Birman exact sequence for Γ by pullback. We have the following question:

Question 1.1 (Birman exact sequence splitting problem)**.** *For which subgroups $\Gamma \leqslant \mathrm{Mod}(\Sigma_g)$ does the Birman exact sequence for Γ*

$$1 \to \pi_1(\Sigma_g) \to \widetilde{\Gamma} \to \Gamma \to 1 \tag{2}$$

split?[1]

For $\Gamma = \mathrm{Mod}(\Sigma_g)$, the full Birman exact sequence (1) does not split for any $g \geq 2$. This is an easy consequence two observations. For one, there exist non-cyclic torsion subgroups of $\mathrm{Mod}(\Sigma_g)$, and secondly, it is simple to show that no such subgroups exist in $\mathrm{Mod}(\Sigma_{g,*})$. While this argument quickly dispatches the case of $\Gamma = \mathrm{Mod}(\Sigma_g)$, it is unsuitable even for general finite-index subgroups, as $\mathrm{Mod}(\Sigma_g)$ is known to be virtually torsion-free (see e.g. [FM12, Theorem 6.9]).

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[1]Recall that a group homomorphism $p : G \to H$ is said to *split* if there is a homomorphism $s : H \to G$ such that $p \circ s = \mathrm{id}$. More generally, such a p is said to *virtually split* if there is a finite-index subgroup H' such that the restriction of p to $p^{-1}(H')$ splits. For a short exact sequence $1 \to A \to B \to C \to 1$, the homomorphism under consideration is always the surjection $B \to C$.

For $g = 2$ the Birman exact sequence *does* virtually split, as follows from the fact that every Riemann surface of genus 2 is hyperelliptic and is therefore equipped with 6 distinguished *Weierstrass points*. The purpose of this paper is to show that a similar phenomenon cannot occur for higher genus surfaces, and moreover to resolve Question 1.1 for a large class of groups that includes all finite-index subgroups. For the definition of the Torelli group $\mathcal{I}(\Sigma_g)$, see Section 2.2.

Theorem A. *For $g \geq 4$, the Birman exact sequence does not virtually split. Moreover, for any subgroup $\Gamma \leqslant \mathcal{I}(\Sigma_g)$ of finite index in the Torelli group, there is no splitting $\sigma : \Gamma \to \mathcal{I}(\Sigma_{g,*})$ of the restriction of the Birman exact sequence to Γ.*

Topological reformulation: multisections. Question 1.1 and Theorem A admit a topological reformulation. Let $p : E \to B$ be a Σ_g-bundle with monodromy representation $\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma_g)$, and define $\Gamma := \mathrm{Im}(\rho)$. The existence of a splitting of the Birman exact sequence for Γ as in (2) is equivalent to the existence of a section of $p : E \to B$, i.e., a continuous map $s : B \to E$ satisfying $p \circ s = \mathrm{id}$.

More generally, a *multisection* (of cardinality n) of p is a continuously-varying choice of n distinct points on each fiber. A multisection is not necessarily an amalgamation of n distinct sections, since the points may be permuted by moving around loops in B. However, this permutation monodromy can be made trivial by pulling back the surface bundle along a well-chosen finite-sheeted cover B' of B. Thus a multisection always gives rise to a *virtual section* of $p : E \to B$, i.e., a finite-sheeted cover B' of B such that the pullback of p to B' admits a section.

The topological reformulation of Theorem A concerns multisections of the *universal curve* $\pi : \mathscr{M}_{g,*} \to \mathscr{M}_g$. Here, $\mathscr{M}_g$ is the moduli space of Riemann surfaces of genus g, and $\mathscr{M}_{g,*}$ is the moduli space of Riemann surfaces of genus g equipped with a marked point. In order to avoid technicalities induced by the orbifold structure on $\mathscr{M}_g$, we consider instead the finite-sheeted cover $\pi : \mathscr{M}_{g,*}[3] \to \mathscr{M}_g[3]$ of (marked) Riemann surfaces equipped with a framing of $\mathbb{Z}/3\mathbb{Z}$-homology; $\mathscr{M}_g[3]$ is a manifold and $\pi : \mathscr{M}_{g,*}[3] \to \mathscr{M}_g[3]$ is a Σ_g-bundle. In the topological setting, the Torelli group corresponds to the *Torelli space* $\mathscr{I}_g$ whose points correspond to Riemann surfaces equipped with a framing of *integral* homology; likewise $\mathscr{I}_{g,*}$ consists of homologically-framed curves equipped with a marked point. There are covering maps $\mathscr{I}_g \to \mathscr{M}_g[3]$ and $\mathscr{I}_{g,*} \to \mathscr{M}_{g,*}[3]$, and these maps are compatible with the bundle projection maps. We thus obtain Theorem B below as an immediate corollary of Theorem A.

Theorem B. *For $g \geq 4$, the universal family $\pi : \mathscr{I}_{g,*} \to \mathscr{I}_g$ does not admit any continuous multisection. A fortiori, for $g \geq 4$, there is no continuous multisection of $\pi : \mathscr{M}_{g,*}[3] \to \mathscr{M}_g[3]$.*

Context for Theorem A: how to study finite-index subgroups? The class of finite-index subgroups of $\mathrm{Mod}(\Sigma_g)$ is famously mysterious, and there are very few nontrivial results known about an arbitrary $\Gamma \leqslant \mathrm{Mod}(\Sigma_g)$. Indeed, many guiding conjectures about the mapping class group ask whether all finite-index subgroups have a particular property:

- The *congruence subgroup conjecture* asks if every finite-index subgroup contains a subgroup of a particular form (a so-called congruence subgroup).

- The *virtual first Betti number problem* asks whether the first rational Betti number of every finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ is 0.
- The *virtual Nielsen realization problem* asks whether any finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ is realizable as a group of homeomorphisms.

A pioneering study of arbitrary finite-index subgroups of $\mathrm{Mod}(\Sigma_g)$ was carried out by Ivanov [Iva84]. Below, the *extended mapping class group* $\mathrm{Mod}(\Sigma_g)^\pm$ is the group of mapping classes that do not necessarily preserve orientation; it contains $\mathrm{Mod}(\Sigma_g)$ as a normal subgroup of index 2.

Theorem 1.2 (Ivanov's rigidity theorem)**.** *Any injective homomorphism from a finite index subgroup of* $\mathrm{Mod}(\Sigma_g)$ *to* $\mathrm{Mod}(\Sigma_g)$ *is induced by conjugation by* $\mathrm{Mod}(\Sigma_g)^\pm$.

Ivanov's method is to extract topological information from the relevant algebraic data. For any finite-index subgroup $\Gamma \leqslant \mathrm{Mod}(\Sigma_g)$ and any simple closed curve c, there exists some N (possibly depending on c) such that the Dehn twist power T_c^N is contained in Γ. By studying the images of these elements under an injective map, Ivanov shows that an injective homomorphism induces an automorphism of a simplicial complex $\mathcal{C}(\Sigma_g)$ known as the *curve complex* for the surface Σ_g. Ivanov also shows that $\mathrm{Aut}(\mathcal{C}(\Sigma_g)) \cong \mathrm{Mod}(\Sigma_g)^\pm$, leading to the result.

Any naïve attempt to apply Ivanov's methods to Question 1.1 is fated to be unsuccessful. Ivanov's methods do extend to show that a virtual splitting of the Birman exact sequence induces an injective map between curve complexes $\mathcal{C}(\Sigma_g) \hookrightarrow \mathcal{C}(\Sigma_{g,*})$, which might appear paradoxical. However, such maps exist in abundance! This follows, for instance, from a theorem of Birman–Series [BS85]. Thus to address the virtual splitting of the Birman exact sequence, we must develop methods beyond Ivanov's in order to look deeper into the structure of a general finite-index subgroup of $\mathrm{Mod}(\Sigma_g)$.

The work of Mess. Theorem A is claimed in the 1990 preprint [Mes90] of G. Mess. Unfortunately, as detailed in the paper [Che17] of the first author, Mess' argument contains a fatal error. In [Che17], the first author proves Theorem A in the special case of the full Torelli group $\Gamma = \mathcal{I}(\Sigma_g)$. The methods therein make essential use of some special relations in $\mathcal{I}(\Sigma_g)$ which disappear upon passing to finite-index subgroups.

In the present note, we return to the outline of the argument as proposed by Mess. We follow his argument to the point where his error occurs; this is essentially the content of Section 2. The core of Mess' idea is to show that a splitting of the Birman exact sequence for Γ as in Theorem A induces (at least up to finite index) a section s of a fibration π of *configuration spaces*

$$\pi : \mathrm{PConf}_2(\Sigma_p) \to \Sigma_p.$$

Here $\mathrm{PConf}_2(\Sigma_p)$ denotes the space of distinct ordered pairs of points on the surface Σ_p and π is the projection onto the first coordinate; the relation of p to the original genus g is explained at the start of Section 3. Roughly speaking, Mess incorrectly assumes that any section s must necessarily be horizontal, i.e., that the projection onto the second coordinate must be constant, and derives a contradiction predicated on this assumption.

Our approach. Unfortunately, π *does* admit continuous sections (e.g. the graph of any fixed-point-free map $f : \Sigma_p \to \Sigma_p$), which seems to spell trouble for Mess' method. However, the section s is far from arbitrary: it arises from a hypothetical section of a surface bundle with a rich monodromy group. We exploit this to extract more properties of a hypothetical section s, both of an algebraic and a geometric nature. The core of our analysis is a study of the following intriguing questions about rigidity properties of homomorphisms between surface groups. Below, we say that an element $c \in H \leqslant \pi_1(\Sigma_g)$ is a *simple curve power* if $i(c) = d^k$ for some $d \in \pi_1(\Sigma_g)$ in the homotopy class of a simple closed curve on Σ_g.

Question 1.3. *Let $H \leqslant \pi_1(\Sigma_g)$ be a finite-index subgroup. Let $i : H \to \pi_1(\Sigma_g)$ denote the inclusion map, and let $p : H \to \pi_1(\Sigma_g)$ be an arbitrary homomorphism.*

    (1) Suppose that $p(c) = i(c)$ for all simple curve powers $c \in H$. Must $p = i$?

