# THE ABELIANIZATION OF THE JOHNSON KERNEL

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Abstract. We prove that the first complex homology of the Johnson subgroup of the Torelli group $T_g$ is a non-trivial, unipotent $T_g$-module for all $g \geq 4$ and give an explicit presentation of it as a $\mathrm{Sym}_{\bullet} H_1(T_g, \mathbb{C})$-module when $g \geq 6$. We do this by proving that, for a finitely generated group $G$ satisfying an assumption close to formality, the triviality of the restricted characteristic variety implies that the first homology of its Johnson kernel $K$ is a nilpotent module over the corresponding Laurent polynomial ring, isomorphic to the infinitesimal Alexander invariant of the associated graded Lie algebra of $G$. In this setup, we also obtain a precise nilpotence test.

## Contents



## 1. Introduction

Fix a closed oriented surface $\Sigma$ of genus $g \geq 2$. The genus $g$ mapping class group $\Gamma_g$ is defined to be the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma$. For a commutative ring $R$, denote $H_1(\Sigma, R)$ by $H_R$. The intersection pairing $\theta : H_R \otimes H_R \to R$ is a unimodular, skew-symmetric bilinear form. Set $\mathrm{Sp}(H_R) = \mathrm{Aut}(H_R, \theta)$. The action of $\Gamma_g$ on $\Sigma$ induces a surjective

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homomorphism $r : \Gamma_g \to \mathrm{Sp}(H_{\mathbb{Z}})$. The Torelli group $T_g$ is defined to be the kernel of $r$. One thus has the extension

$$1 \longrightarrow T_g \longrightarrow \Gamma_g \xrightarrow{r} \mathrm{Sp}(H_{\mathbb{Z}}) \longrightarrow 1. \tag{1.1}$$

Dennis Johnson [12] proved that $T_g$ is finitely generated when $g \geq 3$.

The intersection form $\theta$ spans a copy of the trivial representation in $\wedge^2 H_R$. One therefore has the $\mathrm{Sp}(H_R)$-module

$$V_R := (\wedge^3 H_R)/(\theta \wedge H_R)$$

which is torsion free as an $R$-module for all $R$.

Johnson [11] constructed a surjective morphism (the "Johnson homomorphism") $\tau : T_g \to V_{\mathbb{Z}}$ and proved in [14] that it induces an $\mathrm{Sp}(H_{\mathbb{Z}})$-module isomorphism

$$\bar{\tau} : H_1(T_g)/(\text{2-torsion}) \to V_{\mathbb{Z}}\,.$$

The *Johnson group* $K_g$ is the kernel of $\tau$. By a fundamental result of Johnson [13], it is the subgroup of $\Gamma_g$ generated by Dehn twists on separating simple closed curves.

The goal of this paper is to describe the $\Gamma_g/K_g$-module $H_1(K_g, \mathbb{C})$. The first and third authors [4] proved that $H_1(K_g, \mathbb{C})$ is finite dimensional whenever $g \geq 4$. Our first result is:

**Theorem A.** *If $g \geq 4$, then $H_1(K_g, \mathbb{C})$ is a non-trivial, unipotent $H_1(T_g)$-module and $H_1(T_g, \mathbb{C}_\rho)$ vanishes for all non-trivial characters $\rho$ in the identity component $\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, \mathbb{C}^*)$ of $H^1(T_g, \mathbb{C}^*)$.*

When $g \geq 6$ we find a presentation of $H_1(K_g, \mathbb{C})$ as a $\Gamma_g/K_g$-module. Describing this module structure requires some preparation.

Suppose that $g \geq 3$. Denote the highest weight summand of the second symmetric power of the $\mathrm{Sp}(H_{\mathbb{C}})$-module $\wedge^2 H_{\mathbb{C}}$ by $Q$.[1] There is a unique $\mathrm{Sp}(H_{\mathbb{C}})$-module projection (up to multiplication by a non-zero scalar), $\pi : \wedge^2 V_{\mathbb{C}} \twoheadrightarrow Q$.

Define a left $\mathrm{Sym}_\bullet(V_{\mathbb{C}})$-module homomorphism

$$q : \mathrm{Sym}_\bullet(V_{\mathbb{C}}) \otimes \wedge^3 V_{\mathbb{C}} \to \mathrm{Sym}_\bullet(V_{\mathbb{C}}) \otimes Q$$

by

$$q(f \otimes (a_0 \wedge a_1 \wedge a_2)) = \sum_{i \in \mathbb{Z}/3} f \cdot a_i \otimes \pi(a_{i+1} \wedge a_{i+2})\,.$$

The map $q$ is $\mathrm{Sp}(H_{\mathbb{C}})$-equivariant. Thus, the cokernel of $q$ is both a $\mathrm{Sp}(H_{\mathbb{C}})$-module and a graded $\mathrm{Sym}_\bullet(V_{\mathbb{C}})$-module. We show in Section 4 that $\mathrm{coker}(q)$ is finite-dimensional, when $g \geq 6$. It follows that $\mathrm{coker}(q)$ is a $(\mathrm{Sp}(H_{\mathbb{C}}) \ltimes V_{\mathbb{C}})$-module, where $v \in V_{\mathbb{C}}$ acts via its exponential $\exp v$. One therefore has the $(\mathrm{Sp}(H_{\mathbb{C}}) \ltimes V_{\mathbb{C}})$-module

$$M := \mathbb{C} \oplus \mathrm{coker}(q)\,, \tag{1.2}$$

[1] If $\lambda_1, \ldots, \lambda_g$ is a set of fundamental weights of $\mathrm{Sp}(H_{\mathbb{C}})$, then $Q$ is the irreducible module with highest weight $2\lambda_2$. Alternatively, it is the irreducible module corresponding to the partition $[2, 2]$.

where $\mathbb{C}$ denotes the trivial module.

To relate the $(\mathrm{Sp}(H_\mathbb{C}) \ltimes V_\mathbb{C})$-action on $M$ to the $\Gamma_g/K_g$-action on $H_1(K_g, \mathbb{C})$, we recall that Morita [16] has shown that there is a Zariski dense embedding $\Gamma_g/K_g \hookrightarrow \mathrm{Sp}(H_\mathbb{C}) \ltimes V_\mathbb{C}$, unique up to conjugation by an element of $V_\mathbb{C}$, such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_g/K_g & \longrightarrow & \Gamma_g/K_g & \longrightarrow & \mathrm{Sp}(H_\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \bar\tau} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & V_\mathbb{C} & \longrightarrow & \mathrm{Sp}(H_\mathbb{C}) \ltimes V_\mathbb{C} & \longrightarrow & \mathrm{Sp}(H_\mathbb{C}) & \longrightarrow & 1 \end{array}$$

commutes.

**Theorem B.** *If $g \geq 6$, then there is an isomorphism $H_1(K_g, \mathbb{C}) \cong M$ which is equivariant with respect to a suitable choice of the Zariski dense homomorphism $\Gamma_g/K_g \to \mathrm{Sp}(H_\mathbb{C}) \ltimes V_\mathbb{C}$ described above.*

1.1. **Relative completion.** These results are proved using the *infinitesimal Alexander invariant* introduced in [17] and the *relative completion* of mapping class groups from [8]. Alexander invariants occur as $K_g$ contains the commutator subgroup $T_g'$ of $T_g$ and $K_g/T_g'$ is a finite vector space over $\mathbb{Z}/2\mathbb{Z}$. So one would expect $H_1(K_g, \mathbb{C})$ to be closely related to the complexified Alexander invariant $H_1(T_g', \mathbb{C})$ of $T_g$. A second step is to replace $T_g$ by its Malcev (i.e., unipotent) completion, and $K_g$ by the derived subgroup of the unipotent completion of $T_g$. These groups, in turn, are replaced by their Lie algebras. The resulting module is the infinitesimal Alexander invariant of $T_g$.

The role of relative completion of mapping class groups is that it allows, via Hodge theory, to identify filtered invariants, such as $H_1(T_g', \mathbb{C})$, with their associated graded modules which are, in general, more amenable to computation. For example, the lower central series of $T_g$ induces via conjugation a filtration on $H_1(T_g', \mathbb{C})$, whose first graded piece is identified in [8] with $V(2\lambda_2) \oplus \mathbb{C}$, over $\mathrm{Sp}(H_\mathbb{C})$.

1.2. **Alexander invariants.** The classical Alexander invariant of a group $G$ is the abelianization $G'_{\mathrm{ab}}$ of its derived subgroup $G' := [G, G]$. Conjugation by $G$ endows it with the structure of a module over the integral group ring $\mathbb{Z}G_{\mathrm{ab}}$ of the abelianization of $G$. More generally, if $N$ is a normal subgroup of $G$ that contains $G'$, then one has the $\mathbb{C}G_{\mathrm{ab}}$-module $N_{\mathrm{ab}} \otimes \mathbb{C} = H_1(N, \mathbb{C})$. Our primary example is where $G = T_g$ and $N$ is its Johnson subgroup $K_g$.

