Annals of Mathematics, 118 (1983), 423–442

The structure of the Torelli group I: A finite set of generators for $\mathcal{I}$

By Dennis Johnson

1. Introduction

This is the first of three papers concerning the so-called Torelli group. Let $M = M_{g,n}$ be a compact orientable surface of genus g having n boundary components and let $\mathcal{M} = \mathcal{M}_{g,n}$ be its mapping class group, that is, the group of orientation preserving diffeomorphisms of M which are 1 on the boundary ∂M modulo isotopies which fix ∂M pointwise. This group is also known to the complex analysts as the *Teichmüller group* or *modular group*. If $n = 0$ or 1, let further $\mathcal{I} = \mathcal{I}_{g,n}$ be the subgroup of $\mathcal{M}$ which acts trivially on $H_1(M, Z)$. The topologists have no traditional name for $\mathcal{I}$, but the analysts tell me it was known classically and is called the *Torelli group*. Several interesting problems and conjectures exist concerning $\mathcal{I}$. The principal one can be found in Kirby's problem list [K] and asks if $\mathcal{I}_{g,0}$ is finitely generated. In this first paper we shall answer the question affirmatively for both $\mathcal{I}_{g,0}$ and $\mathcal{I}_{g,1}$ when $g \geq 3$ and shall give a fairly simple set of generators.

Two other conjectures were made by the author. The first involves the subgroup $\mathcal{T}$ of $\mathcal{I}$ which is generated by twists on nulhomologous simple closed curves. [J1] produces a surjective homomorphism τ: $\mathcal{I}_{g,1} \to \Lambda^3 H_1(M, Z)$ which kills $\mathcal{T}$, and it is conjectured there that in fact $\mathcal{T} = \operatorname{Ker} \tau$. The proof of this is the content of the second paper. In the third paper we use the results of the second to compute the abelianization $\mathcal{I}/\mathcal{I}'$ explicitly, thereby verifying another conjecture in [J1].

The first reasonably simple (but infinite) set of generators for $\mathcal{I}_{g,0}$ was produced by Powell in [P]. His generators consist of two types: a) twists on bounding simple closed curves, b) opposite twists on a (bounding) pair of disjoint homologous simple closed curves, each of which are nonbounding. Using his result, the author showed in [J2] that the maps of the second type, which we call *BP maps* (for bounding pair), are actually sufficient to generate both $\mathcal{I}_{g,0}$ and $\mathcal{I}_{g,1}$ for $g \geq 3$ and in fact that we need only those whose two curves bound a genus one subsurface of M (note that for $g = 2$ all BP maps are trivial and hence the result fails in this case). In the finite set of generators produced in this paper only

BP maps are used, but they include maps for which the bounded subsurface has genus greater than one.

Throughout the paper all surfaces will be compact, orientable and *oriented* and all maps will be smooth. A surface of genus g with n boundary components will frequently be described as "an $M_{g,n}$". SCC means simple closed curve; if γ is an SCC then it determines a *twist map* $T_\gamma \in \mathfrak{M}$ in the usual way. The convention here is that T_γ affects an arc crossing γ by causing it to turn *right* as it approaches γ, run once around γ and then proceed on as before. The order of composition for maps is the *functional* one: $T_\beta T_\alpha$ means apply T_α first, then T_β. On the other hand, if α and β are closed curves representing elements of $\pi_1(M)$ then by $\alpha\beta$ we mean that closed curve which traverses α first, then β. Finally, $[x, y]$ means $xyx^{-1}y^{-1}$.

2. Chains and chain maps

For the purpose of describing our generators succinctly we need a different way than by specifying the two twisting curves; the latter is inconvenient, and we take another approach. Since a BP map consists of twists on the boundary of an $M_{k,2}$ subsurface of M, everything could be defined equally well by drawing a spine of the $M_{k,2}$ in M, along with which boundary curve receives the positive twist. Consider for example the BP map of genus k given below:

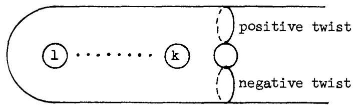


FIGURE 1a

One possible spine of this surface is the union of the k circles numbered above and the $k + 1$ circles below:

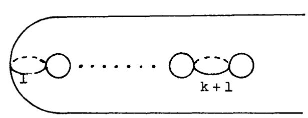


FIGURE 1b

This spine is a kind of "chain" of $2k + 1$ circles. If we give these circles some order, say left to right, and write them $c_1, c_2, \ldots, c_{2k+1}$, then they acquire orientations in a unique way if we require that:

a) The algebraic intersection $c_i \cdot c_{i+1}$ is $+1$ (this determines the orientations up to a simultaneous reversal of them all), and

b) the odd numbered curves $c_1, c_3, \ldots, c_{2k+1}$, which split the $M_{k,2}$ into two $M_{0,k+2}$'s, shall be so oriented that the $M_{0,k+2}$ which is on their left contains the positive twist boundary curve. Conversely, an oriented chain of this kind determines a unique BP map by taking opposite twists on the two boundary curves of a regular neighborhood of the chain, with the positive twist on the "left side" of $c_1, c_3, \ldots, c_{2k+1}$. Of course, many different chains will determine the same map, but for our purpose this will not be a problem. We therefore make the following definitions:

Definition 1. A *chain* in M is an ordered collection $(c_1, c_2, \ldots, c_n)$ of oriented SCC's such that

a) c_i intersects c_{i+1} transversely in a single point and the algebraic intersection $c_i \cdot c_{i+1}$ is $+1$,

b) $c_i \cap c_j$ is empty if $|i - j| > 1$, and

c) the homology classes of the c_i's are linearly independent.

Chains which are isotopic via an ambient isotopy are not considered as distinct (more precisely, we could define a chain to be an isotopy class of the above objects). Note that the regular neighborhood of the above chain has genus $\frac{n-1}{2}$ and two boundary components if n is odd and has genus $\frac{n}{2}$ and one boundary component if n is even. The purpose of condition c) is to insure that, for odd n, each of the two boundary curves is nonbounding; if n is even, condition c) follows from a) and b) alone. In order to call the reader's attention to the fact that $(c_1, \ldots, c_n)$ is a chain, we will frequently write $\mathrm{ch}(c_1, \ldots, c_n)$.