    (2) Suppose that $p(c) = 1$ for all simple curve powers $c \in H$. Must p be the trivial homomorphism?

The first of these is answered in the affirmative in Lemma 3.6. We answer the latter question (again affirmatively) under the additional assumption that p has a certain equivariance property; this is the content of Lemma 3.7. Ultimately this is used to obtain information about the Lefschetz number of certain maps between surfaces which is used to obstruct the existence of sections s satisfying all the properties we show they must.

Splittings over normal subgroups. For an arbitrary subgroup Γ, Question 1.1 is far too broad to be approachable. To better understand the splitting problem for subgroups $\Gamma \leqslant \mathrm{Mod}(\Sigma_g)$, it is best to focus attention on classes of subgroups for which some amount of structure is imposed. Restricting to the class of *normal* subgroups $\Gamma \lhd \mathrm{Mod}(\Sigma_g)$ is one reasonable starting point, but attempting to study Question 1.1 for the class of all normal subgroups is still too audacious – there is nothing even approaching a conjectural classification or taxonomy of normal subgroups of mapping class groups. Moreover, there are known examples of normal subgroups of $\mathrm{Mod}(\Sigma_g)$ that are abstractly isomorphic to free groups, and more generally certain right-angled Artin groups [DGO17, CMM19]. It is trivial to construct splittings of the Birman exact sequence over such subgroups.

In spite of this, it is now known that a broad collection of normal subgroups Γ additionally have the property $\mathrm{Aut}(\Gamma) \cong \mathrm{Mod}(\Sigma_g)^\pm$ (see [BM18]). Examples of such subgroups include the Torelli group, each term of the Johnson filtration, the terms of the Magnus filtration, and the groups generated by k^{th} powers of all Dehn twists - in contrast, the automorphism groups of the right-angled Artin subgroups mentioned above are much larger.

Question 1.4. *Let $\Gamma \lhd \mathrm{Mod}(\Sigma_g)$ be a normal subgroup with $\mathrm{Aut}(\Gamma) \cong \mathrm{Mod}(\Sigma_g)^\pm$. Does the restriction of the Birman exact sequence to Γ split?*

Organization of the paper. Section 2 collects the necessary facts from the theory of mapping class groups, and establishes some preliminary results. Major points of interest are Definition 2.10,

which defines the handle-pushing subgroups at the center of the argument, and Lemma 2.14, which characterizes the behavior of special mapping classes under a splitting of (2). The proof of Theorem A is then carried out in Section 3. The first major intermediate result is Corollary 3.5, which shows that the map s discussed above has one of two very special forms and leads to Question 1.3. After resolving this in Lemmas 3.6 and 3.7, Section 3.4 finishes the argument by a reduction to Lefschetz's fixed point theorem.

2. Mapping class groups

Definitions and conventions. The mapping class group of a surface S is the group $\mathrm{Mod}(S)$ of isotopy classes of orientation-preserving homeomorphisms of S that restrict trivially to the (possibly empty) boundary of S and fix marked points as a set. Before beginning discussion of the theory of canonical reduction systems, we establish some standard conventions that arise when working with homeomorphisms and curves up to isotopy. To avoid cumbersome notation, we will not use brackets to denote isotopy classes, so that e.g. the symbol f could denote both a specific homeomorphism as well as an entire mapping class, and the symbol c could denote both a specific simple closed curve and its isotopy class. Where necessary we will indicate precisely which is meant, but the reader should be aware of some standard abuses of language. For instance, one may say that a mapping class f fixes a curve c: precisely this means that any representative homeomorphism for f fixes any representative curve for c up to isotopy. We will also speak of cutting a surface S along an isotopy class of curve c: by this we mean cutting S along a representative curve for c.

2.1. **Canonical reduction systems.** The central tool for the proof of Theorem A is the notion of a *canonical reduction system*, which can be viewed as an enhancement of the Nielsen–Thurston classification and is originated from work of [BLM83]. We remind the reader that a curve $c \subset S$ is said to be *peripheral* if c is isotopic to a boundary component or a puncture of S. The Nielsen–Thurston classification asserts that each nontrivial element $f \in \mathrm{Mod}(S)$ is of exactly one of the following types: *periodic*, *reducible*, or *pseudo-Anosov*. A mapping class f is *periodic* if for some representative f, there is some $n \geq 1$ such that f^n is isotopic to the identity. A mapping class f is *reducible* if it is of infinite order but for some $n \geq 1$, there is some non-peripheral simple closed curve $c \subset S$ such that $f^n(c)$ is isotopic to c. If neither of these conditions are satisfied, f is said to be *pseudo-Anosov*. In this case, f has a representative homeomorphism of a very special form. We will not need to delve into the theory of pseudo-Anosov mappings, and refer the interested reader to [FM12, Chapter 13] and [FLP12] for more details.

Definition 2.1 (**Reduction systems**). A *reduction system* of a reducible mapping class h in $\mathrm{Mod}(S)$ is a set of disjoint non-peripheral isotopy classes of curves that h fixes as a set (up to isotopy). A

reduction system is *maximal* if it is maximal with respect to the inclusion of reduction systems for h. The *canonical reduction system* $\mathrm{CRS}(h)$ is the intersection of all maximal reduction systems of h.

Canonical reduction systems allow for a refined version of the Nielsen–Thurston classification. For a reducible element f, there exists n such that f^n fixes each element in $\mathrm{CRS}(f)$ and after cutting out $\mathrm{CRS}(f)$, the restriction of f^n on each component is either identity or pseudo-Anosov. See [FM12, Corollary 13.3]. In Propositions 2.2 - 2.5, we list some properties of the canonical reduction systems that will be used later.

For two curves a, b on a surface S, let $i(a, b)$ be the geometric intersection number of a and b (for the definition of the geometric intersection number, see [FM12, Section 1.2.3]). For two sets of curves P and Q, we say that P and Q *intersect* if there exist $a \in P$ and $b \in Q$ such that $i(a, b) \neq 0$. We emphasize that "intersection" here refers to the intersection of curves on S, and not the abstract set-theoretic intersection of P and Q as sets.

Proposition 2.2. *Let h be a reducible mapping class in $\mathrm{Mod}(S)$. If $\{\gamma\}$ and $\mathrm{CRS}(h)$ intersect, then no power of h fixes γ.*

Proof. Suppose that h^n fixes γ. Therefore γ belongs to a maximal reduction system M. By definition, $\mathrm{CRS}(h) \subset M$. However γ intersects some curve in $\mathrm{CRS}(f)$; this contradicts the fact that M is a set of disjoint curves. $\square$

Proposition 2.3. *Suppose that $h, f \in \mathrm{Mod}(S)$ and $fh = hf$. Then $\mathrm{CRS}(h)$ and $\mathrm{CRS}(f)$ do not intersect.*

Proof. Conjugating relation gives $\mathrm{CRS}(hfh^{-1}) = h(\mathrm{CRS}(f))$. Since $hfh^{-1} = f$, it follows that $\mathrm{CRS}(f) = h(\mathrm{CRS}(f))$. Therefore h fixes the whole set $\mathrm{CRS}(f)$. There is some $n \geq 1$ such that h^n fixes all curves element-wise in $\mathrm{CRS}(f)$. By Proposition 2.2, curves in $\mathrm{CRS}(h)$ do not intersect curves in $\mathrm{CRS}(f)$. $\square$

For a curve a on a surface S, denote by T_a the Dehn twist about a. More generally, a *Dehn multitwist* is any mapping class of the form

$$T = \prod T_{a_i}^{k_i}$$

for a collection of pairwise-disjoint simple closed curves $\{a_i\}$ and arbitrary nonzero integers k_i.

Proposition 2.4. *Let*

$$T = \prod T_{a_i}^{k_i}$$

be a Dehn multitwist. Then

$$\mathrm{CRS}(T) = \{a_i | a_i \text{ is non-peripheral}\}.$$

Proof. This follows quickly from [BLM83, Lemma 2.5] and [FM12, Proposition 3.2]. $\square$

The final result we will require appears as [McC82, Theorem 1].

Proposition 2.5 (McCarthy)**.** *Let S be a Riemann surface of finite type, and let $f \in \mathrm{Mod}(S)$ be a pseudo-Anosov element. Then the centralizer subgroup of f in $\mathrm{Mod}(S)$ is virtually cyclic.*

2.2. **The Torelli group, separating twists, and bounding pair maps.** For the duration of this section, let S be a surface with b boundary components and p punctures with $b + p \leq 1$. The action of a mapping class on homology gives rise to the *symplectic representation*

$$\Psi : \mathrm{Mod}(S) \to \mathrm{Aut}(H_1(S; \mathbb{Z})).$$

The *Torelli group* is the kernel $\mathcal{I}(S) := \ker(\Psi)$. There are several classes of elements of the Torelli group that will feature in the proof of Theorem A. For context, background, and proofs of the following assertions, see [FM12, Chapter 6]. A *separating twist* is a Dehn twist T_c, where c is a separating curve on S. Separating twists $T_c \in \mathcal{I}(S)$ are elements of the Torelli group. A pair of curves $\{a, b\} \subset S$ is said to be a *bounding pair* if a, b are individually nonseparating, but $a \cup b$ bounds a subsurface of S of positive genus on both sides. A *bounding pair map* is the Dehn multitwist $T_a T_b^{-1}$; necessarily $T_a T_b^{-1} \in \mathcal{I}(S)$ for any bounding pair $\{a, b\}$.

Purity and the Torelli group. For a general mapping class f, we have remarked above that there exists some $n \geq 1$ such that f^n fixes each element of $\mathrm{CRS}(f)$ as well as each component of $\Sigma_g \setminus \mathrm{CRS}(f)$ and the restriction of f^n to each component of $\Sigma_g \setminus \mathrm{CRS}(f)$ is either pseudo-Anosov or trivial. A mapping class is said to be *pure* if $n = 1$ suffices; a subgroup $\Gamma \leq \mathrm{Mod}(\Sigma_g)$ is said to be *pure* if all elements are pure. In the sequel we will often use the following result without comment.