There is an infinitesimal analog of the Alexander invariant. It is obtained by replacing the group $G$ by its (complex) Malcev completion $\mathcal{G}(G)$ (also known as its unipotent completion). The Malcev completion of $G$ is a prounipotent group, and is thus determined by its Lie algebra $\mathfrak{g}(G)$ via the exponential mapping $\exp : \mathfrak{g}(G) \to \mathcal{G}(G)$, which is a bijection. (Cf. [21, Appendix A].) The first version of the infinitesimal Alexander invariant of $G$ is the abelianization $\mathcal{B}(G) := \mathcal{G}(G)'_{\mathrm{ab}}$ of the derived subgroup $\mathcal{G}(G)' = [\mathcal{G}(G), \mathcal{G}(G)]$ of $\mathcal{G}(G)$. One also has the abelianization $\mathfrak{b}(G)$ of the derived subalgebra $\mathfrak{g}(G)' = [\mathfrak{g}(G), \mathfrak{g}(G)]$ of $\mathfrak{g}(G)$. The exponential mapping induces an isomorphism $\mathfrak{b}(G) \to \mathcal{B}(G)$. When $N/G'$ is finite, one has the diagram

$$N_{\mathrm{ab}} \otimes \mathbb{C} \longrightarrow \mathcal{B}(G) \xleftarrow[\exp]{\simeq} \mathfrak{b}(G) \tag{1.3}$$

where the left most map is induced by the homomorphism $N \to \mathcal{G}(G)'$.

The next step is to replace the Alexander invariant of $\mathfrak{g}(G)$ by a graded module by means of the lower central series. Recall the lower central series of a group $G$,

$$G = G^1 \supseteq G^2 \supseteq G^3 \supseteq \cdots$$

It is defined by $G^{q+1} = [G, G^q]$. One also has the lower central series $\{\mathfrak{g}(G)^q\}_{q \geq 1}$ of its Malcev Lie algebra. There is a natural graded Lie algebra isomorphism

$$\mathfrak{g}_\bullet(G) := \bigoplus_{q \geq 1} (G^q/G^{q+1}) \otimes \mathbb{C} \xrightarrow{\simeq} \bigoplus_{q \geq 1} \mathfrak{g}(G)^q/\mathfrak{g}(G)^{q+1},$$

where the bracket of the left-hand side is induced by the commutator of $G$.

The *infinitesimal Alexander invariant* $\mathfrak{b}_\bullet(G)$ of $G$, as introduced in [17], is the Alexander invariant of this graded Lie algebra, with a degree shift by 2:[2]

$$\mathfrak{b}_\bullet(G) := \mathfrak{g}_\bullet(G)'_{\mathrm{ab}}[2]\,.$$

The adjoint action induces an action of the abelian Lie algebra $\mathfrak{g}_\bullet(G)_{\mathrm{ab}} = G_{\mathrm{ab}} \otimes \mathbb{C}$ on $\mathfrak{b}_\bullet(G)$, and this makes the latter a (graded) module over the polynomial ring $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$. One reason for considering $\mathfrak{b}_\bullet(G)$ is that, in general, it is easier to compute than $\mathfrak{b}(G)$.

The invariant $\mathfrak{b}_\bullet(G)$ is most useful when $G$ is a group whose Malcev Lie algebra $\mathfrak{g}(G)$ is isomorphic to the degree completion $\widehat{\mathfrak{g}_\bullet(G)}$ of its associated graded Lie algebra. Groups that satisfy this condition include the Torelli group $T_g$ when $g \geq 3$, which is proved in [8], and 1-formal groups[3] (such as Kähler groups). An isomorphism of $\mathfrak{g}(G)$ with $\widehat{\mathfrak{g}_\bullet(G)}$ induces an isomorphism of the infinitesimal Alexander invariant $\mathcal{B}(G)$ with the degree completion of $\mathfrak{b}_\bullet(G)$.

When $G$ is finitely generated, $N/G'$ is finite and $H_1(N, \mathbb{C})$ is a finite dimensional nilpotent $\mathbb{C}G_{\mathrm{ab}}$-module, it follows from Proposition 2.4 that all maps in (1.3) are isomorphisms.

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[2]That is, as a graded vector space, $\mathfrak{b}_q(G) = \mathfrak{g}_{q+2}(G)'_{\mathrm{ab}}$, for $q \geq 0$.

[3]In the sense of Dennis Sullivan [24]. Note that $T_g$ is 1-formal when $g \geq 6$, but is not when $g = 3$.

1.3. **Main general result.** To emphasize the key features, it is useful to abstract the situation. Define the *Johnson kernel* $K_G$ of a group $G$ to be the kernel of the natural projection, $G \twoheadrightarrow G_{\mathrm{abf}}$, where $G_{\mathrm{abf}}$ denotes the maximal torsion-free abelian quotient of $G$. Assume from now on that $G$ is finitely generated. For example, when $g \geq 3$, the Torelli group $G = T_g$ is finitely generated, $G_{\mathrm{abf}} = V_{\mathbb{Z}}$ and $K_G = K_g$.

Under additional assumptions, we want to relate the $\mathbb{C}G_{\mathrm{ab}}$-module $H_1(K_G, \mathbb{C})$ to the graded $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$-module $\mathfrak{b}_\bullet(G)$. The first issue is that the rings $\mathbb{C}G_{\mathrm{ab}}$ and $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$ are different. This is not serious as it is well-known that they become isomorphic, after completion. Specifically, denote the augmentation ideal of $\mathbb{C}G_{\mathrm{ab}}$ by $I_{G_{\mathrm{ab}}}$ and the $I_{G_{\mathrm{ab}}}$-adic completion of $\mathbb{C}G_{\mathrm{ab}}$ by $\widehat{\mathbb{C}G_{\mathrm{ab}}}$. The exponential mapping induces a filtered ring isomorphism,

$$\widehat{\exp}\colon \widehat{\mathbb{C}G_{\mathrm{ab}}} \xrightarrow{\simeq} \widehat{\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})}\,, \tag{1.4}$$

with the degree completion of $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$.

Recall that a $\mathbb{C}G_{\mathrm{ab}}$-module is *nilpotent* if it is annihilated by $I^q_{G_{\mathrm{ab}}}$, for some $q$, and *trivial* if it is annihilated by $I_{G_{\mathrm{ab}}}$. When $H_1(K_G, \mathbb{C})$ is nilpotent, it has a *natural* structure of $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$-module. Indeed, $H_1(K_G, \mathbb{C}) = \widehat{H_1(K_G, \mathbb{C})}$ by nilpotence, so we may restrict via (1.4) the canonical $\widehat{\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})}$-module structure of $H_1(K_G, \mathbb{C})$ to $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$. We may now state our main general result.

**Theorem C.** *Suppose that $G$ is a finitely generated group whose Malcev Lie algebra $\mathfrak{g}(C)$ is isomorphic to the degree completion of its associated graded Lie algebra $\mathfrak{g}_\bullet(G)$. If $H_1(G, \mathbb{C}_\rho)$ vanishes for every non-trivial character $\rho : G \to \mathbb{C}^*$ that factors through $G_{\mathrm{abf}}$, then $H_1(K_G, \mathbb{C})$ is a finite-dimensional nilpotent $\mathbb{C}G_{\mathrm{ab}}$-module and there is a $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$-module isomorphism $H_1(K_G, \mathbb{C}) \cong \mathfrak{b}_\bullet(G)$. Moreover, $I^q_{G_{\mathrm{ab}}}$ annihilates $H_1(K_G, \mathbb{C})$ if and only if $\mathfrak{b}_q(G) = 0$.*

The vanishing of $H_1(G, \mathbb{C}_\rho)$ above can be expressed geometrically in terms of the *character group* $\mathbb{T}(G) = \mathrm{Hom}(G_{\mathrm{ab}}, \mathbb{C}^*)$ of $G$. Since $G$ is finitely generated, this is an algebraic torus. Its identity component $\mathbb{T}^0(G)$ is the subtorus $\mathrm{Hom}(G_{\mathrm{abf}}, \mathbb{C}^*)$. The *restricted characteristic variety* $\mathcal{V}(G)$ is the set of those $\rho \in \mathbb{T}^0(G)$ for which $H_1(G, \mathbb{C}_\rho) \neq 0$. It is known that $\mathcal{V}(G)$ is a Zariski closed subset of $\mathbb{T}^0(G)$. (This follows for instance from work by E. Hironaka in [10].) The vanishing hypothesis in Theorem C simply means that $\mathcal{V}(G)$ is trivial, i.e., $\mathcal{V}(G) \subseteq \{1\}$.

Work by Dwyer and Fried [5] (as refined in [18]) implies that $\mathcal{V}(G)$ is finite precisely when $H_1(K_G, \mathbb{C})$ is finite-dimensional. This approach led in [4] to the conclusion that $\dim_{\mathbb{C}} H_1(K_g, \mathbb{C}) < \infty$, for $g \geq 4$. Further analysis (carried out in Section 3) reveals that $\mathcal{V}(G)$ is trivial if and only if $H_1(K_G, \mathbb{C})$ is nilpotent over $\mathbb{C}G_{\mathrm{ab}}$.

We show in Section 2 that the $I_{G_{\mathrm{ab}}}$-adic completions of $H_1(G', \mathbb{C})$ and $H_1(K_G, \mathbb{C})$ are isomorphic. The triviality of $\mathcal{V}(G)$ implies that the finite-dimensional vector space $H_1(K_G, \mathbb{C})$ is isomorphic to its completion. On the other hand, the first hypothesis of Theorem C implies, via a result from [3], that the degree completion of the infinitesimal Alexander invariant $\mathfrak{b}_\bullet(G)$ is isomorphic to the $I_{G_{\mathrm{ab}}}$-adic completion of $H_1(G', \mathbb{C})$. The details appear in Section 4.