Definition 2. a) The *length* of the above chain is n; we refer to such an n-chain as *even* or *odd* in accord with its length.

b) The *basic circles* of the above chain are the SCC's c_i.

Definition 3. If $\mathrm{ch}(c_1, \ldots, c_{2k+1})$ is an odd chain, the unique BP map in $\mathcal{I}$ given by opposite twists on the two boundary curves, the positive twist being taken on that curve lying to the "left" of $c_1, c_3, \ldots, c_{2k+1}$, will be written $[c_1, c_2, \ldots, c_{2k+1}]$ and called the *chain map* of $\mathrm{ch}(c_1, \ldots, c_{2k+1})$.

The mapping class group of M acts on chains in the obvious way: If g is a diffeomorphism representing an element of $\mathfrak{M}$ and $(c_1, \ldots, c_n)$ is a chain, then we write $g * (c_1, \ldots, c_n)$ for the chain $(g(c_1), \ldots, g(c_n))$. This is well defined up to isotopy. The chain map of $g * \mathrm{ch}(c_1, \ldots, c_n)$ is clearly $g[c_1, \ldots, c_n]g^{-1}$ (we are using here the fact that $gT_\gamma g^{-1} = T_{g(\gamma)}$ for any SCC γ). We will usually, by analogy, write this conjugate as $g * [c_1, \ldots, c_n]$, and more generally write $g * f$ for gfg^{-1}; this will simplify many of the formulas in the sequel. If γ is any SCC disjoint from the c_i's, then clearly $T_\gamma * \mathrm{ch}(c_1, \ldots, c_n) = \mathrm{ch}(c_1, \ldots, c_n)$ and so $T_\gamma * [c_1, \ldots, c_n] = [c_1, \ldots, c_n]$. But we note also that if γ equals any one of the c_i's then it is disjoint from the two boundary curves of the regular neighborhood of the chain, and hence we still have $T_\gamma * [c_1, \ldots, c_k] = [c_1, \ldots, c_k]$. In this case however, it is *not* true that $T_\gamma * \mathrm{ch}(c_1, \ldots, c_k) = \mathrm{ch}(c_1, \ldots, c_k)$; for example consider the following 3-chain:

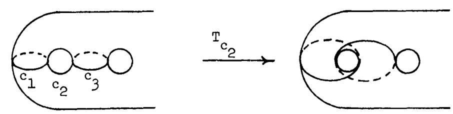


FIGURE 2

The results of [J3] (see Theorem 5, its corollary, and Theorem 6) tell us that the number of generators of $\mathcal{I}$ must be large. In order to construct a large number of chains from a given one $(c_1, \ldots, c_n)$ in a unified way, we now introduce the notion of a *subchain*. One possible definition is obvious: if $c_i, c_{i+1}, \ldots, c_j$ is any consecutive subset of the basic circles, then they clearly form a chain; we call such a subchain a *consecutive subchain*. We will need, however, a more general notion of a subchain.

Let $\gamma_1, \ldots, \gamma_n$ be any collection of mutually transverse oriented SCC's with no triple points, i.e. with $\gamma_i \cap \gamma_j \cap \gamma_k = \varnothing$ for i, j, k distinct. We use them to construct a collection of *disjoint* oriented SCC's as follows: choose disjoint disc neighborhoods of the intersection points and in each of them replace by (and similarly for the mirror image type of crossing). The result is well defined up to isotopy and we call the new collection $\gamma_1 + \cdots + \gamma_n$. Suppose we are given r consecutive blocks of consecutive basic circles of our chain, that is, consecutive subchains $K_1 = \mathrm{ch}(c_{i_1}, c_{i_1+1}, \ldots, c_{i_2-1})$, $K_2 = \mathrm{ch}(c_{i_2}, c_{i_2+1}, \ldots, c_{i_3-1}), \ldots,$ $K_r = \mathrm{ch}(c_{i_r}, \ldots, c_{i_{r+1}-1})$. Then the sum of the curves in each K_i is an SCC k_i and $(k_1, k_2, \ldots, k_r)$ is easily seen to be a chain; this is our general definition for a subchain of $\mathrm{ch}(c_1, \ldots, c_n)$.

Notation. The above subchain will be denoted by $(i_1 i_2 \ldots i_{r+1})$ or $\mathrm{ch}(i_1 i_2 \ldots i_{r+1})$; the BP map determined by this chain will be denoted by $[i_1 i_2 \ldots i_{r+1}]$.

Consider for example the chain $(c_1, \ldots, c_6)$ in $M_{3,1}$ shown below in Figure 3a; the subchains $(1\,3\,6) = (c_1 + c_2, c_3 + c_4 + c_5)$ and $(1\,2\,4\,7) = (c_1, c_2 + c_3, c_4 + c_5 + c_6)$ are depicted in Figures 3b, c, respectively.

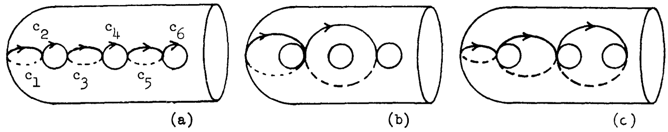


FIGURE 3

In terms of this notation, the full chain $(c_1, \ldots, c_n)$ is just $(1\,2\,3 \ldots n+1)$, and we see that the notation creates a correspondence between subchains of the full chain and subsets of the full index set $1, 2, 3, \ldots, n+1$; this explains our use of the term "subchain". (*Note*: In the notation $(1\,2 \ldots n+1)$ the $(n+1)$-st curve is *not* involved in the chain and may not even exist (if $n = 2g$). The last number serves to note the termination of the chain at the n-th curve.) The notation also makes clear the following.