Lemma 2.6. *For $g \geq 1$, the Torelli group $\mathcal{I}(\Sigma_g)$ is pure.*

Proof. According to [Iva92, Corollary 1.8], for $g \geq 1$ the level-3 mapping class group $\mathrm{Mod}(\Sigma_g)[3]$ is pure in the above sense; purity passes to subgroups. $\qquad\square$

2.3. **Point- and disk-pushing subgroups.** The kernel $\pi_1(\Sigma_g, *)$ of the Birman exact sequence is referred to as the *point-pushing subgroup* of $\mathrm{Mod}(\Sigma_{g,*})$. An element $\alpha \in \pi_1(\Sigma_g, *)$ determines a mapping class $\alpha \in \mathrm{Mod}(\Sigma_{g,*})$ as follows: one "pushes" the marked point $*$ along the loop determined by α. In the course of our argument, we will have occasion to consider two apparently distinct notions of *simplicity* for curves on Σ_g. We pause here to explain that these are actually equivalent. An element $x \in \pi_1(\Sigma_g, *)$ is said to be *simple* if it has a simple closed curve representative as an *unbased* curve on Σ_g, and $x \in \pi_1(\Sigma_g, *)$ is said to be *based-simple* if it has a simple closed curve representative based at $*$.

Lemma 2.7. *Let $x \in \pi_1(\Sigma_g, *)$ be given. Then x is based-simple if and only if it is simple.*

Proof. Certainly if x is based-simple then it is simple. For the converse, we observe that if x is simple, then there is some conjugate of x that is based-simple. We claim that if *some* element of the conjugacy class of x is based-simple, then *every* element of the conjugacy class is based-simple. To see this, suppose that yxy^{-1} is based-simple with based representative ξ. The point-pushing subgroup $\pi_1(\Sigma_g, *)$ of $\mathrm{Mod}(\Sigma_g, *)$ acts by inner automorphisms on $\pi_1(\Sigma_g, *)$. Let Y be a homeomorphism of $(\Sigma_g, *)$ determining the point-push along $y \in \pi_1(\Sigma_g, *)$; then $Y^{-1}(\xi)$ is a simple loop based at $*$ in the based homotopy class of x. $\qquad\square$

Following Lemma 2.7, we will use the term *simple* to refer to both based and unbased simplicity.

There is an analogous notion of a "disk-pushing subgroup". Let $S = \Sigma_{g,1}$ denote a surface of genus g with one boundary component. In this setting, the Birman exact sequence is originally due to Johnson (see [Joh83, Section 3]) and takes the form

$$1 \to \pi_1(UT\Sigma_g) \to \mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_g) \to 1. \tag{3}$$

Here, $UT\Sigma_g$ denotes the *unit tangent bundle* of Σ_g; i.e., the S^1-subbundle of the tangent bundle $T\Sigma_g$ consisting of unit-length tangent vectors (relative to an arbitrarily-chosen Riemannian metric). In this context, the kernel $\pi_1(UT\Sigma_g)$ is known as the *disk-pushing subgroup*. An element $\widetilde{\alpha} \in \pi_1(UT\Sigma_g)$ determines a "disk-pushing" homeomorphism of $\Sigma_{g,1}$ as follows: one treats the boundary component Δ as the boundary of a disk D, and "pushes" D along the path determined by the image $\alpha \in \pi_1(\Sigma_g)$. The extra information of the tangent vector encoded in $\widetilde{\alpha}$ is used to give a consistent framing of ∂D along its path. By convention, if $\widetilde{\alpha} \in \pi_1(UT\Sigma_g)$ is specified, the symbol α will always denote the projection of $\widetilde{\alpha}$ to $\pi_1(\Sigma_g)$.

The proposition below records some basic facts about point- and disk-pushing subgroups. In item 5 below, the *support* of a (not necessarily simple) element $\alpha \in \pi_1(\Sigma_g)$ is defined to be the minimal subsurface $S_\alpha \subset \Sigma_{g,*}$ that contains α for which every component of ∂S_α is *essential*, i.e., non-nullhomotopic and non-peripheral.

Proposition 2.8.

(1) *There are containments $\pi_1(\Sigma_g) \leq \mathcal{I}(\Sigma_{g,*})$ and $\pi_1(UT\Sigma_g) \leq \mathcal{I}(\Sigma_{g,1})$.*

(2) *Let $\alpha \in \pi_1(\Sigma_g)$ be a* simple *element. Viewed as a point-push map, α has an expression as a bounding pair map*

$$\alpha = T_{\alpha_L} T_{\alpha_R}^{-1},$$

where α_L, α_R are the simple closed curves on $\Sigma_{g,*}$ lying to the left (resp. right) of α (by convention, Dehn twists are left-handed).

(3) *Let $\zeta \in \pi_1(UT\Sigma_g)$ be a generator of the kernel of the map $\pi_1(UT\Sigma_g) \to \pi_1(\Sigma_g)$. Viewed as a push map, ζ is equal to T_Δ, the twist about the boundary component of $\Sigma_{g,1}$.*

(4) *Let $\widetilde{\alpha} \in \pi_1(UT\Sigma_g)$ be simple (in the sense that $\alpha \in \pi_1(\Sigma_g)$ can be represented as a simple closed curve). Viewed as a disk-pushing map, there is an expression*

$$\widetilde{\alpha} = T_{\alpha_L} T_{\alpha_R}^{-1} T_\Delta^k$$

for some $k \in \mathbb{Z}$.

(5) *Let $\alpha \in \pi_1(\Sigma_g)$ be an arbitrary (not necessarily simple) element. Then*

$$\mathrm{CRS}(\alpha) = \partial(S_\alpha),$$

the (possibly empty) boundary of the support S_α. Moreover, when α is non-simple as a loop, the push map α is pseudo-Anosov on the subsurface S_α.

Proof. Items (1)- (4) are standard; see [FM12, Chapters 4,6] for details. Item (5) is a reformulation of a theorem of Kra, adapted to the language of canonical reduction systems. See [FM12, Theorem 14.6]. □

In Section 3, we will make use of the following lemma concerning the action of separating twist maps on the underlying fundamental group.

Lemma 2.9. *Let $T_c \in \mathcal{I}(\Sigma_{g,*})$ be a Dehn twist about a separating simple closed curve c. Let $\alpha \in \pi_1(\Sigma_g)$ be an arbitrary element, represented as a (not necessarily simple) curve based at $* \in \Sigma_{g,*}$. If*

$$T_c^k(\alpha) = \alpha$$

for any $k \neq 0$, then there exists a representative of α that is disjoint from c.

Proof. If $T_c^k(\alpha) = \alpha$ for some $k \neq 0$, then T_c^k and α commute as elements of $\mathcal{I}(\Sigma_{g,*})$. By Propositions 2.3, 2.4, and 2.8.5, we have that $\mathrm{CRS}(\alpha) = \partial(S_\alpha)$ and $\mathrm{CRS}(T_c^k) = \{c\}$ must be disjoint. Thus c is either contained in S_α or else in $\Sigma_{g,*} \setminus S_\alpha$.

If α is a non-simple loop then by Lemma 2.8.5, the push map α is pseudo-Anosov on S_α and hence does not fix any isotopy class of curves on S_α. But since α and T_c^k commute, necessarily α fixes c, showing that $c \subset \Sigma_{g,*} \setminus S_\alpha$ and hence α and c admit disjoint representatives as claimed.

In the degenerate case where α is simple and c is contained in S_α the claim still holds, since in this case α and c must actually be disjoint as isotopy classes of curves. □

2.4. **The handle-pushing subgroup.** As in Mess's approach, we will prove Theorem A by showing that certain "handle-pushing" subgroups (contained in any finite-index subgroup of $\mathcal{I}(\Sigma_g)$) do not admit sections to $\mathcal{I}(\Sigma_{g,*})$. To define these, let c be a separating curve on Σ_g. The complement $\Sigma_g \setminus \{c\}$ has closure consisting of two connected components P and Q, with $P \cong \Sigma_{p,1}$ and $Q \cong \Sigma_{q,1}$. Let $\mathcal{I}(c) \leq \mathcal{I}(\Sigma_g)$ be the subgroup consisting of Torelli mapping classes that are a product of mapping classes with supports on either P or Q. The subgroup $\mathcal{I}(c)$ satisfies the following exact sequence (c.f. [FM12, Theorem 3.18]):

$$1 \to \mathbb{Z} \to \mathcal{I}(P) \times \mathcal{I}(Q) \to \mathcal{I}(c) \to 1,$$

where $\mathbb{Z}$ is generated by (T_c, T_c^{-1}).

Definition 2.10. [Handle-pushing subgroup] Let c be a separating curve as in Figure 1, dividing $\Sigma_g \setminus \{c\}$ into surfaces P and Q of genera p and q, respectively. The *handle-pushing subgroup* on P, written $\mathcal{H}(P)$, is defined as

$$\mathcal{H}(P) := \pi_1(UTP) \leq \mathcal{I}(c).$$

More broadly, any finite-index subgroup of $\mathcal{H}(P)$ will also be called a handle-pushing subgroup.

Remark 2.11. Every finite-index subgroup of $\mathcal{H}(P)$, being isomorphic to a finite-index subgroup of $\pi_1(UTP)$, is isomorphic to a non-split extension of a surface group of genus $p' \geq p$ by $\mathbb{Z}$. The persistence of this phenomenon to every finite-index subgroup of $\mathcal{I}(\Sigma_g)$ provides the key family of relations we exploit to prove Theorem A.
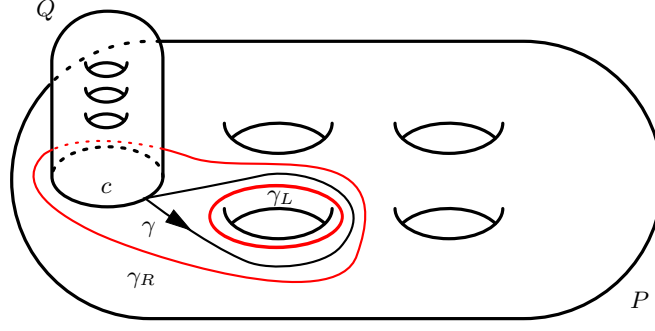
FIGURE 1. The mapping class $T_{\gamma_L} T_{\gamma_R}^{-1}$ is an element of the handle-pushing subgroup for P.