To prove Theorem A we need to check that $\mathcal{V}(T_g) \subseteq \{1\}$. This is achieved in two steps. Firstly, we improve one of the main results from [4], by showing that $\mathcal{V}(T_g)$ is not just finite, but consists only of torsion characters. This is done in a broader context, in Theorem 3.1. In this theorem, the symplectic symmetry plays a key role: the $\mathrm{Sp}(H_{\mathbb{Z}})$-module $(T_g)_{\mathrm{ab}}$ gives a canonical action of $\mathrm{Sp}(H_{\mathbb{Z}})$ on the algebraic group $\mathbb{T}^0(T_g)$. We know from [4] that this action leaves the restricted characteristic variety $\mathcal{V}(T_g)$ invariant. The second step is to infer that actually $\mathcal{V}(T_g) = \{1\}$. We prove this by using a key result due to Putman, who showed in [20] that all finite index subgroups of $T_g$ that contain $K_g$ have the same first Betti number when $g \geq 3$.

A basic result from [8], valid for $g \geq 3$, guarantees that $T_g$ satisfies the assumption on the Malcev Lie algebra in Theorem C. Theorem A follows. Again by [8], the group $T_g$ is 1-*formal*, when $g \geq 6$; equivalently, the graded Lie algebra $\mathfrak{g}_\bullet(T_g)$ has a quadratic presentation. Theorem B follows from a general result in [17], that associates to a quadratic presentation of the Lie algebra $\mathfrak{g}_\bullet(G)$ a finite $\mathrm{Sym}_\bullet(G_{\mathrm{ab}} \otimes \mathbb{C})$-presentation for the infinitesimal Alexander invariant $\mathfrak{b}_\bullet(G)$. When $g \geq 6$, we use the quadratic presentation of $\mathfrak{g}_\bullet(T_g)$ obtained in [8].

## 2. Completion

We start in this section by establishing several general results, related to $I$-adic completions of Alexander-type invariants. We refer the reader to the books by Eisenbud [6, Chapter 7] and Matsumura [15, Chapter 9], for background on completion techniques in commutative algebra. Throughout the paper, we work with $\mathbb{C}$-coefficients, unless otherwise specified. The *augmentation ideal* of a group $G$, $I_G$, is the kernel of the $\mathbb{C}$-algebra homomorphism, $\mathbb{C}G \to \mathbb{C}$, that sends each group element to 1.

Let $N$ be a normal subgroup of $G$. Note that $G$-conjugation endows $H_\bullet N$ with a natural structure of (left) module over the group algebra $\mathbb{C}(G/N)$, and similarly for cohomology. If $N$ contains the derived subgroup $G'$, both $H_\bullet N$ and $H^\bullet N$ may be viewed as $\mathbb{C}(G/G')$-modules, by restricting the scalars via the ring epimorphism $\mathbb{C}(G/G') \twoheadrightarrow \mathbb{C}(G/N)$. When $G$ is finitely generated, $H_1 N$ is a finitely generated module over the commutative Noetherian ring $\mathbb{C}(G/N)$.

An important particular case arises when $N = G'$. Denoting abelianization by $G_{\mathrm{ab}} := G/G'$, set $B(G) := H_1 G' = G'_{\mathrm{ab}} \otimes \mathbb{C} = (G'/G'') \otimes \mathbb{C}$, and call $B(G)$ the *Alexander invariant* of $G$. These constructions are functorial, in the following sense. Given a group homomorphism, $\varphi : \overline{G} \to G$, it induces a $\mathbb{C}$-linear map, $B(\varphi)$ :

$B(\overline{G}) \to B(G)$, and a ring homomorphism $\mathbb{C}\varphi : \mathbb{C}\overline{G}_{\mathrm{ab}} \to \mathbb{C}G_{\mathrm{ab}}$. Moreover, $B(\varphi)$ is $\mathbb{C}\varphi$-*equivariant*, i.e., $B(\varphi)(\bar{a} \cdot \bar{x}) = \mathbb{C}\varphi(\bar{a}) \cdot B(\varphi)(\bar{x})$, for $\bar{a} \in \mathbb{C}\overline{G}_{\mathrm{ab}}$ and $\bar{x} \in B(\overline{G})$.

The *$I$-adic filtration* of the $\mathbb{C}G_{\mathrm{ab}}$-module $B(G)$, $\{I^q_{G_{\mathrm{ab}}} \cdot B(G)\}_{q\geq 0}$, gives rise to the *completion* map $B(G) \to \widehat{B(G)}$, and to the $I$-adic associated graded, $\mathrm{gr}_\bullet B(G)$. By $\mathbb{C}\varphi$-equivariance, $B(\varphi)$ respects the $I$-adic filtrations. Consequently, there is an induced filtered map, $\widehat{B(\varphi)} : \widehat{B(\overline{G})} \to \widehat{B(G)}$, compatible with the completion maps. One knows that $\widehat{B(\varphi)}$ is a filtered isomorphism if and only if $\mathrm{gr}_\bullet(B\varphi) : \mathrm{gr}_\bullet B(\overline{G}) \to \mathrm{gr}_\bullet B(G)$ is an isomorphism.

A useful related construction (see [22]) involves the lower central series of a group $G$. The (complex) *associated graded Lie algebra*

$$\mathfrak{g}_\bullet(G) := \bigoplus_{q\geq 1} (G^q/G^{q+1}) \otimes \mathbb{C}$$

is generated as a Lie algebra by $\mathfrak{g}_1(G) = H_1 G$. Each group homomorphism $\varphi : \overline{G} \to G$ gives rise to a graded Lie algebra homomorphism, $\mathrm{gr}_\bullet(\varphi) : \mathfrak{g}_\bullet(\overline{G}) \to \mathfrak{g}_\bullet(G)$.

Malcev completion (over $\mathbb{C}$), as defined by Quillen [21, Appendix A], is a useful tool. It associates to a group $G$ a complex prounipotent group $\mathcal{G}(G)$, and a homomorphism $G \to \mathcal{G}(G)$. The *Malcev Lie algebra* of $G$ is the Lie algebra $\mathfrak{g}(G)$ of $\mathcal{G}(G)$. It is pronilpotent. The exponential mapping $\exp : \mathfrak{g}(G) \to \mathcal{G}(G)$ is a bijection.

The lower central series filtrations

$$\begin{gathered} G = G^1 \supseteq G^2 \supseteq G^3 \supseteq \cdots \\ \mathcal{G}(G) = \mathcal{G}(G)^1 \supseteq \mathcal{G}(G)^2 \supseteq \mathcal{G}(G)^3 \supseteq \cdots \\ \mathfrak{g}(G) = \mathfrak{g}(G)^1 \supseteq \mathfrak{g}(G)^2 \supseteq \mathfrak{g}(G)^3 \supseteq \cdots \end{gathered}$$

of $G$, $\mathcal{G}(G)$ and $\mathfrak{g}(G)$ are preserved by the canonical homomorphism $G \to \mathcal{G}(G)$ and the exponential mapping $\exp : \mathfrak{g}(G) \to \mathcal{G}(G)$. They induce Lie algebra isomorphisms of the associated graded objects:

$$\mathrm{gr}_\bullet(G) \otimes \mathbb{C} \xrightarrow{\simeq} \mathrm{gr}_\bullet \mathcal{G}(G) \xleftarrow{\simeq} \mathrm{gr}_\bullet \mathfrak{g}(G)$$

(cf. [21, Appendix A]).

We will need the following basic fact, which is a straightforward generalization of a result [23] of Stallings: if a group homomorphism $\varphi : \overline{G} \to G$ induces an isomorphism $\varphi^1 : H^1 G \xrightarrow{\simeq} H^1\overline{G}$ and a monomorphism $\varphi^2 : H^2 G \hookrightarrow H^2\overline{G}$, then

$$\mathfrak{g}(\varphi) : \mathfrak{g}(\overline{G}) \xrightarrow{\simeq} \mathfrak{g}(G) \tag{2.1}$$

is a filtered Lie isomorphism. A proof can be found in [9, Corollary 3.2].

With these preliminaries, we may now state and prove our first result.

**Proposition 2.1.** *Suppose that $\overline{G}$ is a finite index subgroup of a finitely generated group $G$. If $\varphi_1 : H_1\overline{G} \to H_1 G$ is a an isomorphism, then $\widehat{B(\varphi)} : \widehat{B(\overline{G})} \to \widehat{B(G)}$ is a filtered isomorphism, where $\varphi : \overline{G} \hookrightarrow G$ is the inclusion map.*

*Proof.* Since $[G : \overline{G}]$ is finite, $\varphi^\bullet : H^\bullet G \to H^\bullet \overline{G}$ is a monomorphism. So, $\varphi^1$ is an isomorphism and $\varphi^2$ is injective. Hence, the filtered Lie isomorphism (2.1) holds.

Proposition 5.4 from [3] guarantees that the filtered vector space $\widehat{B(G)}$ is functorially determined by the filtered Lie algebra $\mathfrak{g}(G)$. This completes the proof. □

Consider now a group extension

$$1 \to N \xrightarrow{\psi} \pi \to Q \to 1\,. \tag{2.2}$$

Denote by $p_\bullet : H_\bullet N \twoheadrightarrow (H_\bullet N)_Q$ the canonical projection onto the co-invariants. Clearly, $\psi_\bullet : H_\bullet N \to H_\bullet \pi$ factors through $p_\bullet$, giving rise to a map

$$q_\bullet : (H_\bullet N)_Q \to H_\bullet \pi\,. \tag{2.3}$$

When $Q$ is finite, $q_\bullet$ is an isomorphism; see Brown's book [1, Chapter III.10].