LEMMA 1. a) *A consecutive subchain is notated by a consecutive set of indices.*

b) *The length of a subchain is one less than the number of its indices.*

c) *The basic circles of the subchain* $(i_1 i_2 \ldots i_{r+1})$ *are the* 1*-chains* $(i_1 i_2) = c_{i_1} + c_{i_1+1} + \cdots + c_{i_2-1}, (i_2 i_3)$, *etc.*

d) *The number of k-subchains of an n-chain is* $\binom{n+1}{k+1}$.

e) *If p is a subchain of q and q of r then p is a subchain of r.*

f) *If* $C_j = T_{c_j}$ *then* C_j *commutes with the subchain map* $[i_1 i_2 \ldots]$ *if and only if* j *and* $j+1$ *are either both contained in or are disjoint from the* i*'s. If* $j = i_m$ *but* $j+1 \neq i_{m+1}$, *that is if* $j = i_m \neq i_{m+1} - 1$, *we get* $C_{i_m}^{-1} * [i_1 \ldots] = [i_1 \ldots i_{m-1}, i_m + 1, i_{m+1} \ldots]$ *and likewise, if* $j+1 = i_m$ *but* $j \neq i_{m-1}$, *that is if* $j + 1 = i_m \neq i_{m-1} + 1$, *we get*

$$C_{i_m - 1} * [i_1 \ldots] = [i_1 \ldots i_{m-1}, i_m - 1, i_{m+1}, \ldots].$$

We need only prove the last statement f). If j, $j+1$ are either contained in or disjoint from the index set, then c_j is either a basic circle of the subchain or disjoint from it, and the commutativity follows. The other two cases follow easily by a comparison with the following pictures:

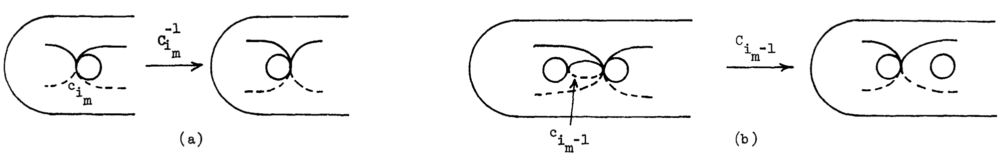


FIGURE 4

We shall single out two specific chains on our surface $M = M_{g,1}$. The basic circles $c_1, \dots, c_{2g}$ and c_β are shown in Figure 5,

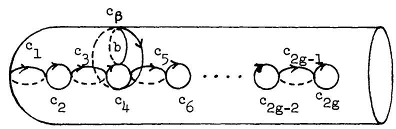


FIGURE 5

and the two chains we want are the $2g$-chain $(c_1, c_2, \dots, c_{2g})$, consisting of all c_i $(i = 1, 2, \dots, 2g)$, and also the $(2g-3)$-chain $(c_\beta, c_5, c_6, \dots, c_{2g})$ consisting of c_β and all c_i for $i \geq 5$. The curve b is not in either of these chains, but we note that if we put $B = T_b$, then $c_\beta = B(c_4)$.

Now any subchain of $\mathrm{ch}(c_\beta \dots c_{2g})$ which does not involve the first circle c_β is also a subchain of $\mathrm{ch}(c_1, \dots, c_{2g})$. Thus we would like a notation for the subchains of $\mathrm{ch}(c_\beta, c_5, \dots)$ which is consistent with the notation given to the subchains of $\mathrm{ch}(c_1, \dots, c_{2g})$. This is easily done by using the index set $\beta, 5, 6, \dots, 2g+1$ for notating the subchains of $\mathrm{ch}(c_\beta, c_5, \dots)$. Thus $\mathrm{ch}(\beta i_1 i_2 \dots)$, where $i_1 \geq 5$, denotes the chain whose basic circles are $(\beta i_1) = c_\beta + c_5 + \cdots + c_{i_1 - 1}, (i_1 i_2) =$ etc., and any chain $(i_1 \dots)$ with $i_1 \geq 5$ denotes the same subchain of either $\mathrm{ch}(c_1, \dots, c_{2g})$ or $\mathrm{ch}(c_\beta, \dots, c_{2g})$. The two original chains are denoted by $\mathrm{ch}(1\,2\,3 \dots 2g+1)$ and $\mathrm{ch}(\beta\,5\,6 \dots 2g+1)$. Clearly we have $(\beta i_1) = B * (4 i_1)$, and more generally $\mathrm{ch}(\beta i_1 i_2 \dots) = B * \mathrm{ch}(4 i_1 i_2 \dots)$ whenever $i_1 \geq 5$. The circles $(\beta 5), (\beta 6), (\beta 7)$, etc., are depicted below.

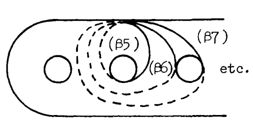


FIGURE 6

We are now in a position to state the principal theorem of this paper.

MAIN THEOREM. *For* $g \geq 3$ *the odd subchain maps of the two chains* $\mathrm{ch}(1\,2\,3 \dots 2g+1)$ *and* $\mathrm{ch}(\beta\,5\,6 \dots 2g+1)$ *generate* $\mathcal{I}_{g,1}$.

We are thus interested in subchains of $\mathrm{ch}(1\,2\,3 \dots 2g+1)$, which will be called *straight chains*, and subchains of the form $\mathrm{ch}(\beta i_1 \dots)$, which will be called *$\beta$-chains*.

Before getting involved in the technicalities of the proof it might be useful to give a brief description of our approach. By results mentioned earlier, for

$g \geq 3$, $\mathcal{I}_{g,1}$ is generated by all BP maps of genus 1, or in our present terminology by all possible 3-chain maps. Since all BP maps of genus 1 are conjugate in $\mathfrak{M}_{g,1}$, this can be rephrased in the following way: *A subgroup of* $\mathcal{I}_{g,1}$ ($g \geq 3$) *which is normal in* $\mathfrak{M}_{g,1}$ *and contains a* 3*-chain map must be* $\mathcal{I}_{g,1}$ *itself.* Our aim will be then to prove that the group generated by straight- and β-chain maps is normal in $\mathfrak{M}_{g,1}$ (true for $g \geq 2$). For this it suffices to choose generators G_i for $\mathfrak{M}_{g,1}$ and show that, for all i, the conjugate of each of our chain maps by $G_i^{\pm 1}$ is still in the group they generate. Hence it would pay us to choose the G_i carefully. The most convenient set of generators for our purposes are those given by Humphries; in [H] he reduces Lickorisch's original set of $3g - 1$ twist generators (see [L]) to just $2g + 1$, namely $C_i = T_{c_i}$ ($i = 1, \ldots, 2g$) and $B = T_b$. These generators are nicely "aligned" with our chains and make the computation of the required conjugates particularly simple. One technical hitch is that Humphries' original theorem shows only that the above maps generate $\mathfrak{M}_{g,0}$, but we will show below that they generate $\mathfrak{M}_{g,1}$ also. We begin now the proof in earnest.