Denote by $A \leq \mathcal{I}(c)$ the group generated by the disk-pushing subgroups on both subsurfaces P and Q. Then A satisfies the following exact sequence:

$$1 \to \mathbb{Z} \to \pi_1(UTP) \times \pi_1(UTQ) \xrightarrow{\pi} A \to 1. \tag{4}$$

Lemma 2.12. *For p, q both at least 2, the exact sequence (4) does not virtually split.*

Proof. We begin with the following claim.

Claim. *Let G be a group and let*

$$1 \to \mathbb{Z} \to \widetilde{G} \xrightarrow{\alpha} G \to 1 \tag{5}$$

be a $\mathbb{Z}$-central extension of G. Let $b_1(G)$ denote the rational first Betti number of G, and define $b_1(\widetilde{G})$ similarly. Then if $b_1(G) = b_1(\widetilde{G})$, the sequence (5) does not virtually split.

To prove the claim, let $G' \leq G$ be a finite-index subgroup determining a pullback

$$1 \to \mathbb{Z} \to \widetilde{G'} \xrightarrow{\alpha'} G' \to 1 \tag{6}$$

of (5). The five-term exact sequences for (5) and (6) with rational coefficients (c.f. [Bro94, Corollary VII.6.4]) fit together to give the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
H_2(G') & \xrightarrow{d} & \mathbb{Q} & \longrightarrow & H_1(\widetilde{G'}) & \xrightarrow{\widetilde{\alpha}_*} & H_1(G') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H_2(G) & \xrightarrow{d} & \mathbb{Q} & \longrightarrow & H_1(\widetilde{G}) & \xrightarrow{\alpha_*} & H_1(G) & \longrightarrow & 0
\end{array}
\tag{7}
$$

According to the theory of the Euler class for group extensions (c.f. [Bro94, Sections IV.3 and VII.6]), if $d : H_2(G') \to \mathbb{Q}$ is nonzero, then the sequence (6) does not split. Since $G' \leq G$ is a subgroup of finite index, the theory of the transfer map implies that the map $H_2(G') \to H_2(G)$ is a surjection, and thus it suffices to show that $d : H_2(G) \to \mathbb{Q}$ is nonzero. By exactness of the bottom row of (7), d is nonzero if and only if α_* is an isomorphism, or equivalently if and only if $b_1(G') = b_1(G)$.

By the Claim, to prove Lemma 2.12 we only need to show that $b_1(A) = b_1(\pi_1(UTP) \times \pi_1(UTQ))$. However, since $p \geq 2$ and $q \geq 2$ by assumption, $b_1(\pi_1(UTP)) = b_1(\pi_1(P))$ (and similarly replacing P with Q), and hence

$$b_1(\pi_1(UTP) \times \pi_1(UTQ)) = b_1(\pi_1(P) \times \pi_1(Q)).$$

Since there is a surjective map $A \to \pi_1(P) \times \pi_1(Q)$, it follows that

$$b_1(A) \geq b_1(\pi_1(P) \times \pi_1(Q)) = b_1(\pi_1(UTP) \times \pi_1(UTQ)).$$

On the other hand, π gives a surjective map $\pi_1(UTP) \times \pi_1(UTQ) \to A$, and hence

$$b_1(A) \leq b_1(\pi_1(UTP) \times \pi_1(UTQ)).$$

Thus $b_1(A) = b_1(\pi_1(UTP) \times \pi_1(UTQ))$ as desired. $\qquad\qquad\qquad\qquad\qquad$ $\square$

2.5. **Lifts of some special mapping classes.** The foundation of the proof of Theorem A is an analysis of the possible images of bounding pair maps and separating twists under a hypothetical section. This is recorded as Lemma 2.14. Throughout this subsection, fix a finite-index subgroup $\Gamma \leq \mathcal{I}(\Sigma_g)$ and a hypothetical section $\sigma : \Gamma \to \mathrm{Mod}(\Sigma_{g,*})$ of the Birman exact sequence for Γ. We first record a useful preliminary observation.

Lemma 2.13. *With Γ and σ fixed as above, necessarily $\sigma(\Gamma) \leq \mathcal{I}(\Sigma_{g,*})$.*

Proof. Proposition 2.8.1 observes that $\pi_1(\Sigma_g) \leq \mathcal{I}(\Sigma_{g,*})$. Thus the restriction of the Birman exact sequence to $\mathcal{I}(\Sigma_g)$ takes the form

$$1 \to \pi_1(\Sigma_g) \to \mathcal{I}(\Sigma_{g,*}) \to \mathcal{I}(\Sigma_g) \to 1,$$

and consequently any section $\sigma : \mathcal{I}(\Sigma_g) \to \mathrm{Mod}(\Sigma_{g,*})$ is valued in $\mathcal{I}(\Sigma_{g,*})$. *A fortiori* the same holds for any subgroup $\Gamma \leq \mathcal{I}(\Sigma_g)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Since Γ is a finite-index subgroup of $\mathcal{I}(\Sigma_g)$, there is no assumption that a given separating twist T_c or bounding pair map $T_a T_b^{-1}$ is an element of Γ. However, the assumption that Γ is of finite index in $\mathcal{I}(\Sigma_g)$ does imply that each separating twist T_c and each bounding pair map $T_a T_b^{-1}$ has some power in Γ. In the following lemma and throughout, for a curve $\widetilde{c}$ on $\Sigma_{g,*}$ (resp. $\Sigma_{g,1}$), when we say $\widetilde{c}$ is *unmarked-isotopic* to a curve c on Σ_g, we mean that $\widetilde{c}$ is isotopic to c after forgetting the marked point (resp. boundary component).

Lemma 2.14.

(1) Let $\{a, b\}$ be a bounding pair, and fix $k > 0$ such that $(T_a T_b^{-1})^k \in \Gamma$. Up to a swap of a and b, we have that $\sigma((T_a T_b^{-1})^k) = (T_{a'} T_{b'}^{-1})^k (T_{a'}^{-1} T_{a''})^n$, where n is an integer (possibly zero) and a', a'', b' are three disjoint curves on $\Sigma_{g,*}$ such that a', a'' are unmarked-isotopic to a and b' is unmarked-isotopic to b.

(2) Let $c \subset \Sigma_g$ be a separating curve such that each component of $\Sigma_g \setminus \{c\}$ has genus at least 2, and let $k > 0$ be such that $T_c^k \in \Gamma$. Then there exists a curve $c' \subset \Sigma_{g,*}$ unmarked-isotopic to c such that $\sigma(T_c^k) = T_{c'}^k$.

Proof. We decompose the proof into the following five steps.

Claim 2.15 (**Step 1**). *Let a, b form a bounding pair on Σ_g. Then $\sigma((T_a T_b^{-1})^k)$ is reducible, i.e., $CRS(\sigma((T_a T_b^{-1})^k))$ is nonempty. Similarly if c is an arbitrary separating curve on Σ_g, then $\sigma(T_c^k)$ is reducible as well.*

Proof. The proofs of the two assertions are functionally identical; we describe the bounding-pair case. Let $(T_a T_b^{-1})^k \in \Gamma$ be a power of a bounding pair map. Since the centralizer of $(T_a T_b^{-1})^k$ contains a copy of $\mathbb{Z}^{2g-3}$ as a subgroup of $\mathcal{I}(\Sigma_g)$ (see Figure 2), the centralizer of $(T_a T_b^{-1})^k$ as a subgroup of Γ contains a copy of $\mathbb{Z}^{2g-3}$ as well. By the injectivity of σ, the centralizer of $\sigma(T_a T_b^{-1}) \in \mathcal{I}(\Sigma_{g,*})$ contains a copy of $\mathbb{Z}^{2g-3}$. When $g > 3$, we have that $2g - 3 > 3$. Therefore $\sigma((T_a T_b^{-1})^k) \in \mathcal{I}(\Sigma_{g,*})$ cannot be pseudo-Anosov because the centralizer of a pseudo-Anosov element is a virtually cyclic group by Proposition 2.5. $\qquad\square$
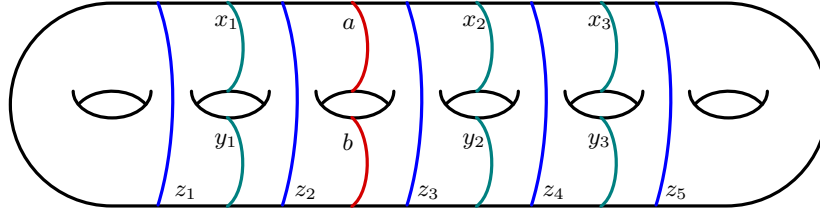

FIGURE 2. The $g-2$ bounding pair maps $T_a T_b^{-1}$ and $T_{x_i} T_{y_i}^{-1}$, and the $g-1$ separating twists T_{z_i}, illustrated for $g = 6$. Altogether this determines a copy of $\mathbb{Z}^{2g-3}$ centralizing $(T_a T_b^{-1})^k$.

In the remaining steps, we adopt the following notational convention. For any curve γ' on $\Sigma_{g,*}$, denote by γ the same curve on Σ_g.

Claim 2.16 (**Step 2**).
  (1) *$CRS(\sigma((T_a T_b^{-1})^k))$ only contains curves that are unmarked-isotopic to a or b.*
  (2) *If c is a separating curve that bounds a subsurface of genus at least 2 on both sides, then $CRS(\sigma(T_c^k))$ only contains curves that are unmarked-isotopic to c.*
  (3) *If c is a separating curve that bounds a surface $S \subset \Sigma_g$ of genus 1, then $CRS(\sigma(T_c^k))$ only contains curves that lie in S up to unmarked isotopy.*

Proof. We begin by formulating an assertion which implies all three statements. Let f denote either $(T_a T_b^{-1})^k$ or T_c^k. Either $\{a, b\}$ or $\{c\}$ separates Σ_g into two subsurfaces C_1, C_2.