Given a $\mathbb{C}\pi$-module $M$, note that $I_N \cdot M$ is a $\mathbb{C}\pi$-submodule of $M$; see [1, Chapter II.2]. Consequently, the natural projection onto the $N$-co-invariants, $p : M \twoheadrightarrow M_N$, is $\mathbb{C}\pi$-linear and induces a filtered map, $\hat{p} : \widehat{M} \to \widehat{M_N}$, between $I_\pi$-adic completions.

We will need the following probably known result. For the reader's convenience, we sketch a proof.

**Lemma 2.2.** *Suppose that $N$ is a finite subgroup of a finitely generated abelian group $\pi$. If $M$ is a finitely generated $\mathbb{C}\pi$-module, then $\hat{p} : \widehat{M} \to \widehat{M_N}$ is a filtered isomorphism.*

*Proof.* We start with a simple remark: if $R$ is a finitely generated commutative $\mathbb{C}$-algebra and $I \subseteq R$ is a maximal ideal, then the roots of unity $u$ from $1 + I$ act as the identity on $M/I^q \cdot M$, for all $q$, when $M$ is a finitely generated $R$-module. Indeed, $u - 1$ annihilates $I^s \cdot M/I^{s+1} \cdot M$ for all $s$, so the $u$-action on the finite-dimensional $\mathbb{C}$-vector space $M/I^q \cdot M$ is both unipotent and semisimple, hence trivial.

Now, consider the exact sequence of finitely generated $R$-modules,

$$0 \to I_N \cdot M \to M \to M_N \to 0\,,$$

where $R = \mathbb{C}\pi$. Tensoring it with $R/I_\pi^q$, we infer that our claim is equivalent to $I_N \cdot M \subseteq \cap_q I_\pi^q \cdot M$. This in turn follows from the remark. □

**Remark 2.3.** Let $M$ be a module over a group ring $\mathbb{C}\pi$, and $\overline{\pi} \twoheadrightarrow \pi$ a group epimorphism, giving $M$ a structure of $\mathbb{C}\overline{\pi}$-module, by restriction via $\mathbb{C}\overline{\pi} \twoheadrightarrow \mathbb{C}\pi$. Plainly, $I_{\overline{\pi}}^q \cdot M = I_\pi^q \cdot M$, for all $q$. In particular, the $I_{\overline{\pi}}$-adic and $I_\pi$-adic completions of $M$ are filtered isomorphic, and $M$ is nilpotent (or trivial) over $\mathbb{C}\overline{\pi}$ if and only if this happens over $\mathbb{C}\pi$.

Given a group $G$, set $G_{\rm abf} := G_{\rm ab}/(\text{torsion})$. The *Johnson kernel*, $K_G$, is the kernel of the canonical projection $G \twoheadrightarrow G_{\rm abf}$. When $G = T_g$ and $g \geq 3$, Johnson's fundamental results from [11, 14] show that $K_G = K_g$, whence our terminology.

More generally, consider an extension

$$1 \to G' \xrightarrow{\psi} K \to F \to 1\,, \tag{2.4}$$

with $F$ finite. Plainly, $\psi_\bullet : H_\bullet G' \to H_\bullet K$ is $\mathbb{C}G_{\rm ab}$-linear. Let $\widehat{\psi_\bullet}$ be the induced map on $I_{G_{\rm ab}}$-adic completions. (When $K = K_G$, note that $H_\bullet K_G$ is actually a $\mathbb{C}G_{\rm abf}$-module, with $\mathbb{C}G_{\rm ab}$-module structure induced by restriction, via $\mathbb{C}G_{\rm ab} \twoheadrightarrow \mathbb{C}G_{\rm abf}$. By Remark 2.3, its $I_{G_{\rm ab}}$-adic and $I_{G_{\rm abf}}$-adic completions coincide.) Here is our second main result in this section.

**Proposition 2.4.** *If $G$ is a finitely generated group and $K$ is a subgroup like in (2.4), then $\widehat{\psi_1} : \widehat{H_1 G'} \to \widehat{H_1 K}$ is a filtered isomorphism.*

*Proof.* We apply Lemma 2.2 to $F \subseteq G_{\rm ab}$ and $M = H_1 G'$, to obtain a filtered isomorphism $\hat{p} : \widehat{H_1 G'} \xrightarrow{\simeq} (\widehat{H_1 G'})_F$ between $I_{G_{\rm ab}}$-adic completions. We conclude by noting that the isomorphism (2.3) coming from (2.4), $q : (H_1 G')_F \xrightarrow{\simeq} H_1 K$, is $\mathbb{C}G_{\rm ab}$-linear. The last claim is easy to check: plainly, the $\mathbb{C}F$-module structure on $H_1 G'$ coming from (2.4) is the restriction to $\mathbb{C}F$ of the canonical $\mathbb{C}G_{\rm ab}$-structure. □

## 3. Characteristic varieties

We show that the (restricted) characteristic variety of $T_g$ is trivial for all $g \geq 4$, as stated in Theorem A, thus improving one of the main results in [4]. Fix a symplectic basis of the first homology $H_{\mathbb{Z}}$ of the reference surface $\Sigma$. This gives an identification of $\mathrm{Sp}(H_R)$ with $\mathrm{Sp}_g(R)$ for all rings $R$.

We start by reviewing a couple of definitions and relevant facts. Let $G$ be a finitely generated group. The *character torus* $\mathbb{T}(G) = \mathrm{Hom}(G_{\rm ab}, \mathbb{C}^*)$ is a linear algebraic group with coordinate ring $\mathbb{C}G_{\rm ab}$. The connected component of $1 \in \mathbb{T}(G)$ is denoted $\mathbb{T}^0(G) = \mathrm{Hom}(G_{\rm abf}, \mathbb{C}^*)$ and has coordinate ring $\mathbb{C}G_{\rm abf}$.

The *characteristic varieties* of $G$ are defined for (degree) $i \geq 0$, (depth) $k \geq 1$ by

$$\mathcal{V}^i_k(G) = \{\rho \in \mathbb{T}(G) \mid \dim_{\mathbb{C}} H_i(G, \mathbb{C}_\rho) \geq k\}\,. \tag{3.1}$$

Here $\mathbb{C}_\rho$ denotes the $\mathbb{C}G$-module $\mathbb{C}$ given by the change of rings $\mathbb{C}G \to \mathbb{C}$ corresponding to $\rho$. Their *restricted* versions are the intersections $\mathcal{V}^i_k(G) \cap \mathbb{T}^0(G)$. The restricted characteristic variety $\mathcal{V}^1_1(G) \cap \mathbb{T}^0(G)$ is denoted $\mathcal{V}(G)$. As explained in [4, Section 6], it follows from results in [10] about finitely presented groups that both $\mathcal{V}^1_k(G)$ and $\mathcal{V}^1_k(G) \cap \mathbb{T}^0(G)$ are Zariski closed subsets, for all $k$.

When $G = T_g$ and $g \geq 3$, these constructions acquire an important symplectic symmetry; see [4]. We recall that the linear algebraic group $\mathrm{Sp}_g(\mathbb{C})$ is defined over $\mathbb{Q}$, simple, with positive $\mathbb{Q}$-rank, and contains $\mathrm{Sp}_g(\mathbb{Z})$ as an arithmetic subgroup.

The $\Gamma_g$-conjugation in the defining extension (1.1) for $T_g$ induces representations of $\mathrm{Sp}_g(\mathbb{Z})$ in the finitely generated abelian groups $(T_g)_{\mathrm{ab}}$ and $(T_g)_{\mathrm{abf}}$. They give rise to natural $\mathrm{Sp}_g(\mathbb{Z})$-representations in the algebraic groups $\mathbb{T}(T_g)$ and $\mathbb{T}^0(T_g)$, for which the inclusion $\mathbb{T}^0(T_g) \subseteq \mathbb{T}(T_g)$ becomes $\mathrm{Sp}_g(\mathbb{Z})$-equivariant. Furthermore, $\mathcal{V}(T_g) \subseteq \mathbb{T}^0(T_g)$ is $\mathrm{Sp}_g(\mathbb{Z})$-invariant.

By Johnson's work [11, 14], we also know that the $\mathrm{Sp}_g(\mathbb{Z})$-action on $(T_g)_{\mathrm{abf}}$ extends to a rational, irreducible and non-trivial $\mathrm{Sp}_g(\mathbb{C})$-representation in $(T_g)_{\mathrm{abf}} \otimes \mathbb{C}$.

We will need the following refinement of a basic result on propagation of irreducibility, proved by Dimca and Papadima in [4]. This refinement is closely related to an open question formulated in [19, Section 10], on outer automorphism groups of free groups.

**Theorem 3.1.** *Let $L$ be a $D$-module which is finitely generated and free as an abelian group. Assume that $D$ is an arithmetic subgroup of a simple $\mathbb{C}$-linear algebraic group $S$ defined over $\mathbb{Q}$, with $\mathrm{rank}_{\mathbb{Q}}(S) \geq 1$. Suppose also that the $D$-action on $L$ extends to an irreducible, non-trivial, rational $S$-representation in $L \otimes \mathbb{C}$. Let $W \subset \mathbb{T}(L)$ be a $D$-invariant, Zariski closed, proper subset of $\mathbb{T}(L)$. Then $W$ is a finite set of torsion elements in $\mathbb{T}(L)$.*

*Proof.* According to one of the main results from [4] (which needs no non-triviality assumption on the $S$-representation $L \otimes \mathbb{C}$), $W$ must be finite. We have to show that any $t \in W$ is a torsion point of $\mathbb{T} = \mathbb{T}(L)$. We know that the stabilizer of $t$, $D_t$, has finite index in $D$. By Borel's density theorem, $D_t$ is Zariski dense in $S$.