3. Generating the kernel of $\mathcal{I}_{g,1} \to \mathcal{I}_{g,0}$

Given the surface $M = M_{g,1}$ we may fill in the boundary curve with a disc D and get a closed surface $\overline{M}$ of genus g. If we think of $\mathfrak{M}_{g,1}$ as all maps of $\overline{M}$ which are 1 on D modulo isotopies which are pointwise fixed on D, then we get a homomorphism $\mathfrak{M}_{g,1} \to \mathfrak{M}_{g,0}$. That this homomorphism is surjective follows from the fact that every (orientation preserving) diffeomorphism of $\overline{M}$ is isotopic to one which equals 1 on D. The kernel K of this map may thus be thought of as diffeomorphisms of $\overline{M}$ which are 1 on D and isotopic to 1 on $\overline{M}$, modulo isotopies which leave D pointwise fixed. Clearly K is contained in $\mathcal{I}_{g,1}$, so we get an exact sequence $0 \to K \to \mathcal{I}_{g,1} \to \mathcal{I}_{g,0} \to 0$, and we see that to show $\mathcal{I}_{g,0}$ is finitely generated (f.g.), it suffices to show that $\mathcal{I}_{g,1}$ is f.g.

By a *maximal odd chain* on M we mean any $(2g - 1)$-chain (q). The boundary curves of a regular neighborhood of (q) then split $\overline{M} = M \cup D$ into an $M_{g-1,2}$ and an annulus, the latter containing D. The chain map $[q]$ is thus isotopic to 1 in the annulus, i.e., $[q] \in K$. Our first goal is to produce a convenient set of maximal odd chain maps which generate K. This bears on our problem as follows: in order to prove the main theorem we will be faced with showing that the conjugate of some $(2k - 1)$-chain map by some twist map T_c remains in the group of (straight and β-) chain maps. In most cases, the union of the $(2k - 1)$-chain and the curve c will lie in a subsurface of type $M_{k,1}$, and in this surface the conjugated chain map is maximal and hence in the kernel of $\mathfrak{M}_{k,1} \to \mathfrak{M}_{k,0}$. A detailed knowledge concerning this kernel will then enable us to conclude that the conjugated chain map still lies in our group of chain maps.

To begin with, we need an alternate description of K. This kernel is closely related to the braid group of $\bar{M}$ (compare [B], pp. 5–15). For future use we will need this alternate description for surfaces of the form $M_{g,n}$ with $n > 1$ as well. Let then $M = M_{g,n}$, and if ∂ is one of the boundary curves of M, fill it in with a disc D to get a surface $\bar{M}$ of type $M_{g,n-1}$. As in the above paragraph, we think of maps of $\mathfrak{M}_{g,n}$ as diffeomorphisms of $\bar{M}$ which are 1 on D, etc., and we still have a surjective homomorphism $\mathfrak{M}_{g,n} \to \mathfrak{M}_{g,n-1}$ with kernel which we still denote by K.

If now $f \in K$, it is isotopic to 1 in $\bar{M}$; so let f_t be an isotopy with $f_0 = f$ and $f_1 = 1$. The restriction of f_t to D, $f_t\colon D \to \bar{M}$, gives us then a homotopy class of framed curves on $\bar{M}$ in the obvious way: we fix a base point $d \in \operatorname{Int} D$ and a frame v at d once and for all; then for each t we get a point $f_t(d)$ and frame $df_t(v)$. We call this homotopy class $\varphi(t)$; since a framed curve is completely determined up to homotopy by its first (unit) vector, $\varphi(t)$ lies in $\tilde{\pi} = \pi_1(U\bar{M}, (d, v))$, where $U\bar{M}$ is the unit tangent bundle of $\bar{M}$ and (d, v) is the "base point" in $U\bar{M}$ (v is now just a unit vector).

So far $\varphi(f)$ depends not only on the specific diffeomorphism but also on the choice of isotopy f_t connecting f to 1. Note however that since $\pi_1(\operatorname{Diff} \bar{M}) = 0$ for $g \geq 2$ (see [G], Theorem 2) any two isotopies f_t and f_t' connecting f to 1 can be "filled in" by a map $\bar{M} \times D^2 \to \bar{M}$. The latter map, restricted to the base frame, gives us thereby a homotopy connecting the two framed curves defined by f_t and f_t'. Thus φ is at least a function from $\operatorname{Ker}(\operatorname{Diff} M \to \mathfrak{M}_{g,n-1})$ to $\tilde{\pi}$.

LEMMA 2. *φ is a homomorphism. It is zero on maps isotopic to* 1 rel D *and so is actually a homomorphism from K to $\tilde{\pi}$.*

Proof. Choose isotopies f_t, g_t of $\bar{M}$ connecting f, g to 1. The result of the isotopy f_t is to move each point x to $f^{-1}(x)$ and hence move $fg(x)$ to $g(x)$. Following this by the isotopy g_t results in x; thus f_t followed by g_t is an isotopy connecting fg to 1. Restricting this composite isotopy to the base frame gives us $\varphi(fg)$ as the composite of two framed paths $\varphi(f) \cdot \varphi(g)$, proving the first statement. If f is isotopic to 1 on M then the isotopy leaves D pointwise fixed and so $\varphi(f) = 1$ in $\tilde{\pi}$.