Claim. *Suppose that C_i has Euler characteristic at most -2. Fix $\gamma' \in CRS(\sigma(f))$. If the associated γ is supported on C_i, then γ' is unmarked-isotopic to a component of ∂C_i.*

To prove the claim, suppose otherwise. Since $\chi(C_i) \leq -2$ and C_i has positive genus, there exists a separating curve d on C_i such that $i(d, \gamma) \neq 0$. Choose m such that $T_d^m \in \Gamma$. Since f and T_d^m commute in Γ, the two mapping classes $\sigma(f)$ and $\sigma(T_d^m)$ commute in $\mathcal{I}(\Sigma_{g,*})$. Therefore a power of $\sigma(T_d^m)$ fixes

$\mathrm{CRS}(\sigma(f))$; more specifically a power of T_d^m fixes γ. However by Lemma 2.2, no power of T_d fixes γ. This is a contradiction. □

Step 2 establishes the following picture for the canonical reduction system of $\sigma((T_a T_b^{-1})^k)$, shown in Figure 3. There is a similar picture for $\mathrm{CRS}(\sigma(T_c^k))$ which we omit.
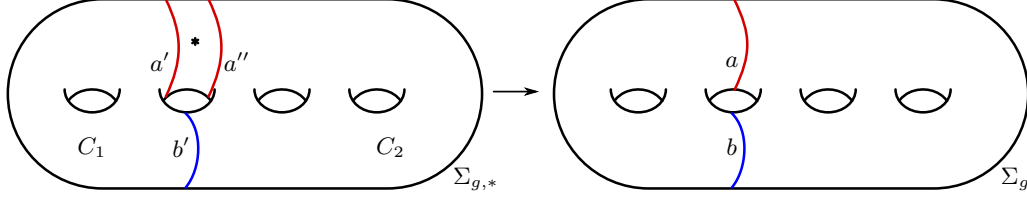

FIGURE 3. The canonical reduction system for $\sigma((T_a T_b^{-1})^k)$. Since the curves must be disjoint, there can be at most two curves a', a'' (resp. b', b'') that are unmarked-isotopic to a (resp. b), and moreover there can be at most three total curves. Without loss of generality we eliminate b''. We further note that following Step 2 we only know that the canonical reduction system is nonempty; prior to Step 3 it is possible that one or more of a', a'', b' does not appear.

Claim 2.17 (Step 3). *$CRS(\sigma((T_a T_b^{-1})^k))$ must contain curves a' and b' that are unmarked-isotopic to a and b, respectively. Similarly, if c is a separating curve that bounds subsurfaces of genus at least 2 on both sides, then $CRS(\sigma(T_c^k))$ must contain a curve c' that is unmarked-isotopic to c.*

Proof. For the case of a separating curve c this follows by combining Steps 1 and 2. Suppose that $\mathrm{CRS}(\sigma((T_a T_b^{-1})^k))$ does not contain a curve a' unmarked-isotopic to a. Then by Step 2, $\mathrm{CRS}(\sigma((T_a T_b^{-1})^k))$ either consists of one curve b' unmarked-isotopic to b or two curves b' and b'' both unmarked-isotopic to b. After cutting $\Sigma_{g,*}$ along $\mathrm{CRS}(\sigma((T_a T_b^{-1})^k))$, there is exactly one component C that is not a punctured annulus. Ignoring the marked point, C is homeomorphic to the complement of b in Σ_g.

Since the Torelli group is pure (Lemma 2.6), $\sigma((T_a T_b^{-1})^k)$ is either pseudo-Anosov on C or else is the identity on C. If $\sigma((T_a T_b^{-1})^k)$ is pseudo-Anosov on C, then the centralizer of $\sigma((T_a T_b^{-1})^k)|_C$ is virtually cyclic by Proposition 2.5. Combining with $T_{b'}$ and $T_{b''}$, the centralizer of $\sigma((T_a T_b^{-1})^k)$ in $\mathcal{I}(\Sigma_{g,*})$ is then virtually an abelian group of rank at most 3. This contradicts the fact that the centralizer of $\sigma((T_a T_b^{-1})^k)$ contains a subgroup $\mathbb{Z}^{2g-3}$, since $g \geq 4$ and hence $2g - 3 > 3$. Therefore $\sigma((T_a T_b^{-1})^k)$ is the identity on C. However, $(T_a T_b^{-1})^k$ is not the identity on C (here we view C as a subsurface of Σ_g by forgetting the marked point). Since σ is a section, $\sigma((T_a T_b^{-1})^k)$ does not act as the identity on C either, a contradiction. □

Claim 2.18 (Step 4). *$\sigma((T_a T_b^{-1})^k) = (T_{a'} T_{b'}^{-1})^k (T_{a'}^{-1} T_{a''})^n$, where n is an integer and a', a'', b' are three disjoint curves on $\Sigma_{g,*}$ such that a', a'' are unmarked-isotopic to a and b' is unmarked-isotopic to b. Similarly $\sigma(T_c^k) = T_{c'}^k (T_{c'}^{-1} T_{c''})^n$, where c' and c'' are disjoint and unmarked-isotopic to c.*

Proof. With reference to Figure 3, it suffices to show that $\sigma((T_a T_b^{-1})^k)$ cannot be pseudo-Anosov on either positive-genus component C_i of

$$\Sigma_{g,*} \setminus \mathrm{CRS}(\sigma((T_a T_b^{-1})^k)).$$

Since C_i has positive genus and Euler characteristic at most -2, there exists a curve s on C_i that is separating on Σ_g. Thus $\sigma(T_s^m)$ commutes with $\sigma((T_a T_b^{-1})^k)$ in $\sigma(\Gamma)$. It follows that $\sigma((T_a T_b^{-1})^k)$ fixes $\mathrm{CRS}(\sigma(T_s^m))$. By Step 1, $\mathrm{CRS}(\sigma(T_s^m))$ is nonempty and by Step 2 each curve is either unmarked-isotopic to s or else contained in a surface of genus 1 bounded by s. In either case, $\mathrm{CRS}(\sigma(T_c^k))$ includes some non-peripheral curve on C_i, and so $\sigma((T_a T_b^{-1})^k)$ is not pseudo-Anosov on C_i. It follows that $\sigma((T_a T_b^{-1})^k)$ must be a product of Dehn twists about the curves in $\mathrm{CRS}(\sigma((T_a T_b^{-1})^k))$. Since $\sigma((T_a T_b^{-1})^k)$ is a lift of $(T_a T_b^{-1})^k$, the claim holds.   □

Observe that at this point, Lemma 2.14.1 has been established.

Claim 2.19 (Step 5). *Let $c \subset \Sigma_g$ be a separating curve such that each component of $\Sigma_g \setminus \{c\}$ has genus at least 2, and let $k > 0$ be such that $T_c^k \in \Gamma$. Then there exists a curve $c' \subset \Sigma_{g,*}$ unmarked-isotopic to c such that $\sigma(T_c^k) = T_{c'}^k$.*

Proof. If this is not the case, then $\sigma(T_c^k) = T_{c'}^l T_{c''}^m$ where c', c'' bound an annulus and $l \neq 0, m \neq 0$. Let A be the subgroup constructed in (4) above, relative to the separating curve c. Since A centralizes T_c, the image $\sigma(A \cap \Gamma)$ must be contained in the centralizer of $T_{c'}^l T_{c''}^m$. Observe that the centralizer of $T_{c'}^l T_{c''}^m$ in $\mathcal{I}(\Sigma_g)$ must necessarily fix each of c' and c'', since any mapping class exchanging c' and c'' must exchange the subsurfaces bounded by c' and c'' and hence act nontrivially on $H_1(\Sigma_{g,*})$. It follows that $\sigma(A \cap \Gamma)$ must be contained in the disk-pushing subgroups on the sides of c' and c'' not bounding the annulus. This gives a virtual splitting of exact sequence (4), contradicting Lemma 2.12.   □

                                                                                    □

3. Proof of Theorem A

Beginning the proof. For the sake of obtaining a contradiction, we assume that there exists $\Gamma \leq \mathcal{I}(\Sigma_g)$ a subgroup of finite index for which $\sigma : \Gamma \to \mathcal{I}(\Sigma_{g,*})$ is a section. Our first goal is to give the construction of a pair of diagrams (8) and (10); following this, we will use these to construct the map s at the heart of our argument.

By the hypothesis that $g \geq 4$, there exists a separating simple closed curve $c \subset \Sigma_g$ that divides Σ_g into subsurfaces P and Q with $p, q \geq 2$. Let T_c denote the corresponding Dehn twist. Choosing k such that $T_c^k \in \Gamma$, Lemma 2.14.2 asserts that $\sigma(T_c^k) = T_{\widetilde{c}}^k$ for some separating curve $\widetilde{c} \subset \Sigma_{g,*}$. The curve $\widetilde{c}$ divides $\Sigma_{g,*}$ into two subsurfaces $\widetilde{P}$ and $\widetilde{Q}$, respectively homeomorphic to P and Q after forgetting $*$. Without loss of generality, we assume that the marked point $*$ lies in $\widetilde{P}$.