Suppose that $t \in W$ has infinite order, and let $\mathbb{T}_t \subseteq \mathbb{T}$ be the Zariski closure of the subgroup generated by $t$. By our assumption, the closed subgroup $\mathbb{T}_t$ is positive-dimensional. Since $\mathbb{T}_t$ is fixed by $D_t$, the Lie algebra $T_1\mathbb{T}_t \subseteq T_1\mathbb{T} = \mathrm{Hom}(L \otimes \mathbb{C}, \mathbb{C})$ is $D_t$-fixed as well. By Zariski density, $T_1\mathbb{T}_t$ is then a non-zero, $S$-fixed subspace of $T_1\mathbb{T}$, contradicting the non-triviality hypothesis on $L \otimes \mathbb{C}$. □

We know from [4] that $\mathcal{V}(T_g)$ is finite, for $g \geq 4$. We may apply Theorem 3.1 to $D = \mathrm{Sp}_g(\mathbb{Z})$ acting on $L = (T_g)_{\mathrm{abf}}$, $S = \mathrm{Sp}_g(\mathbb{C})$ and $W = \mathcal{V}(T_g)$. We infer that $\mathcal{V}(T_g)$ consists of $m$-torsion elements in $\mathbb{T}^0(T_g)$, for some $m \geq 1$.

To derive the triviality of $\mathcal{V}(T_g)$ from this fact, we will use another standard tool from commutative algebra. For an affine $\mathbb{C}$-algebra $A$, let $\mathrm{Specm}(A)$ be its maximal spectrum. For a $\mathbb{C}$-algebra map between affine algebras, $f : A \to B$, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ stands for the induced map, that sends $\mathrm{Specm}(B)$ into $\mathrm{Specm}(A)$. For a finitely generated $A$-module $M$, the *support* $\mathrm{supp}_A(M)$ is the Zariski closed subset of $\mathrm{Spec}(A)$ $V(\mathrm{ann}(M)) = V(E_0(M))$, where $E_0(M)$ is the ideal generated by the codimension zero minors of a finite $A$-presentation for $M$; see [6, Chapter 20].

There is a close relationship between characteristic varieties and supports of Alexander-type invariants. Let $N$ be a normal subgroup of a finitely generated group $G$, with abelian quotient. Denote by $\nu : G \twoheadrightarrow G/N$ the canonical projection, and let $\nu^* : \mathbb{T}(G/N) \hookrightarrow \mathbb{T}(G)$ be the induced map on maximal spectra of corresponding abelian group algebras. It follows for instance from Theorem 3.6 in [18] that $\nu^*$ restricts to an identification (away from 1)

$$\mathrm{Specm}(\mathbb{C}(G/N)) \cap \mathrm{supp}_{\mathbb{C}(G/N)}(H_1 N) \equiv \mathrm{im}(\nu^*) \cap \mathcal{V}_1^1(G)\,. \tag{3.2}$$

We will need the following (presumably well-known) result on supports. For the sake of completeness, we include a proof.

**Lemma 3.2.** *If $f : A \hookrightarrow B$ is an integral extension of affine $\mathbb{C}$-algebras and $M$ is a finitely generated $B$-module, then $M$ is finitely generated over $A$ and* $\mathrm{supp}_A(M) = f^*(\mathrm{supp}_B(M))$, $\mathrm{Specm}(A) \cap \mathrm{supp}_A(M) = f^*(\mathrm{Specm}(B) \cap \mathrm{supp}_B(M))$.

*Proof.* The reader may consult [6, Chapter 4], for background on integral extensions. Clearly, $\mathrm{ann}_A(M) = A \cap \mathrm{ann}_B(M)$, and the extension $\bar{f} : A/\mathrm{ann}_A(M) \hookrightarrow B/\mathrm{ann}_B(M)$ is again integral. The inclusion $f^*(\mathrm{supp}_B(M)) \subseteq \mathrm{supp}_A(M)$ follows from the definitions. For the other inclusion, pick any prime ideal $\mathfrak{p}$ containing $\mathrm{ann}_A(M)$. Since $\bar{f}$ induces a surjection on prime spectra, there is a prime ideal $\mathfrak{q}$ containing $\mathrm{ann}_B(M)$, such that $f^*(\mathfrak{q}) = \mathfrak{p}$, and $\mathfrak{q}$ is maximal, if $\mathfrak{p}$ is maximal. □

The interpretation (3.2) for the closed points of the support of Alexander-type invariants leads to the following key nilpotence test.

**Lemma 3.3.** *Let $\nu : G \twoheadrightarrow H$ be a group epimorphism with finitely generated source, abelian image and kernel $N$. Then the following are equivalent.*

(1) *The $\mathbb{C}H$-module $H_1 N$ is nilpotent.*
(2) *The inclusion* $\mathrm{Specm}(\mathbb{C}H) \cap \mathrm{supp}_{\mathbb{C}H}(H_1 N) \subseteq \{1\}$ *holds.*
(3) *The intersection $\nu^*(\mathbb{T}(H)) \cap \mathcal{V}_1^1(G)$ is contained in* $\{1\}$.

*Proof.* Note that $1 \in \mathbb{T}(H)$ corresponds to the maximal ideal $I_H \subseteq \mathbb{C}H$. With this remark, the equivalence (1) $\Longleftrightarrow$ (2) becomes an easy consequence of the Hilbert Nullstellensatz. The equivalence (2) $\Longleftrightarrow$ (3) follows directly from (3.2). □

For a group $G$, we denote by $p_G : G \twoheadrightarrow G_{\mathrm{abf}}$ the canonical projection. Assume $G$ is finitely generated and fix an integer $m \geq 1$. Denoting by $\iota_m : G_{\mathrm{abf}} \hookrightarrow G_{\mathrm{abf}}$ the multiplication by $m$, note that its extension to group algebras, $\mathbb{C}\iota_m : \mathbb{C}G_{\mathrm{abf}} \hookrightarrow \mathbb{C}G_{\mathrm{abf}}$, is integral, and the associated map on maximal spectra, $\mathbb{C}\iota_m^* : \mathbb{T}(G_{\mathrm{abf}}) \to \mathbb{T}(G_{\mathrm{abf}})$, is the $m$-power map of the character group $\mathbb{T}(G_{\mathrm{abf}})$.

Let $p_m : G(m) \twoheadrightarrow G_{\mathrm{abf}}$ be the pull-back of $p_G$ via $\iota_m$. Clearly, $G(m)$ is a normal, finite index subgroup of $G$ containing the Johnson kernel $K_G$, with inclusion denoted $\varphi_m : G(m) \hookrightarrow G$, and

$$p_G \circ \varphi_m = \iota_m \circ p_m\,. \tag{3.3}$$

**Lemma 3.4.** *Assume that all finite index subgroups of $G$ containing $K_G$ have the same first Betti number. Then the following hold.*

(1) *The map induced by $\varphi_m$ on $I$-adic completions, $\widehat{B(\varphi_m)} : \widehat{B(G(m))} \xrightarrow{\simeq} \widehat{B(G)}$, is a filtered isomorphism.*
(2) *The inclusion $K_{G(m)} \subseteq K_G$ is actually an equality.*

*Proof.* Our assumption implies that $\varphi_m$ induces an isomorphism $H_1G(m) \xrightarrow{\simeq} H_1G$. Property (1) follows then from Proposition 2.1. The second claim is a consequence of the fact that $p_m$ may be identified with $p_{G(m)}$. To obtain this identification, we apply to (3.3) the functor abf. By construction, $\mathrm{abf}(p_G)$ is an isomorphism and $\mathrm{abf}(\iota_m) = \iota_m$ is a rational isomorphism. We also know that $\mathrm{abf}(\varphi_m) \otimes \mathbb{Q} : H_1(G(m), \mathbb{Q}) \xrightarrow{\simeq} H_1(G, \mathbb{Q})$ is an isomorphism. We infer that $\mathrm{abf}(p_m)$ is a rational isomorphism, hence an isomorphism. □

**Proof of Theorem A (except for the non-triviality assertion).** We first prove that $\mathcal{V}(T_g) = \{1\}$. According to a recent result of Putman [20], the group $G = T_g$ satisfies for $g \geq 3$ the hypothesis of Lemma 3.4. Denote by $\psi : G' \hookrightarrow K_G$ and $\psi_m : G(m)' \hookrightarrow K_{G(m)} = K_G$ the inclusions. According to Proposition 2.4, they induce filtered isomorphisms, $\widehat{B(G)} \xrightarrow{\simeq} \widehat{H_1 K_G}$ and $\widehat{B(G(m))} \xrightarrow{\simeq} \widehat{H_1 K_G}$, between the corresponding $I$-adic completions. In these isomorphisms, the $\mathbb{C}G_{\mathrm{abf}}$-module structure of $M = H_1 K_G$ comes from the group extension associated to $p_G$, respectively $p_m$. Denote the second $\mathbb{C}G_{\mathrm{abf}}$-module by ${}^mM$, and note that ${}^mM$ is obtained from $M$ by restriction of scalars, via $\mathbb{C}\iota_m : \mathbb{C}G_{\mathrm{abf}} \hookrightarrow \mathbb{C}G_{\mathrm{abf}}$.