LEMMA 3. *$\varphi\colon K \to \tilde{\pi}$ is an isomorphism.*

Proof. Since K is all maps of $\bar{M}$ which are 1 on D and isotopic to 1 on $\bar{M}$, modulo isotopies which are pointwise fixed on D, we can form a quotient K' of K by further dividing out by those isotopies which leave $d \in D$ fixed; K' is known as the (one-point) *braid group* of $\overline{M}$. Given an f in the kernel of $K \to K'$, the isotopy of f to 1 which fixes d will rotate the frame v at d a certain integral number of times n (counterclockwise rotation = positive n). In particular, for the boundary curve ∂, the boundary twist map $T_\partial \in K$ is connected to 1 by a single clockwise rotation of D. But then $f \cdot T_\partial^n$ clearly can be connected to 1 by an isotopy which leaves D pointwise fixed; that is, $fT_\partial^n = 1$ in K. Hence the kernel of $K \to K'$ is generated by T_∂ and is obviously isomorphic to the integers $\mathbf{Z}$.

By throwing away the framing, the map φ: $K \to \tilde{\pi}$ passes to the quotient map φ': $K' \to \pi = \pi_1(\overline{M}, d)$. The projection map $\tilde{\pi} \to \pi$ also has kernel $\mathbf{Z}$ with generator given by the "fiber class" z obtained by rotating the frame at d counterclockwise one turn; hence $\varphi(T_\partial) = z^{-1}$. We have then a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z} & \to & K & \to & K' & \to & 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \varphi' & & \\ 0 & \to & \mathbf{Z} & \to & \tilde{\pi} & \to & \pi & \to & 0 \end{array}$$

and the first vertical map is an isomorphism. But φ' is also well known and easily seen to be an isomorphism (see [B], Theorem 1.7, p. 17 or [G], Prop. 1, p. 55). This implies that φ is an isomorphism too. Q.E.D.

Since $h \in \mathfrak{M}_{g,n}$ fixes D pointwise, it acts on $\tilde{\pi} = \pi_1(U\overline{M}, (d, v))$. $\mathfrak{M}_{g,n}$ also acts on its normal subgroup K by conjugation. That φ preserves this action is given by:

LEMMA 4. *For* $h \in \mathfrak{M}_{g,n}$, $f \in K$, *we have* $\varphi(hfh^{-1}) = h(\varphi(f))$; *that is,* $\varphi(h * f) = h(\varphi(f))$.

Proof. Let f_t connect f to 1 in $\overline{M}$; then hf_th^{-1} connects hfh^{-1} to 1. Restricting to D and noting that $h = 1$ on D we see that $\varphi(h * f)$ is given by the framed path defined by $h \circ f_t$ on D, which is just $h(\varphi(f))$.

We return to the case of primary interest to us, namely surfaces $M = M_{g,1}$ with a single boundary component. We are now in a position to produce generators and relations in K by doing the same for $\tilde{\pi}$. One convenient way of representing elements of $\tilde{\pi}$ is by means of smooth curves beginning at d and tangent to v there, the framing being given by the tangent vector (this tangent vector may point either forwards or backwards along the curve, but is determined throughout by continuity once it starts at v). As an added computational convenience we shall permit the use of curves which are smooth everywhere excepting a finite and even number of cusps. The framing is uniquely defined by the conditions that it begin at v, is continuous, and is tangential at a smooth point. We distinguish two kinds of framed cusps:

FIGURE 7a

The arrows indicate the direction in which the curve is traversed; the direction of the tangent vector is irrelevant. Figure 7b shows some basic homotopies in $U\overline{M}$ between framed curves with cusps:

FIGURE 7b

Note also that cusps and loops can be moved freely along the curve, and hence two positive cusps anywhere on the curve can be replaced by smooth arcs and an extra z factor.

We apply these ideas to produce generators of K. Starting with our standard straight $2g$-chain of M we form its $2g + 1$ maximal odd subchain maps $W_1 = [2\,3\,4 \ldots 2g + 1]$, $W_2 = [1\,3\,4 \ldots 2g + 1]$, $W_3 = [1\,2\,4\,5 \ldots 2g + 1], \ldots,$ $W_{2g+1} = [1\,2\,3\,4 \ldots 2g]$ and we put $\varphi(W_i) = w_i \in \tilde{\pi}$. Since W_{2g+1} is given by Figure $8a$,

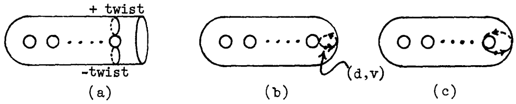


FIGURE 8

we see that w_{2g+1} is the tangentially framed curve shown in Figure $8b$. Conjugating W_{2g+1} by C_{2g}^{-1} gives $C_{2g}^{-1} * W_{2g+1} = W_{2g}$, so $w_{2g} = C_{2g}^{-1}(w_{2g+1})$ is as in Figure $8c$. Likewise we get the other w_i's as in Figure 9:

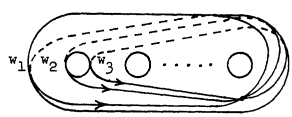


FIGURE 9

LEMMA 5. *K is generated by the maximal odd subchain maps $W_1, \ldots, W_{2g}$ and $B * W_1$.*

Proof. We are asked to show that $\tilde{\pi}$ is generated by the $2g + 1$ elements $w_1, \ldots, w_{2g}$ and $B(w_1)$. Now the images of w_1 through w_{2g} in $\pi = \pi_1(\overline{M}, d)$ certainly generate there (for example, it is easy to see that the complement of these curves is a disc so that any curve in $\overline{M}$ can be homotoped into their union). Thus if $\tilde{x} \in \tilde{\pi}$ has image $x \in \pi$, there is a word in $w_1, \ldots, w_{2g}$ which projects to x and so is of the form $\tilde{x} \cdot z^n$ in $\tilde{\pi}$; our problem reduces then to the generation of z. The curve $w_4 w_3^{-1} w_2$ is shown in Figure 10a and has two positive cusps (the cusp on the back looks negative because of orientation reversal):

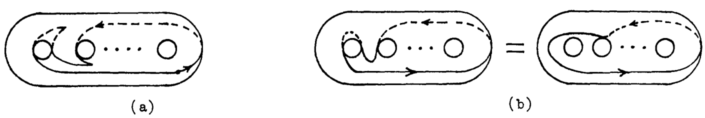


FIGURE 10

It is thus z times the curve of Figure 10b, which is easily seen to be $B(w_1)$. In other words, $B(w_1) \cdot z = w_4 w_3^{-1} w_2$, Q.E.D.