Let $\overline{P}$ be the (closed) surface obtained from P by capping the boundary component with a disk. The group $\mathrm{PB}_{1,1}(\overline{P})$ is then defined as the fundamental group of the configuration space $\mathrm{PConf}_{1,1}(\overline{P})$,

where

$$\mathrm{PConf}_{1,1}(\overline{P}) := \{(x,v) \mid x \in \overline{P}, v \in T^1_y(\overline{P}), \ x \neq y\}.$$

Here, $T^1_y(\overline{P})$ denotes the space of unit-length tangent vectors in the tangent space $T_y(\overline{P})$, relative to an arbitrarily-chosen Riemannian metric. Projection onto either factor realizes $\mathrm{PConf}_{1,1}(\overline{P})$ as a fibration in two ways. Below P' denotes the surface obtained from $\overline{P}$ by removing a point, so that $P' \cong \Sigma_{p,*}$.

$$\begin{array}{ccc}
& P' & \quad (8) \\
& \downarrow & \\
UTP' \longrightarrow \mathrm{PConf}_{1,1}(\overline{P}) \xrightarrow{\;p_1\;} \overline{P} \\
& \Big\downarrow{\scriptstyle p_2} & \\
& UT\overline{P} &
\end{array}$$

The commutative diagram (9) below relates the Birman exact sequences for $\widetilde{P}$ and P, restricted to their respective Torelli groups. One subtlety here is in the definition of the Torelli group for $\widetilde{P}$: we define $\mathcal{I}(\widetilde{P})$ to simply be the full preimage $\pi^{-1}(\mathcal{I}(P))$ under the projection $\pi : \mathrm{Mod}(\widetilde{P}) \to \mathrm{Mod}(P)$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{PB}_{1,1}(\overline{P}) & \longrightarrow & \mathcal{I}(\widetilde{P}) & \longrightarrow & \mathcal{I}(\overline{P}) & \longrightarrow & 1 \qquad (9) \\
& & \Big\downarrow{\scriptstyle (p_2)_*} & & \Big\downarrow{\scriptstyle \pi} & & \Big\| & & \\
1 & \longrightarrow & \pi_1(UT\overline{P}) & \longrightarrow & \mathcal{I}(P) & \longrightarrow & \mathcal{I}(\overline{P}) & \longrightarrow & 1
\end{array}$$

We wiill need to understand how the section $\sigma : \Gamma \to \mathcal{I}(\Sigma_{g,*})$ restricts to $\mathcal{I}(P)$.

Lemma 3.1. *Given $f \in \mathrm{Mod}(P) \cap \Gamma$, the lift $\sigma(f)$ is supported on $\widetilde{P}$. Consequently the restriction of σ to $\mathrm{Mod}(P) \cap \Gamma$ is valued in $\mathcal{I}(\widetilde{P})$.*

Proof. We begin with the following claim.

Claim. *Let $f \in \mathrm{Mod}(P) \cap \Gamma$ be given. Then $\sigma(f)$ preserves $\widetilde{P}$ and $\widetilde{Q}$.*

To prove the claim, let k be given so that $T_c^k \in \Gamma$. Since f and T_c^k commute, so do $\sigma(f)$ and $\sigma(T_c^k) = T_{\widetilde{c}}^k$, the latter equality holding by Lemma 2.14.2. Thus $\sigma(f)$ preserves $\mathrm{CRS}(\sigma(T_{\widetilde{c}}^k)) = \{\widetilde{c}\}$. As $\sigma(f) \in \mathcal{I}(\Sigma_{g,*})$ by Lemma 2.13, it follows that $\sigma(f)$ moreover preserves the subsurfaces $\widetilde{P}$ and $\widetilde{Q}$ bounded by $\widetilde{c}$.

Following the claim, it remains to see that $\sigma(f)$ restricts trivially to $\widetilde{Q}$. To see this, let $g \in \Gamma$ be any element supported on Q. Applying the claim to f and g, it follows that both $\sigma(f)$ and $\sigma(g)$ restrict to $\widetilde{Q}$. As f and g commute, so do $\sigma(f)$ and $\sigma(g)$. Since this holds for *arbitrary* g, we conclude that $\sigma(f)$ must restrict trivially to $\widetilde{Q}$ as claimed. $\qquad\square$

Lemma 3.1 and diagram (9) can be combined into the diagram below, where the dashed arrow indicates that σ is only defined on the finite-index subgroup $\Gamma \leq \mathcal{I}(P)$.

$$1 \longrightarrow \mathrm{PB}_{1,1}(\overline{P}) \longrightarrow \mathcal{I}(\widetilde{P}) \longrightarrow \mathcal{I}(\overline{P}) \longrightarrow 1 \qquad (10)$$

$$1 \longrightarrow \pi_1(UT\overline{P}) \longrightarrow \mathcal{I}(P) \longrightarrow \mathcal{I}(\overline{P}) \longrightarrow 1$$

(with vertical maps $(p_2)_*$, σ on the left; π, σ in the middle; and the identity on the right)

3.1. **The map** s. We now come to the central object of study in the argument. Let $\mathcal{H} = \mathcal{H}(P) \cap \Gamma$ denote the handle-pushing subgroup inside $\pi_1(UT\overline{P})$ (c.f. Definition 2.10). Combining diagrams (10) and (8), we obtain a homomorphism

$$\widetilde{s} := (p_1)_* \circ \sigma : \mathcal{H} \to \pi_1(\overline{P}). \qquad (11)$$

We will see that $\widetilde{s}$ has paradoxical properties, leading to a contradiction that establishes the non-existence of the section σ. A first observation, to be recorded in Lemma 3.2 below, is that we can replace $\widetilde{s}$ by a map between surface groups. Let $\varpi : \pi_1(UT\overline{P}) \to \pi_1(\overline{P})$ denote the projection, and define $\overline{\mathcal{H}} := \varpi(\mathcal{H})$. By construction, $\overline{\mathcal{H}}$ is a finite-index subgroup of $\pi_1(\overline{P})$ (c.f. Remark 2.11).

Lemma 3.2. *There is a homomorphism*

$$s : \overline{\mathcal{H}} \to \pi_1(\overline{P}) \qquad (12)$$

such that $\widetilde{s}$ factors as $\widetilde{s} = s \circ \varpi$.

Proof. As noted in Remark 2.11, $\mathcal{H}$ has the structure of a cyclic central extension of a finite-index subgroup $\overline{\mathcal{H}} \le \pi_1(\overline{P})$. Viewed as a subgroup of $\mathcal{I}(P)$, the center of $\mathcal{H}$ consists of elements of the form T_c^k. Thus it will suffice to show that $(p_1)_*(\sigma(T_c^k)) = 1$. By Lemma 2.14.2, $\sigma(T_c^k) = T_{\widetilde{c}}^k$, where $\widetilde{c}$ is the boundary of the subsurface $\widetilde{P} \subset \Sigma_{g,*}$. The map $(p_1)_* : \mathrm{PB}_{1,1}(\overline{P}) \to \pi_1(\overline{P})$ is induced from the boundary-capping map $\widetilde{P} \to \overline{P}$, and so $(p_1)_*(T_{\widetilde{c}}^k) = 1$ as required. $\qquad\qquad \square$

The construction of s allows us to continue the analysis of σ begun in Lemma 2.14, giving a complete description of σ on (powers of) bounding-pair maps.

Lemma 3.3. *Let a, b form a bounding pair on Σ_g. Then there exists a bounding pair $\widetilde{a}, \widetilde{b}$ on $\Sigma_{g,*}$ such that $\sigma(T_a^k T_b^{-k}) = T_{\widetilde{a}}^k T_{\widetilde{b}}^{-k}$ for any k such that $T_a^k T_b^{-k} \in \Gamma$.*

Proof. Since $g \ge 4$, given any bounding pair a, b on Σ_g, it is possible to choose a separating curve c such that (1) c separates Σ_g into subsurfaces P, Q as above, each of genus at least 2 and (2) the curves a, b, c form a pair of pants. (Such a triple a, b, c is depicted in Figure 1; the bounding pair there consists of the curves γ_L, γ_R.) We continue to use the suite of notation $(\widetilde{c}, \widetilde{P}, \widetilde{Q}, \overline{P}$ etc.) introduced above.

Choose ℓ such that $T_c^\ell \in \Gamma$. As $T_a^k T_b^{-k}$ commutes with T_c^ℓ, the same is true for the lifts $\sigma(T_a^k T_b^{-k})$ and $\sigma(T_c^\ell) = T_{\widetilde{c}}^\ell$. In particular, $\sigma(T_a^k T_b^{-k})$ is supported on exactly one component $\widetilde{P}, \widetilde{Q}$ of the surface $\Sigma_{g,*} \setminus \{\widetilde{c}\}$. There are thus two possibilities to consider, depending on whether this component is $\widetilde{P}$ (which also contains $*$) or $\widetilde{Q}$.

According to Lemma 2.14.1, there are simple closed curves $\widetilde{a}, \widetilde{a}', \widetilde{b} \subset \Sigma_{g,*}$ and an integer m such that

$$\sigma(T_a^k T_b^{-k}) = T_{\widetilde{a}}^{k-m} T_{\widetilde{a}'}^m T_{\widetilde{b}}^{-k}. \tag{13}$$

The curves $\widetilde{a}$ and $\widetilde{a}'$ are both unmarked-isotopic to a, but may not be isotopic on $\Sigma_{g,*}$, i.e., $\widetilde{a} \cup \widetilde{a}'$ can bound an annulus A containing the marked point $*$. If this is not the case, then $\widetilde{a}, \widetilde{a}'$ determine the same isotopy class on $\Sigma_{g,*}$, and the result follows. Note that in the case where $A \subset \widetilde{Q}$, this must necessarily hold.

We therefore assume that $A \subset \widetilde{P}$. Since a, b, c form a pair of pants on Σ_g, it follows that $T_a^k T_b^{-k}$ is an element of $\mathcal{H}$, the handle-pushing subgroup. As before, we let $\overline{P}$ denote the closed surface obtained by capping off P. Then there is a *one-to-one correspondence* between elements of $\overline{\mathcal{H}}$ represented by (a power of) a *simple* closed curve on $\overline{P}$, and the set of bounding pairs a, b under consideration. We write $\alpha(a, b) \in \pi_1(\overline{P})$ for the element of $\overline{\mathcal{H}}$ corresponding to the bounding pair $T_a T_b^{-1}$. Our proof now proceeds by analyzing s on such elements of $\overline{\mathcal{H}}$.

As observed above, $*$ may or may not be contained in the annulus A. If $*$ is not, we can reformulate the above argument by observing that $s(\alpha(a, b)^k) = 1$. In the remaining case, we aim to show that either $m = 0$ or $m = k$ in (13). As (without loss of generality) $\widetilde{a}'$ becomes isotopic to $\widetilde{b}$ upon capping c by a disk, it follows that

$$s(\alpha(a, b)^k) = (p_1)_*(T_{\widetilde{a}}^{k-m} T_{\widetilde{a}'}^m T_{\widetilde{b}}^{-k}) = T_{\widetilde{a}}^{k-m} T_{\widetilde{b}}^{m-k} = \alpha(a, b)^{m-k}. \tag{14}$$

To summarize, we have shown that for all bounding pairs a, b under consideration, there is an integer $m(a, b, k)$ such that

$$s(\alpha(a, b)^k) = \alpha(a, b)^{m(a,b,k)}.$$

The desired assertion $m = 0$ or $m = k$ now follows from Lemma 3.4 below. $\qquad \square$

Lemma 3.4. *Let $G \leq \pi_1(\overline{P})$ be a subgroup of finite index, and let $f : G \to \pi_1(\overline{P})$ be an arbitrary homomorphism. Suppose that for all simple elements $\alpha \in \pi_1(\overline{P})$, there is an integer $m(\alpha, k)$ such that*

$$f(\alpha^k) = \alpha^{m(\alpha,k)}.$$

Then either $m(\alpha, k) = 0$ or else $m(\alpha, k) = k$, independent of α.