Taking into account the isomorphism from Lemma 3.4(1), it follows that $\mathrm{id}_M : {}^mM \to M$, viewed as a $\mathbb{C}\iota_m$-equivariant map, induces an isomorphism between the corresponding $I$-adic completions.

Let $\rho \in \mathbb{T}(G_{\mathrm{abf}})$ be a closed point of $\mathrm{supp}_{\mathbb{C}G_{\mathrm{abf}}}(M)$. By (3.2), applied to $N = K_G$, $\mathbb{C}\iota_m^*(\rho) = 1$, since $\mathcal{V}(G)$ consists of $m$-torsion points, for $g \geq 4$. We infer from Lemma 3.2, applied to $f = \mathbb{C}\iota_m : \mathbb{C}G_{\mathrm{abf}} \hookrightarrow \mathbb{C}G_{\mathrm{abf}}$, that $\mathrm{Specm}(\mathbb{C}G_{\mathrm{abf}}) \cap \mathrm{supp}_{\mathbb{C}G_{\mathrm{abf}}}({}^mM) \subseteq \{1\}$. Take $\nu = p_m : G(m) \twoheadrightarrow G_{\mathrm{abf}}$ in Lemma 3.3, whose kernel is $K_{G(m)} = K_G$. We deduce that ${}^mM$ is nilpotent over $\mathbb{C}G_{\mathrm{abf}}$, that is, $I^q \cdot {}^mM = 0$ for some $q$, where $I \subseteq \mathbb{C}G_{\mathrm{abf}}$ is the augmentation ideal.

Denote by $\kappa_m : M \to \widehat{{}^mM}$ and $\kappa : M \to \widehat{M}$ the completion maps, with kernels $\cap_{r\geq 0} I^r \cdot {}^mM$ and $\cap_{r\geq 0} I^r \cdot M$. It follows from naturality of completion that $\kappa$ is injective, since $\kappa_m$ is injective and $\widehat{\mathrm{id}_M} : \widehat{{}^mM} \xrightarrow{\simeq} \widehat{M}$ is an isomorphism.

We also know from [4] that $\dim_{\mathbb{C}} M < \infty$. It follows that the $I$-adic filtration of $M$ stabilizes to $I^q \cdot M = \cap_{r\geq 0} I^r \cdot M = 0$, for $q$ big enough. Applying Lemma 3.3 to $\nu = p_G : G \twoheadrightarrow G_{\mathrm{abf}}$, with kernel $K_G$, we infer that $\mathcal{V}(G) = \{1\}$.

We extract from the preceding argument the following corollary. Together with the triviality of $\mathcal{V}(T_g)$, this completes the proof of Theorem A (except for the non-triviality assertion) via Remark 2.3.

**Corollary 3.5.** *If $g \geq 4$, then $H_1(K_g, \mathbb{C})$ is a nilpotent module, over both $\mathbb{C}(T_g)_{\rm ab}$ and $\mathbb{C}(T_g)_{\rm abf}$.*

## 4. Infinitesimal Alexander invariant

Our next task is to prove Theorem B and the non-triviality assertion of Theorem A. These follow from general results about infinitesimal Alexander invariants.

Let $\mathfrak{h}_\bullet$ be a positively graded Lie algebra. Consider the exact sequence of graded Lie algebras

$$0 \to \mathfrak{h}'_\bullet/\mathfrak{h}''_\bullet \to \mathfrak{h}_\bullet/\mathfrak{h}''_\bullet \to \mathfrak{h}_\bullet/\mathfrak{h}'_\bullet \to 0\,. \tag{4.1}$$

The universal enveloping algebra of the abelian Lie algebra $\mathfrak{h}_\bullet/\mathfrak{h}'_\bullet$ is the graded polynomial algebra $\mathrm{Sym}_\bullet(\mathfrak{h}_{\rm ab})$. (When the Lie algebra $\mathfrak{h}_\bullet$ is generated by $\mathfrak{h}_1$, $\mathrm{Sym}_\bullet(\mathfrak{h}_{\rm ab}) = \mathrm{Sym}_\bullet(\mathfrak{h}_1)$, with the usual grading.) The adjoint action in (4.1) yields a natural graded $\mathrm{Sym}_\bullet(\mathfrak{h}_{\rm ab})$-module structure on $\mathfrak{h}'_{\rm ab}$. It will be convenient to shift degrees and define the *infinitesimal Alexander invariant* $\mathfrak{b}_\bullet(\mathfrak{h}) := \mathfrak{h}'_{\rm ab}[2]$, by analogy with the Alexander invariant of a group. The graded vector space $\mathfrak{b}_\bullet(\mathfrak{h}) = \oplus_{q\geq 0}\mathfrak{b}_q(\mathfrak{h})$, where $\mathfrak{b}_q(\mathfrak{h}) = \mathfrak{h}'_{q+2}/\mathfrak{h}''_{q+2}$, becomes in this way a graded module over $\mathrm{Sym}_\bullet(\mathfrak{h}_{\rm ab})$.

When $\mathfrak{h}_\bullet = \mathfrak{g}_\bullet(G)$, we denote the graded $\mathrm{Sym}_\bullet(G_{\rm ab}\otimes\mathbb{C})$-module $\mathfrak{b}_\bullet(\mathfrak{h})$ by $\mathfrak{b}_\bullet(G)$. Note that the *degree filtration* of $\mathfrak{b}_\bullet(G)$, $\{\mathfrak{b}_{\geq q}(G)\}_{q\geq 0}$, coincides with its $(G_{\rm ab}\otimes\mathbb{C})$-adic filtration, where $(G_{\rm ab}\otimes\mathbb{C})$ is the ideal of $\mathrm{Sym}(G_{\rm ab}\otimes\mathbb{C})$ generated by $G_{\rm ab}\otimes\mathbb{C}$.

The infinitesimal Alexander invariant, introduced and studied in [17], is functorial in the following sense. A graded Lie map $f : \mathfrak{h} \to \mathfrak{k}$ obviously induces a degree zero map $\mathfrak{b}_\bullet(f) : \mathfrak{b}_\bullet(\mathfrak{h}) \to \mathfrak{b}_\bullet(\mathfrak{k})$, equivariant with respect to the graded algebra map $\mathrm{Sym}(f_{\rm ab}) : \mathrm{Sym}(\mathfrak{h}_{\rm ab}) \to \mathrm{Sym}(\mathfrak{k}_{\rm ab})$.

Let $\mathbb{L}_\bullet(V)$ be the free graded Lie algebra on a finite-dimensional vector space $V$, graded by bracket length. Use the Lie bracket to identify $\mathbb{L}_2(V)$ and $\wedge^2 V$. For a subspace $R \subseteq \wedge^2 V$, consider the (quadratic) graded Lie algebra $\mathfrak{g} = \mathbb{L}(V)/\ \mathrm{ideal}\ (R)$, with grading inherited from $\mathbb{L}_\bullet(V)$. Denote by $\iota : R \hookrightarrow \wedge^2 V$ the inclusion.

Theorem 6.2 from [17] provides the following finite, free $\mathrm{Sym}_\bullet(V)$-presentation for the infinitesimal Alexander invariant: $\mathfrak{b}_\bullet(\mathfrak{g}) = \mathrm{coker}(\nabla)$, where

$$\nabla := \mathrm{id}\otimes\iota + \delta_3\colon\ \mathrm{Sym}_\bullet(V)\otimes(R\oplus\wedge^3 V) \to \mathrm{Sym}_\bullet(V)\otimes\wedge^2 V\,, \tag{4.2}$$

$R$, $\wedge^3 V$ and $\wedge^2 V$ have degree zero, and the $\mathrm{Sym}(V)$-linear map $\delta_3$ is given by $\delta_3(a\wedge b\wedge c) = a\otimes b\wedge c + b\otimes c\wedge a + c\otimes a\wedge b$, for $a, b, c \in V$.

We begin by simplifying the presentation (4.2). To this end, let $\beta : \wedge^2\mathfrak{g}_1 \twoheadrightarrow \mathfrak{g}_2$ be the Lie bracket.

**Lemma 4.1.** *For any quadratic graded Lie algebra* $\mathfrak{g}$, $\mathfrak{b}_\bullet(\mathfrak{g}) = \operatorname{coker}(\overline{\nabla})$, *as graded* $\operatorname{Sym}(V)$*-modules, where the* $\operatorname{Sym}(V)$*-linear map*

$$\overline{\nabla} : \operatorname{Sym}(V) \otimes \wedge^3 \mathfrak{g}_1 \to \operatorname{Sym}(V) \otimes \mathfrak{g}_2$$

*is defined by* $\overline{\nabla} = (\operatorname{id} \otimes \beta) \circ \delta_3$.