COROLLARY . $B * W_1 = W_4 W_3^{-1} W_2 T_\partial$ *in* $\mathcal{G}_{g,1}$.

LEMMA 6. *For genus* 2, *K is generated by the* 5 *chain maps* $W_1, W_2, \ldots, W_5$.

Proof. If we compute $w_5 w_4^{-1} w_3 w_2^{-1} w_1$ we get the curve of Figure 11, with 4 positive cusps:

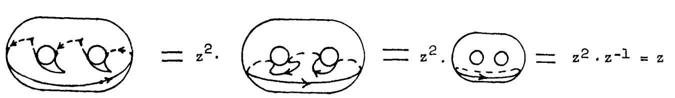


FIGURE 11

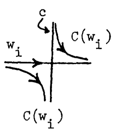
The result follows as in the preceding lemma.

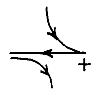
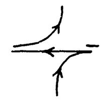
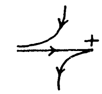
COROLLARY . $W_5 W_4^{-1} W_3 W_2^{-1} W_1 = T_\partial^{-1}$ *in* $\mathcal{G}_{2,1}$.

It is not true that $w_1, \ldots, w_{2g+1}$ generate $\tilde{\pi}$ for $g \geq 3$ (in fact, it can be shown that the smallest positive power of z in the generated group is z^{g-1}). We have, however, the following useful lemma, which will be a basic tool in proving the main theorem.

LEMMA 7. *The group generated by* $W_1, \ldots, W_{2g+1}$ *is normalized by the twist maps* $C_1, C_2, \ldots, C_{2g}$.

Proof. We are asked to prove that the subgroup of $\tilde{\pi}$ generated by $w_1, w_2, \ldots, w_{2g+1}$ is *invariant* under the action of the C_j's; that is, $C_j^{\pm 1}(w_i)$ is in this subgroup for all i, j. If we examine Figure 9, however, it is clear that C_j leaves w_i invariant unless $j = i$ or $i - 1$ and further that $C_i(w_i) = w_{i+1}$ and $C_{i-1}^{-1}(w_i) = w_{i-1}$; so we must show only that $C_i^{-1}(w_i)$ and $C_{i-1}(w_i)$ are in the group. Note that c_i and c_{i-1} intersect w_i transversely in a single point. If we put $c = c_i$ or c_{i-1} and $C = T_c$, we can draw the following schematic representation of the action of C:



and we see that $C(w_i)w_i^{-1} =$  (the second cusp is at the base point d). Likewise we get $C^{-1}(w_i)w_i^{-1} =$ . But note that in the picture for $C(w_i)w_i^{-1}$, we may move the cusp along the curve so as to get: .

Comparing this with the picture for $C^{-1}(w_i)w_i^{-1}$, we get: $(C(w_i)w_i^{-1})^{-1} = C^{-1}(w_i)w_i^{-1}$. Now we have already seen that one of the paths $C^{\pm 1}(w_i)$ is a w_j and so is in the group generated by all the w_i's. The above equation shows that the other path is also in the group. This finishes the proof.

We can now extend Humphries' theorem to $M_{g,1}$.

THEOREM 1. *C_i $(1 \leq i \leq 2g)$ and B generate $\mathfrak{M}_{g,1}$.*

Proof. We have an exact sequence $0 \to K \to \mathfrak{M}_{g,1} \to \mathfrak{M}_{g,0} \to 0$ and Humphries' theorem tells us that the above maps generate $\mathfrak{M}_{g,0}$; so we only need show that in $\mathfrak{M}_{g,1}$ they generate K also. But it suffices to show that the chain map $W_{2g+1} = [1\,2\,3 \ldots 2g]$ is so generated, since from it we can produce consecutive conjugates by $C_{2g}^{-1}, C_{2g-1}^{-1}, \ldots, C_1^{-1}$ and B to get $W_{2g}, W_{2g-1}, \ldots, W_1$ and $B * W_1$, which generate K by Lemma 5. Now $W_{2g+1} = T_x T_{x'}^{-1}$ where x, x' are the curves shown in Figure 8a. Recall that $B = T_b$ where b is as shown in Figure 5. The key to Humphries' proof is to show that in the $M_{3,2}$ of Figure 12, the curve γ_1 can be moved to β' via a sequence of twists using only the curves $\beta, \gamma_2, \ldots, \gamma_6$.

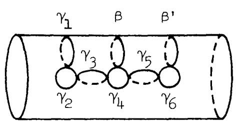


FIGURE 12

In this way, starting with $\gamma_i = c_i$, $\beta = b$, Humphries acquires successively the curves $d, e, \ldots, x$ of Figure 13 using only the generating twists $C_1, \ldots, C_{2g}$ and

B, and concludes that $T_d, T_e, \ldots, T_x$ are also in the generated group. These moves are clearly all carried on our $M_{g,1}$ and thus T_x is also in the generated subgroup of $\mathfrak{M}_{g,1}$.

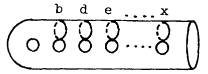


FIGURE 13

Consider next the sequence of moves depicted in Figure 14:

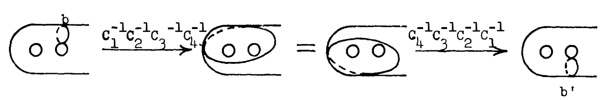


FIGURE 14

Applying Humphries' moves to b' gives the curve x', and so $T_{x'}$ is in the generated group as well. Hence $T_x T_{x'}^{-1} = W_{2g+1}$ is also, Q.E.D.

4. Proof of the main theorem

Before we begin the proof of the theorem, we shall need a generalization of Lemma 7.