Proof. Suppose α, β are simple elements. Then for any ℓ, the conjugate $\beta^\ell \alpha \beta^{-\ell}$ is also simple (Lemma 2.7). Choose k, ℓ such that α^k and β^ℓ are both elements of G. Then definitionally,

$$f(\beta^\ell \alpha^k \beta^{-\ell}) = (\beta^\ell \alpha \beta^{-\ell})^{m(\beta^\ell \alpha \beta^{-\ell}, k)}. \tag{15}$$

On the other hand, since f is a homomorphism, it follows that $m(\beta, -\ell) = -m(\beta, \ell)$ and so

$$f(\beta^\ell \alpha^k \beta^{-\ell}) = f(\beta^\ell) f(\alpha^k) f(\beta^{-\ell}) = \beta^{m(\beta,\ell)} \alpha^{m(\alpha,k)} \beta^{-m(\beta,\ell)}. \tag{16}$$

For an arbitrary nontrivial element $\gamma \in \pi_1(\overline{P})$ and integers m, n, the elements γ^m and γ^n are conjugate if and only if $m = n$. It follows that $m(\alpha, k) = m(\beta^\ell \alpha \beta^{-\ell}, k)$. Thus,

$$(\beta^\ell \alpha \beta^{-\ell})^{m(\alpha,k)} = \beta^{m(\beta,\ell)} \alpha^{m(\alpha,k)} \beta^{-m(\beta,\ell)},$$

and so
$$\beta^{\ell-m(\beta,\ell)}\alpha^{m(\alpha,k)}\beta^{m(\beta,\ell)-\ell} = \alpha^{m(\alpha,k)}.$$

Nontrivial elements $x, y \in \pi_1(\overline{P})$ commute if and only if there are nonzero integers c, d such that $x^c = y^d$. As α, β were assumed to be simple, we conclude that one of three conditions must hold: (1) $\alpha = \beta^{\pm 1}$, or (2) $\ell = m(\beta, \ell)$ or else (3) $m(\alpha, k) = 0$.

Case (1) provides no further information; we henceforth assume that $\alpha \neq \beta^{\pm 1}$. To finish the argument, we must show that if $m(\alpha, k) = 0$, then $m(\beta, \ell) = 0$ for all β, ℓ. Suppose to the contrary that there is some β such that $m(\beta, \ell) \neq 0$. Reversing the roles of α and β in the above argument, we see that (2) must hold and so $k = m(\alpha, k)$, but this contradicts the assumption $m(\alpha, k) = 0$.   □

Translated into the setting of the homomorphism $s : \overline{\mathcal{H}} \to \pi_1(\overline{P})$, Lemmas 3.3 and 3.4 combine to give the following immediate but crucial corollary.

Corollary 3.5. *The homomorphism $s : \overline{\mathcal{H}} \to \pi_1(\overline{P})$ has one of the following properties:*

(A) $s(\alpha^k) = \alpha^k$ for all elements $\alpha^k \in \overline{\mathcal{H}}$ such that $\alpha \in \pi_1(\overline{P})$ is simple.
(B) $s(\alpha^k) = 1$ for all elements $\alpha^k \in \overline{\mathcal{H}}$ such that $\alpha \in \pi_1(\overline{P})$ is simple.

The next step of the argument considers cases (A) and (B) separately. In both cases, we will see that the formula defining s on simple elements extends to all of $\overline{\mathcal{H}}$.

3.2. Case (A).

Lemma 3.6. *Suppose s has property (A) of Corollary 3.5. Then $s : \overline{\mathcal{H}} \to \pi_1(\overline{P})$ is given by the inclusion map.*

Proof. This follows easily from the method of proof of Lemma 3.4. Let $\beta \in \overline{\mathcal{H}}$ be an arbitrary element, let $\alpha \in \pi_1(\overline{P})$ be simple, and let $\alpha^k \in \overline{\mathcal{H}}$. Then $\beta\alpha\beta^{-1}$ is also simple by Lemma 2.7, and $\beta\alpha^k\beta^{-1} \in \overline{\mathcal{H}}$. As $\beta\alpha\beta^{-1}$ is simple,
$$f(\beta\alpha^k\beta^{-1}) = \beta\alpha^k\beta^{-1};$$

on the other hand,
$$f(\beta\alpha^k\beta^{-1}) = f(\beta)\alpha^k f(\beta)^{-1}.$$

Arguing as in Lemma 3.4, this implies $f(\beta) = \beta$ as desired.   □

3.3. Case (B).

Lemma 3.7. *Suppose s has property (B) of Corollary 3.5. Then $s : \overline{\mathcal{H}} \to \pi_1(\overline{P})$ is the trivial homomorphism.*

We do not know whether property (B) of Corollary 3.5 already implies the fact that s is trivial, without the assumption that s is induced from a splitting of the Birman exact sequence for Γ. This seems to be a harder problem worthy of further study. The extra structure present in our situation is an equivariance property described below in Lemma 3.9. Before turning to the proof of Lemma 3.7, our first objective is to formulate and prove this.

By passing to a further finite-index subgroup of Γ if necessary, we can assume that $\mathcal{H} \le \pi_1(UT\overline{P})$ is *characteristic* and hence the conjugation action of $\mathcal{I}(P)$ on $\pi_1(UT\overline{P})$ preserves $\mathcal{H}$. Let P' denote the surface obtained from P by replacing the boundary component with a marked point. Then the action of $\mathcal{I}(P)$ on $\pi_1(UT\overline{P})$ descends to an action of $\mathcal{I}(P')$ on $\overline{\mathcal{H}}$ by conjugation. Thus there is a homomorphism

$$\lambda : \mathcal{I}(P') \to \mathrm{Aut}(\overline{\mathcal{H}}).$$

Consider now the images $\Gamma' \le \mathcal{I}(P')$ and $\overline{\Gamma} \le \mathcal{I}(\overline{P})$ of $\Gamma \cap \mathrm{Mod}(P)$ induced by the capping maps $P \to P'$ and $P \to \overline{P}$. By construction, $\overline{\Gamma} \cap \pi_1(\overline{P}) = \overline{\mathcal{H}}$. As conjugation by $\overline{\mathcal{H}}$ on itself is an inner automorphism, λ descends to a homomorphism

$$\overline{\lambda} : \overline{\Gamma} \to \mathrm{Out}(\overline{\mathcal{H}}). \tag{17}$$

Remark 3.8. $\overline{\mathcal{H}}$ is a finite-index subgroup of $\pi_1(\overline{P})$, and as such, corresponds to a finite-sheeted covering $R \to \overline{P}$. From a topological point of view, $\overline{\lambda}$ is the action of $\overline{\Gamma} \le \mathrm{Mod}(\overline{P})$ on R induced by *lifting* mapping classes from $\overline{P}$ to R.

Lemma 3.9. *The homomorphism s is $\overline{\Gamma}$-equivariant with respect to the action $\overline{\lambda}$ on $\overline{\mathcal{H}}$ and the standard outer action of $\overline{\Gamma}$ on $\pi_1(\overline{P})$. That is, for any outer automorphism $[\alpha] \in \overline{\Gamma}$ and any $x \in \overline{\mathcal{H}}$, the conjugacy classes of $s(\alpha \cdot x)$ and $\alpha \cdot s(x)$ in $\pi_1(\overline{P})$ coincide.*

Proof. Let $a \in \overline{\Gamma}$ be given. Choose an element $\alpha \in \Gamma$ descending to the outer automorphism class a. By construction, for $x \in \overline{\mathcal{H}}$, the image $s(x)$ is given by $((p_1)_* \circ \sigma)(\widetilde{x})$, where $\widetilde{x} \in \mathcal{H}$ is any lift. On $\mathcal{H}$, the action of $\overline{\Gamma}$ is induced by the conjugation action $\widetilde{x} \mapsto \alpha\widetilde{x}\alpha^{-1}$. Thus

$$s(a \cdot x) = (p_1)_*(\sigma(\alpha\widetilde{x}\alpha^{-1})) = (p_1)_*(\sigma(\alpha)) \, s(x) \, (p_1)_*(\sigma(\alpha))^{-1}.$$

Here we exploit the fact that $(p_1)_* : \mathrm{PB}_{1,1}(\overline{P}) \to \pi_1(\overline{P})$ is the restriction of the forgetful homomorphism

$$(p_1)_* : \mathcal{I}(\widetilde{P}) \to \mathcal{I}(\overline{P}).$$

To finish the argument, it suffices to show that $[(p_1)_*(\sigma(\alpha))] = a$ as elements of $\mathcal{I}(\overline{P})$. This follows from the fact that $\sigma : \Gamma \to \mathcal{I}(\widetilde{P})$ is a section of the map $(p_2)_* : \mathcal{I}(\widetilde{P}) \to \mathcal{I}(P)$ in combination with the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{I}(\widetilde{P}) & \xrightarrow{\ (p_1)_*\ } & \mathcal{I}(P') \\ {\scriptstyle (p_2)_*}\big\downarrow & & \big\downarrow \\ \mathcal{I}(P) & \longrightarrow & \mathcal{I}(\overline{P}). \end{array}$$

$\square$

Proof. (of Lemma 3.7) Let $x \in \overline{\mathcal{H}}$ be an arbitrary element, and let d be an arbitrary separating curve on P. Taking k such that $T_d^k \in \Gamma$ and applying Lemma 3.9, there is an equality

$$s(T_d^k(x)) = T_d^k(s(x))$$

of conjugacy classes in $\pi_1(\overline{P})$. To proceed, we will analyze the conjugacy class of $T_d^k(x)$ in $\overline{\mathcal{H}}$. This is complicated by the fact that in this expression, T_d^k acts on x by the action $\overline{\lambda}$ of (17). Following Remark 3.8, T_d^k acts *not* as a separating twist on $\overline{P}$, but rather as the *lift* of such a twist to a finite-sheeted cover $R \to \overline{P}$ corresponding to the finite-index subgroup $\overline{\mathcal{H}} \le \pi_1(\overline{P})$.