*Proof.* It is straightforward to check that the degree zero $\operatorname{Sym}(V)$-linear map $\operatorname{id} \otimes \beta$ induces an isomorphism $\operatorname{coker}(\nabla) \xrightarrow{\simeq} \operatorname{coker}(\overline{\nabla})$. □

**Proof of Theorem C.** In Lemma 3.3, let $\nu$ be the canonical projection $G \twoheadrightarrow G_{\mathrm{abf}}$, with kernel $K_G$. By our hypothesis on $\mathcal{V}(G)$ and Remark 2.3, we infer that the module $H_1K_G$ is nilpotent, over both $\mathbb{C}G_{\mathrm{abf}}$ and $\mathbb{C}G_{\mathrm{ab}}$. Therefore, $\dim_{\mathbb{C}} H_1K_G < \infty$ (since $H_1K_G$ is finitely generated over $\mathbb{C}G_{\mathrm{abf}}$) and the $I_{G_{\mathrm{ab}}}$-adic completion map

$$H_1K_G \xrightarrow{\simeq} \widehat{H_1K_G} \tag{4.3}$$

is a filtered isomorphism. Proposition 2.4 provides another filtered isomorphism,

$$\widehat{B(G)} \xrightarrow{\simeq} \widehat{H_1K_G}\,, \tag{4.4}$$

between $I_{G_{\mathrm{ab}}}$-adic completions. A third filtered isomorphism is a consequence of our assumption on $\mathfrak{g}(G)$:

$$\widehat{B(G)} \xrightarrow{\simeq} \widehat{\mathfrak{b}_\bullet(G)}\,, \tag{4.5}$$

where the completion of $\mathfrak{b}_\bullet(G)$ is taken with respect to the degree filtration; see [3, Proposition 5.4]. Since $\dim_{\mathbb{C}} \widehat{\mathfrak{b}_\bullet(G)} < \infty$, we deduce that $\dim_{\mathbb{C}} \mathfrak{b}_\bullet(G) < \infty$. Hence, the degree filtration is finite, and the completion map

$$\mathfrak{b}_\bullet(G) \xrightarrow{\simeq} \widehat{\mathfrak{b}_\bullet(G)} \tag{4.6}$$

is a filtered isomorphism.

By construction, the isomorphism (4.3) is equivariant with respect to the $I_{G_{\mathrm{ab}}}$-adic completion homomorphism, $\mathbb{C}G_{\mathrm{ab}} \to \widehat{\mathbb{C}G_{\mathrm{ab}}}$. Again by construction, the isomorphism (4.4) is $\widehat{\mathbb{C}G_{\mathrm{ab}}}$-linear. By Proposition 5.4 from [3], the isomorphism (4.5) is $\widehat{\exp}$-equivariant, where $\widehat{\exp}\colon \widehat{\mathbb{C}G_{\mathrm{ab}}} \xrightarrow{\simeq} \operatorname{Sym}\widehat{(G_{\mathrm{ab}} \otimes \mathbb{C})}$ is the identification (1.4). Since the degree filtration of $\mathfrak{b}_\bullet(G)$ coincides with its $(G_{\mathrm{ab}} \otimes \mathbb{C})$-adic filtration, as noted earlier, the isomorphism (4.6) is plainly equivariant with respect to the $(G_{\mathrm{ab}} \otimes \mathbb{C})$-adic completion homomorphism, $\operatorname{Sym}(G_{\mathrm{ab}} \otimes \mathbb{C}) \to \operatorname{Sym}\widehat{(G_{\mathrm{ab}} \otimes \mathbb{C})}$. Putting these facts together, we deduce from (4.3)-(4.6) that the natural $\operatorname{Sym}(G_{\mathrm{ab}} \otimes \mathbb{C})$-module structure of the nilpotent $\mathbb{C}G_{\mathrm{ab}}$-module $H_1K_G$, explained in the Introduction, is isomorphic to $\mathfrak{b}_\bullet(G)$ over $\operatorname{Sym}(G_{\mathrm{ab}} \otimes \mathbb{C})$, as stated in Theorem C.

To finish the proof of Theorem C, we have to show that $I^q_{G_{\mathrm{ab}}} \cdot H_1K_G = 0$ if and only if $\mathfrak{b}_q(G) = 0$, for any $q \geq 0$. This assertion will follow from the easily checked remark that, given a vector space $M$ endowed with a decreasing Hausdorff filtration

$\{F_r\}_{r\geq 0}$ ( i.e., $\cap_r F_r = 0$), $F_q = 0$ if and only if $\mathrm{gr}_{\geq q}(M) = \oplus_{r\geq q} F_r/F_{r+1} = 0$. Plainly, all maps (4.3)-(4.6) induce isomorphisms at the associated graded level, and all filtrations are Hausdorff. We deduce that $I^q_{G_{\mathrm{ab}}} \cdot H_1 K_G = 0$ if and only if $\mathfrak{b}_r(G) = 0$ for $r \geq q$. Since $\mathfrak{b}_\bullet(G)$ is generated in degree zero over $\mathrm{Sym}(G_{\mathrm{ab}} \otimes \mathbb{C})$, this is equivalent to $\mathfrak{b}_q(G) = 0$. The proof of Theorem C is complete. □

**Proof of the non-triviality assertion of Theorem A.** The group $G = T_g$ satisfies the hypotheses of Theorem C when $g \geq 4$. Consequently, if $H_1 K_g$ is a trivial $\mathbb{C}(T_g)_{\mathrm{ab}}$-module, then $\mathfrak{b}_1(T_g) = \mathfrak{g}_3(T_g) = 0$. This implies that $\mathfrak{g}_{\geq 3}(T_g) = 0$, since the Lie algebra $\mathfrak{g}_\bullet(T_g)$ is generated in degree 1. In particular, $\dim_{\mathbb{C}} \mathfrak{g}_\bullet(T_g) < \infty$, which contradicts Proposition 9.5 from [8]. □

For the proof of Theorem B, we need to recall the main result of Hain from [8], that gives an explicit presentation of the graded Lie algebra $\mathfrak{g}_\bullet(T_g)$, for $g \geq 6$, in representation-theoretic terms. For representation theory, we follow the conventions from Fulton and Harris [7], like in [8].

The conjugation action in (1.1) induces an action of $\mathrm{Sp}_g(\mathbb{Z})$ on $\mathfrak{g}_\bullet(T_g)$, by graded Lie algebra automorphisms. By Johnson's work, the $\mathrm{Sp}_g(\mathbb{Z})$-action on $\mathfrak{g}_1(T_g)$ extends to an irreducible rational representation of $\mathrm{Sp}_g(\mathbb{C})$. It follows that the $\mathrm{Sp}_g(\mathbb{Z})$-action on $\mathfrak{g}_\bullet(T_g)$ extends to a degree-wise rational representation of $\mathrm{Sp}_g(\mathbb{C})$, by graded Lie algebra automorphisms. By naturality, the symplectic Lie algebra $\mathfrak{sp}_g(\mathbb{C})$ acts on $\mathfrak{b}_\bullet(T_g)$.

The fundamental weights of $\mathfrak{sp}_g(\mathbb{C})$ are denoted $\lambda_1, \dots, \lambda_g$. The irreducible finite-dimensional representation with highest weight $\lambda = \sum_{i=1}^{g} n_i \lambda_i$ is denoted $V(\lambda)$. By Johnson's work, $\mathfrak{g}_1(T_g) = V(\lambda_3) := V$. The irreducible decomposition of the $\mathfrak{sp}_g(\mathbb{C})$-module $\wedge^2 V(\lambda_3)$ is of the form $\wedge^2 V(\lambda_3) = R \oplus V(2\lambda_2) \oplus V(0)$, with all multiplicities equal to 1. For $g \geq 6$, $\mathfrak{g}_\bullet := \mathfrak{g}_\bullet(T_g) = \mathbb{L}_\bullet(V)/$ ideal $(R)$, as graded Lie algebras with $\mathfrak{sp}_g(\mathbb{C})$-action. In particular, $\beta : \wedge^2 \mathfrak{g}_1 \twoheadrightarrow \mathfrak{g}_2$ is identified with the canonical $\mathfrak{sp}_g(\mathbb{C})$-equivariant projection $\wedge^2 V(\lambda_3) \twoheadrightarrow V(2\lambda_2) \oplus V(0)$.

Set $V(0) = \mathbb{C}\cdot z$, $\tilde{R} = R + \mathbb{C}\cdot z$, and denote by $\pi : \wedge^2 V(\lambda_3) \twoheadrightarrow V(2\lambda_2)$ the canonical $\mathfrak{sp}_g(\mathbb{C})$-equivariant projection. Note that both $\mathrm{id} \otimes \pi : \mathrm{Sym}(V(\lambda_3)) \otimes \wedge^2 V(\lambda_3) \to \mathrm{Sym}(V(\lambda_3)) \otimes V(2\lambda_2)$ and the map $\delta_3 : \mathrm{Sym}(V(\lambda_3)) \otimes \wedge^3 V(\lambda_3) \to \mathrm{Sym}(V(\lambda_3)) \otimes \wedge^2 V(\lambda_3)$ from (4.2) are $\mathfrak{sp}_g(\mathbb{C})$-linear. Consequently,

$$\widetilde{\nabla} := (\mathrm{id} \otimes \pi) \circ \delta_3 \colon \mathrm{Sym}(V(\lambda_3)) \otimes \wedge^3 V(\lambda_3) \to \mathrm{Sym}(V(\lambda_3)) \otimes V(2\lambda_2) \tag{4.7}$$

is both $\mathrm{Sym}(V(\lambda_3))$-linear and $\mathfrak{sp}_g(\mathbb{C})$-equivariant. We are going to view the $\mathfrak{sp}_g(\mathbb{C})$-trivial module $\mathbb{C} \cdot z$ as a trivial $\mathrm{Sym}_\bullet(V(\lambda_3))$-module concentrated in degree zero, and assign degree 0 to both $\wedge^3 V(\lambda_3)$ and $V(2\lambda_2)$.