LEMMA 8. *Let $\Gamma = \text{ch}(c_1, c_2, \ldots, c_n)$ be a chain of M, and for $k \geq 2$ let G be the subgroup of $\mathcal{I} = \mathcal{I}_{g,1}$ generated by all $(2k-1)$-subchain maps of Γ; then G is normalized in $\mathfrak{M}$ by the twist maps $C_i = T_{c_i}$. The same then holds for the group generated by all odd subchain maps of Γ.*

Proof. Let $p = \text{ch}(i_1 i_2 \ldots i_{2k})$ be any $(2k-1)$-subchain, $[p]$ its chain map and C_j any of the basic twist maps. Recall again that C_j commutes with $[p]$ unless exactly one of the j or $j+1$ is an index of p. In this case we can enlarge p to include *both* j and $j+1$, thereby getting a subchain q of length $2k$. Note that

a) the regular neighborhood of q is an $M_{k,1}$,

b) c_j is a basic circle of q, and

c) p is a maximal odd subchain of q.

Hence by Lemma 7, $C_j^{\pm 1} * [p]$ is a product of maximal subchain maps of q. But all maximal subchains of q are $(2k-1)$-subchains of Γ and so $C_j^{\pm 1} * [p]$ is also in G, Q.E.D.

We now begin the proof of the main theorem in the following form:

THEOREM 2. *For $g \geq 2$ the subgroup J_g of $\mathcal{I}_{g,1}$ generated by all odd subchain maps of the standard chains* $\text{ch}(1\,2\,3 \ldots 2g+1)$ *and* $\text{ch}(\beta\,5\,6 \ldots 2g+1)$ *is* *normal in* $\mathcal{M}_{g,1}$. (*As previously pointed out, the theorem that* $\mathcal{I}$ *is* f.g. *is a corollary of this one*: *in fact*, $\mathcal{I} = J_g$.)

Proof. For $g = 2$ the above maps are just $W_1, \ldots, W_5$ which generate the normal subgroup $K = \operatorname{Ker}(\mathcal{M}_{2,1} \to \mathcal{M}_{2,0})$ by Lemma 6. We assume hence that $g \geq 3$ and inductively that the theorem is true for J_{g-1}. If we think of $M_{g-1,1}$ to be imbedded in $M_{g,1}$ as shown in Figure 15,

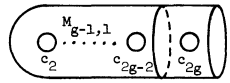


FIGURE 15

we see that among our generators for J_g are found the generators for J_{g-1}, namely all subchain maps of ch$(1\,2\,3 \ldots 2g-1)$ and ch$(\beta\,5 \ldots 2g-1)$ (the latter is empty for $g = 3$); by hypothesis this group J_{g-1} is normalized by $C_1, \ldots, C_{2g-2}$ and B, and in particular by B and C_1 through C_4 since $g \geq 3$. (*Note*: We are implicitly assuming here that $J_{g-1} \subset J_g$. More precisely, $J_{g-1} \subset \mathcal{M}_{g-1,1}$ and $J_g \subset \mathcal{M}_{g,1}$ and generators of J_{g-1} can be identified with some of those of J_g by extending the map on $M_{g-1,1}$ by the identity to $M_{g,1}$. The homomorphism $J_{g-1} \to J_g$ thus produced is not clearly injective (although this is true), but it *is* clear that any diffeomorphism of $M_{g,1}$ which is carried by $M_{g-1,1}$ (that is, is the identity on its complement) and which when restricted to $M_{g-1,1}$ is in J_{g-1} is itself a map in J_g. The reader will encounter below loose statements such as "$f \in J_{g-1}$, so $f \in J_g$," or even "$f \in J_{g-1} \subset J_g$;" these stand as abbreviations for the more precise statement given above.)

Part I. J_g is normalized by C_1, C_2 and C_k for $k \geq 5$.

Proof. C_1, C_2 commute with β-chain maps and normalize the group of straight chain maps by Lemma 8; so they normalize J_g. For $k \geq 5$, c_k is a basic circle of both ch$(1\,2\,3 \ldots 2g+1)$ and ch$(\beta\,5 \ldots 2g+1)$ so that C_k normalizes both their subchain groups, again by Lemma 8, and thus also normalizes J_g.

It remains to prove that C_3, C_4 and B normalize J_g. At this point we need a certain relation between our chain maps. We begin with a relation between twist maps on an $M_{0,4}$ which was exploited in [J2]; for the curves α, β, γ, ε_i shown in Figure 16:

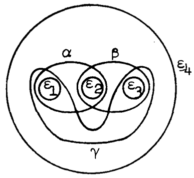


FIGURE 16

we have the following relation in $\mathcal{M}_{0,4}$:

$$T_\gamma T_\beta T_\alpha = T_{\varepsilon_1} T_{\varepsilon_2} T_{\varepsilon_3} T_{\varepsilon_4}.$$

We call this equation "the $M_{0,4}$ relation" and apply it to prove:

LEMMA 9. $[2\,3\,4\,5]^{-1}[4\,5\ldots 2g+1]^{-1}[\beta\,5\ldots 2g+1]B*[2\,3\,4\,5] = [2\,3\,4\ldots 2g+1]^{-1}B*[2\,3\,4\ldots 2g+1]$.