Lemma 3.10. *Let T_d be a Dehn twist on $\overline{P}$, and let $x \in \overline{\mathcal{H}}$ be an arbitrary element. Then there exists some $k \ge 1$, simple elements $\gamma_1, \ldots, \gamma_N$ of $\pi_1(\overline{P})$ and integers $f_1, \ldots, f_N$, such that $\gamma_i^{f_i} \in \overline{\mathcal{H}}$ for all i, and there is an expression*

$$T_d^k(x) = \gamma_1^{f_1} \ldots \gamma_N^{f_N} x$$

of elements of $\overline{\mathcal{H}}$.

Proof. Let $\pi : R \to \overline{P}$ be the covering map associated to the containment $\overline{\mathcal{H}} \le \pi_1(\overline{P})$. For k sufficiently large, T_d^k lifts to a mapping class on R. This lift is not unique, but there is a unique lift up to the action of the deck group of π. Since T_d is a Dehn twist on $\overline{P}$, there is a distinguished lift

$$\widetilde{T_d^k} = \prod T_{\widetilde{d_i}}^{k_i} \tag{18}$$

of T_d^k as a multitwist on R, for certain integers k_i. Here, the set $\{\widetilde{d_i}\}$ consists of all components of the preimage $\pi^{-1}(d)$. Observe that each curve $\widetilde{d_i}$ is contained in the $\pi_1(\overline{P})$-conjugacy class of d^{e_i} for some e_i, and that also the conjugacy class of d^{e_i} is contained in $\overline{\mathcal{H}}$. As the deck group is finite, we can assume that T_d^k acts on $\overline{\mathcal{H}}$ as in (18), possibly after further increasing k.

Choose representative curves for each $\widetilde{d_i}$, and represent $x \in \overline{\mathcal{H}}$ as a map $x(t) : [0,1] \to R$, chosen so as to intersect the set $\{\widetilde{d_i}\}$ in minimal position. This determines a sequence of arcs $\alpha_1, \ldots, \alpha_{N+1}$ as follows. The points of intersection between x and $\{\widetilde{d_i}\}$ can be enumerated via $0 < t_1 < \cdots < t_N < t_{N+1} = 1$ such that $x(t)$ intersects the multicurve $\{\widetilde{d_i}\}$ if and only if $t = t_m$ for some $1 \le m \le N$. The arc α_m is then defined as the image of x restricted to the interval $[0, t_m]$ (so in particular, $\alpha_{N+1} = x$).

Each arc α_m connects $*$ to one of the curves $\widetilde{d_i}$, and thus determines an element γ_m' of $\overline{\mathcal{H}}$ in the conjugacy class of the appropriate $\widetilde{d_i}$. The geometric description of T_d^k as a multitwist allows one to obtain an expression for $T_d^k(x)$ of the desired form. The curve $T_d^k(x)$ can be described as follows: first $T_d^k(x)$ follows α_1 to the first point of intersection with $\{\widetilde{d_i}\}$; this is the curve corresponding to γ_1'. Then $T_d^k(x)$ winds around γ_1' a number of times f_1' as specified by (18). Then $T_d^k(x)$ continues along the portion of α_2 running from $t = t_1$ to $t = t_2$, and continues, winding around each γ_i' some number of times f_i' in succession.

By construction, after each crossing of γ_m', the curve $T_d^k(x)$ traverses the portion of α_{m+1} from t_m to t_{m+1}. This can be replaced by first backtracking along α_m, and then traversing the entirety of α_{m+1}. Written as an element of $\pi_1(R) = \overline{\mathcal{H}}$, this analysis produces an expression

$$T_d^k(x) = \gamma_1'^{f_1'} \ldots \gamma_N'^{f_N'} x.$$

The claim now follows from the observation that each γ_m' is a based loop on R corresponding to a curve $\widetilde{d_i}$. Each $\widetilde{d_i}$ is a component of the preimage of d. As an element of $\pi_1(\overline{P})$, each γ_m' is thus of the form $\gamma_m' = \gamma_m^{e_m}$ for some *simple* curve $\gamma_m \in \pi_1(\overline{P})$ in the conjugacy class of d. Taking $f_m = e_m f_m'$, the result follows. $\square$

Applying Lemma 3.10, there is an equality

$$s(T_d^k(x)) = s(\gamma_1^{f_1} \ldots \gamma_N^{f_n} x) = s(\gamma_1^{f_1}) \ldots s(\gamma_N^{f_N}) s(x) = s(x), \tag{19}$$

with the last equality holding by Corollary 3.5(B) since all the γ_i are simple. We conclude that there is an equality of $\pi_1(\overline{P})$-conjugacy classes

$$T_d^k(s(x)) = s(x).$$

By Lemma 2.9, this implies that $s(x)$ is disjoint from d as curves on $\overline{P}$. As this argument applies for *every* separating curve on $\overline{P}$, we conclude that $s(x)$ must be disjoint from every separating curve d on $\overline{P}$. Since $p \geq 2$, the change-of-coordinates principle implies that any nontrivial element $y \in \pi_1(\overline{P})$ must intersect some separating curve d. This shows that $s(x)$ must be trivial as claimed. $\qquad\square$

3.4. **Finishing the argument.** The final stage of the argument exploits the fact that the existence of a section $\sigma : \overline{\mathcal{H}} \to \mathrm{PB}_{1,1}(\overline{P})$ places strong homological constraints on the map s. Throughout this section, our cohomology groups will implicitly have rational coefficients. To simplify matters further, we forget the (inessential) tangential data encoded in the space $\mathrm{PConf}_{1,1}(\overline{P})$, and consider instead the induced section

$$\overline{\sigma} : \overline{\mathcal{H}} \to \mathrm{PB}_2(\overline{P});$$

here $\mathrm{PB}_2(\overline{P}) = \pi_1(\mathrm{PConf}_2(\overline{P}))$ is the fundamental group of the configuration space of two ordered points on $\overline{P}$. The space $\mathrm{PConf}_2(\overline{P})$ is, by definition, given as

$$\mathrm{PConf}_2(\overline{P}) := \overline{P} \times \overline{P} \setminus \Delta,$$

where Δ is the diagonal locus. In this setting, there is a factorization

$$s = (p_2)_* \circ \overline{\sigma}.$$

A crucial consequence of this is that $s^* : H^*(\overline{P}) \to H^*(\overline{\mathcal{H}})$ factors through $H^*(\mathrm{PB}_2(\overline{P}))$. The following lemma is proved by a standard argument using the formulation of Poincaré duality via Thom spaces.

Lemma 3.11. *Let $[\Delta] \in H^2(\overline{P} \times \overline{P})$ denote the Poincaré dual class of Δ, and let*

$$\iota : \mathrm{PConf}_2(\overline{P}) \to \overline{P} \times \overline{P}$$

denote the inclusion map. Then $\iota^*([\Delta]) = 0 \in H^2(\mathrm{PConf}_2(\overline{P}))$.

Concluding the proof. Let $i : \overline{\mathcal{H}} \to \pi_1(\overline{P})$ denote the inclusion. Consider the product homomorphism

$$i \times s : \overline{\mathcal{H}} \to \pi_1(\overline{P}) \times \pi_1(\overline{P}) \cong \pi_1(\overline{P} \times \overline{P}).$$

Observe that this coincides with the section map $\overline{\sigma} : \overline{\mathcal{H}} \to \mathrm{PB}_2(\overline{P})$, so that there is a factorization

$$i \times s = \iota_* \circ \overline{\sigma}.$$

By Lemma 3.11, it follows that $(i \times s)^*([\Delta]) = (\iota \circ \overline{\sigma})^*([\Delta]) = 0 \in H^2(\overline{\mathcal{H}})$.

Let $x_1, y_1, \ldots, x_p, y_p \in H^1(\overline{P})$ denote a symplectic basis with respect to the cup product form; let also $[\overline{P}]$ denote the fundamental class. Then

$$[\Delta] = 1 \otimes [\overline{P}] + [\overline{P}] \otimes 1 + \sum_{i=1}^{p} x_i \otimes y_i - y_i \otimes x_i$$

as a class in

$$H^2(\overline{P} \times \overline{P}) \cong (H^0(\overline{P}) \otimes H^2(\overline{P})) \oplus (H^2(\overline{P}) \otimes H^0(\overline{P})) \oplus (H^1(\overline{P}) \otimes H^1(\overline{P})).$$

Thus

$$0 = (i \times s)^*([\Delta]) = i^*(1)s^*([\overline{P}]) + i^*([\overline{P}])s^*(1) + \sum_{j=1}^{p} i^*(x_j)s^*(y_j) - i^*(y_j)s^*(x_j).$$

We will see that in both cases (A) and (B), this is a contradiction. Lemma 3.6 asserts that in Case (A), $s^* = i^*$ in degree 1. Since $H^*(\overline{P})$ is generated as an algebra in degree 1, this implies that $s^* = i^*$ in degree 2 as well. Then a basic calculation shows that in this case,

$$(i \times s)^*([\overline{P}]) = (i \times i)^*([\overline{P}]) = \chi(\overline{P})[\overline{\mathcal{H}}],$$

where $\chi(\overline{P})$ denotes the Euler characteristic and $[\overline{\mathcal{H}}]$ denotes the fundamental class of the surface group $\overline{\mathcal{H}}$. As this is nonzero, we have arrived at a contradiction. Similarly, Lemma 3.7 asserts that in Case (B), $s^* = 0$ in positive degrees. Then $(i \times s)^*([\overline{P}]) = i^*([\overline{P}]) = \chi(\overline{P})[\overline{\mathcal{H}}] \neq 0$, again a contradiction. □

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