Consider the canonical, $\mathfrak{sp}_g(\mathbb{C})$-equivariant graded Lie epimorphism

$$f : \mathfrak{g}_\bullet = \mathbb{L}_\bullet(V)/\text{ ideal } (R) \twoheadrightarrow \mathbb{L}_\bullet(V)/\text{ ideal } (\tilde{R}) = \mathfrak{k}_\bullet\,. \tag{4.8}$$

**Lemma 4.2.** *The induced* $\mathrm{Sym}(V)$*-linear,* $\mathfrak{sp}_g(\mathbb{C})$*-equivariant map* $\mathfrak{b}_\bullet(f)$ *is onto, with* 1*-dimensional kernel* $\mathbb{C}\cdot z$*.*

*Proof.* The first three claims are obvious. It is equally clear that $\mathfrak{b}_0(f)$ has kernel $\mathbb{C}\cdot z$. To prove injectivity in degree $q \geq 1$, start with the class $\bar{x}$ of an arbitrary element $x \in \mathbb{L}_{q+2}(V)$. If $\mathfrak{b}(f)(\bar{x}) = 0$, then $x$ is equal, modulo $\mathbb{L}''(V)$, with a linear combination of Lie monomials of the form $\mathrm{ad}_{v_1}\cdots\mathrm{ad}_{v_q}(\tilde{r})$, with $\tilde{r} \in \tilde{R}$.

Therefore, $\bar{x}$ belongs to the $\mathbb{C}$-span of elements of the form $\overline{\mathrm{ad}_{v_1}\cdots\mathrm{ad}_{v_q}(z)}$. As shown in [8], the class of $z$ is a central element of the Lie algebra $\mathbb{L}_\bullet(V)/$ ideal $(R)$, and so we are done, since $q \geq 1$. □

**Proof of Theorem B.** By Theorem C, $H_1 K_g = \mathfrak{b}(\mathfrak{g})$, over $\mathrm{Sym}(V)$, with $\mathfrak{g}$ as in (4.8). By Lemma 4.2, $\mathfrak{b}(\mathfrak{g}) = \mathfrak{b}(\mathfrak{k}) \oplus \mathbb{C}\cdot z$, as graded $\mathrm{Sym}(V)$-modules, where $\mathbb{C}\cdot z$ is $\mathrm{Sym}(V)$-trivial (since $z$ is central in $\mathfrak{g}$), with degree 0.

By Lemma 4.1, the graded $\mathrm{Sym}(V)$-module $\mathfrak{b}_\bullet(\mathfrak{g})$ has presentation (1.2); see (4.7). Note also that the identification $\mathfrak{b}(\mathfrak{g}) = \mathfrak{b}(\mathfrak{k}) \oplus V(0)$ is compatible with the natural $\mathfrak{sp}_g(\mathbb{C})$-symmetry of $\mathfrak{b}(\mathfrak{g}) = \mathfrak{b}(T_g)$.

It remains to prove the assertion about the action of $\Gamma_g/K_g$ on $H_1(K_g,\mathbb{C})$. For this we use the theory of relative completion of mapping class groups developed and studied in [8]. Denote the completion of the mapping class group with respect to the standard homomorphism $\Gamma_g \to \mathrm{Sp}_g(\mathbb{C})$ by $\mathcal{R}(\Gamma_g)$. Right exactness of relative completion implies that there is an exact sequence

$$\mathcal{G}(T_g) \to \mathcal{R}(\Gamma_g) \to \mathrm{Sp}_g(\mathbb{C}) \to 1$$

such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_g & \longrightarrow & \Gamma_g & \longrightarrow & \mathrm{Sp}_g(\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{G}(T_g) & \longrightarrow & \mathcal{R}(\Gamma_g) & \longrightarrow & \mathrm{Sp}_g(\mathbb{C}) & \longrightarrow & 1 \end{array}$$

commutes, where $\mathcal{G}(T_g)$ denotes the Malcev completion of $T_g$.

The conjugation action of $\Gamma_g$ on $T_g$ induces an action of $\Gamma_g$ on the Malcev Lie algebra $\mathfrak{g}(T_g)$ of the Torelli group. Basic properties of relative completion imply that this action factors through the natural homomorphism $\Gamma_g \to \mathcal{R}(\Gamma_g)$. This action descends to an action of $\mathcal{R}(\Gamma_g)$ on the Alexander invariant $\mathfrak{b}(T_g)$ of $\mathfrak{g}(T_g)$. Its kernel contains the image of $\mathcal{G}(T_g)'$ in $\mathcal{R}(\Gamma_g)$. Basic facts about the Lie algebra of $\mathcal{R}(\Gamma_g)$ given in [8] imply that, when $g \geq 3$, $\mathcal{R}(\Gamma_g)/\operatorname{im}\mathcal{G}(T_g)'$ is an extension

$$1 \to V \to \mathcal{R}(\Gamma_g)/\operatorname{im}\mathcal{G}(T_g)' \to \mathrm{Sp}_g(\mathbb{C}) \to 1.$$

Levi's theorem implies that this sequence is split. However, we have to choose compatible splittings of the lower central series of $\mathfrak{g}(T_g)$ and this sequence. The existence of such compatible splittings is a consequence of the existence of the mixed Hodge structures on $\mathfrak{g}(T_g)$ and on the Lie algebra of $\mathcal{R}(\Gamma_g)$, and the fact that the weight filtration of $\mathfrak{g}(T_g)$ (suitably renumbered) is its lower central series. Such compatible mixed Hodge structures are determined by the choice of a complex structure on the reference surface $\Sigma$. With such compatible splittings, one obtains a commutative diagram

$$\begin{array}{ccccc} \Gamma_g/K_g & \longrightarrow & \mathcal{R}(\Gamma_g)/\operatorname{im}\mathcal{G}(T_g)' & \xleftarrow{\simeq} & \mathrm{Sp}_g(\mathbb{C}) \ltimes V \\ & & \downarrow & & \downarrow \\ & & \mathrm{Aut}(\mathfrak{b}(T_g)) & \xleftarrow{\simeq} & \mathrm{Aut}(\mathfrak{b}_\bullet(T_g)) \end{array}$$

when $g \geq 4$. This completes the proof of Theorem B. □

**Example 4.3.** Let us examine the simple case when $G = F_n$, the non-abelian free group on $n$ generators. In this case, $H_1(K_G, \mathbb{C}) = B(G) \otimes \mathbb{C}$. Since $G$ is 1-formal, Theorem 5.6 from [3] identifies the $I$-adic completion $\widehat{H_1 K_G}$ with the degree completion $\widehat{\mathfrak{b}_\bullet(\mathfrak{g})}$, where $\mathfrak{g} = \mathfrak{g}_\bullet(G) = \mathbb{L}_\bullet(V)$, and $V = H_1(F_n, \mathbb{C}) = \mathbb{C}^n$.

On the other hand, $\mathcal{V}^1_1(G) = \mathcal{V}^1_1(G) \cap \mathbb{T}^0(G) = (\mathbb{C}^*)^n$ is infinite, in contrast with the setup from Theorem C. It follows from Corollary 6.2 in [18] that $\dim_{\mathbb{C}} H_1 K_G = \infty$. It is also well-known that $\dim_{\mathbb{C}} \mathfrak{b}_\bullet(\mathfrak{g}) = \infty$ when $n > 1$.

This non-finiteness property of $\mathfrak{b}_\bullet(\mathfrak{g})$ can be seen concretely by using the exact Koszul complex, $\{\delta_i : P_\bullet \otimes \wedge^i V \to P_\bullet \otimes \wedge^{i-1} V\}$, where $P_\bullet = \mathrm{Sym}(V)$. Indeed, we infer from (4.2) that, for every $q \geq 0$,

$$\mathfrak{b}_q(\mathfrak{g}) = \operatorname{coker}(\delta_3 : P_{q-1} \otimes \wedge^3 V \to P_q \otimes \wedge^2 V) \cong \ker(\delta_1 : P_{q+1} \otimes V \twoheadrightarrow P_{q+2})$$

has dimension $\binom{q+n}{q+2}(q+1)$, a computation that goes back to Chen's thesis [2]. Note also that each $\mathfrak{b}_q(\mathfrak{g})$ is an $\mathfrak{sl}_n(\mathbb{C})$-module. It turns out that these modules are irreducible, as we now explain.

Let $\{\lambda_1, \dots, \lambda_{n-1}\}$ be the set of fundamental weights of $\mathfrak{sl}_n(\mathbb{C})$ associated to the ordered basis $e_1, \dots, e_n$ of $V$, as in [7]. One can easily check that, for each $q \geq 0$, the image $v$ of the vector

$$u = e_1^q \otimes (e_1 \wedge e_2) \in P_q \otimes \wedge^2 V$$

in $\mathfrak{b}_q(\mathfrak{g})$ is non-zero. Since $u$ is a highest weight vector of weight $q\lambda_1 + \lambda_2$, it follows that $v$ generates a copy of the irreducible $\mathfrak{sl}_n(\mathbb{C})$-module $V(q\lambda_1 + \lambda_2)$ in $\mathfrak{b}_q(\mathfrak{g})$. Since $\dim V(q\lambda_1 + \lambda_2) = \dim \mathfrak{b}_q(\mathfrak{g})$, we conclude that

$$\mathfrak{b}_q(\mathfrak{g}) = V(q\lambda_1 + \lambda_2).$$

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