Proof. Consider the curves α, β, γ, δ, ε of Figure 17:

FIGURE 17

The curves c_2, c_4, δ, ε bound an $M_{0,4}$ and α, β, γ lie in it in the pattern of the $M_{0,4}$ relation. We get thereby:

$$T_\gamma T_\beta T_\alpha = C_2 C_4 T_\delta T_\varepsilon.$$

Now reflect Figure 17 in the plane of the paper, reversing orientation and front and back parts of the surface. If the transforms of α, β, etc. are denoted by α', β', etc., then right twists transform into left ones and the above relation transforms into

$$T_{\gamma'}^{-1} T_{\beta'}^{-1} T_{\alpha'}^{-1} = C_2^{-1} C_4^{-1} T_{\delta'}^{-1} T_{\varepsilon'}^{-1}.$$

Multiplying these two relations together, we then put $T_\gamma T_{\gamma'}^{-1} = f$. Noting that

$$T_\beta T_{\beta'}^{-1} = [4\,5\,6\ldots 2g+1], \qquad T_\alpha T_{\alpha'}^{-1} = [2\,3\,4\,5],$$

$$T_\delta T_{\delta'}^{-1} = [6\,7\ldots 2g+1], \qquad T_\varepsilon T_{\varepsilon'}^{-1} = [2\,3\,4\ldots 2g+1],$$

and also that twists on disjoint curves commute, we get

$$f[4\,5\ldots 2g+1][2\,3\,4\,5] = [6\,7\ldots 2g+1][2\,3\ldots 2g+1].$$

Now we conjugate the above relation by B; since B commutes with both f and $[6\,7\ldots 2g+1]$, we get

$$f[\beta\,5\ldots 2g+1]B*[2\,3\,4\,5] = [6\,7\ldots 2g+1]B*[2\,3\ldots 2g+1].$$

Finally we invert both sides of the previous relation and multiply it on the right by the above; the result is the desired relation.

COROLLARY 1. *The normal subgroup* $K = \operatorname{Ker}(\mathcal{M}_{g,1} \to \mathcal{M}_{g,0})$ *is contained in* J_g.

Proof. Since $[2\,3\,4\,5] \in J_{g-1}$, $B*[2\,3\,4\,5]$ is also in J_{g-1} and so in J_g. The relation then tells us that $B*[2\,3\,4\ldots 2g+1]$ is also in J_g; in the notation of

Lemma 5, this is the map $B * W_1$. But J_g contains all the maximal straight chain maps $W_1, W_2, \ldots, W_{2g}$ as well, so contains K by Lemma 5.

COROLLARY 2. *J_g contains $B^{\pm 1} * [45\ldots 2g+1]$ and also $X * [\beta 5 \ldots 2g+1]$ for X to be any of $C_3^{\pm 1}$, $C_4^{\pm 1}$ or $B^{\pm 1}$.*

Proof. $B^{-1} * [\beta 5\ldots 2g+1] = [45\ldots 2g+1]$ and $B * [45 \ldots 2g+1] = [\beta 5 \ldots 2g+1]$ are certainly in J_g. Applying B^{-1} to the relation of the lemma gives

$$B^{-1} * [2345]^{-1} B^{-1} * [45\ldots 2g+1]^{-1}[45\ldots 2g+1][2345] = B^{-1} * [234\ldots 2g+1]^{-1}[23\ldots 2g+1].$$

But $B^{-1} * [2345]^{-1}$ is in $J_{g-1} \subset J_g$ and the right side is in $K \subset J_g$ by Corollary 1. Hence $B^{-1} * [45\ldots 2g+1] \in J_g$ as well. Showing that $B * [\beta 5 \ldots 2g+1]$ is in J_g is entirely analogous. If $X = C_3^{\pm 1}$ or $C_4^{\pm 1}$ and we apply X to the relation then the right side is still in $K \subset J_g$, and the left side is

$$X * [2345]^{-1} X * [45\ldots 2g+1]^{-1} X * [\beta 5 \ldots 2g+1](XB) * [2345].$$

Again, the first and last factors are in J_{g-1} and the second is in J_g since X normalizes the group of straight chains. Thus $X * [\beta 5\ldots 2g+1] \in J_g$ also, Q.E.D.

We now return to the problem of showing that C_3, C_4 and B normalize J_g. We can use the following "reduction" process to simplify our work: Let X be any of $C_3^{\pm 1}$, $C_4^{\pm 1}$ or $B^{\pm 1}$ and suppose that $h \in \mathfrak{M}_{g,1}$ commutes with X and is known to normalize J_g. Then for $f \in J_g$, the question of whether $X * f$ is in J_g is equivalent to the question of whether $h * (X * f) = X * (h * f)$ is in J_g. By choosing h carefully we can reduce $h * f$ to a small number of cases. We will use this reduction procedure repeatedly in the following arguments.

Part II. C_3 normalizes J_g.

Proof. C_3 normalizes the group of straight chains, so we need only examine $C_3^{\pm 1} * [\beta i_1 \ldots]$. Since all C_k with $k \geq 5$ commute with C_3 and also normalize J_g we may choose an h involving only the $C_{k \geq 5}$ such that $h * \mathrm{ch}(\beta i_1 \ldots)$ is a *consecutive* chain $\mathrm{ch}(\beta 56\ldots)$. As an example, consider $\mathrm{ch}(\beta 7 8 10)$; applying the rules of Lemma 1f, we get

$$(\beta 7 8 10) \xrightarrow{C_6} (\beta 6 8 10) \xrightarrow{C_5} (\beta 5 8 10) \xrightarrow{C_7} (\beta 5 7 10) \xrightarrow{C_6} (\beta 5 6 10) \xrightarrow{C_9} (\beta 569) \xrightarrow{C_8} (\beta 568) \xrightarrow{C_7} (\beta 567).$$

Hence we need only show that $C_3^{\pm 1} * [\beta 5 \ldots 2k+1] \in J_g$. If $k < g$,

$[\beta 5 \ldots 2k+1]$ is in J_{g-1} and we are done by the induction hypothesis. But $C_3^{\pm 1} * [\beta 5 \ldots 2g+1] \in J_g$ by Corollary 2 to the previous lemma, Q.E.D.

Part III. C_4 normalizes J_g.

Proof. C_4 normalizes the group of straight chains. Furthermore C_4 commutes with β-chain maps $[\beta i_1 \ldots]$ for which $i_1 \geq 6$: for the circle (i.e., 1-chain) $(\beta 6)$ shown in Figure 6 is disjoint from C_4, and similarly for (βi), any $i \geq 6$. As for $C_4^{\pm 1} * [\beta 5\, i_2 \ldots]$, we may "reduce" ch$(5\, i_2 \ldots)$ to a consecutive chain by means of a word h in $C_6, C_7, \ldots, C_{2g+1}$, since such an h commutes with C_4 and normalizes J_g. We now proceed as in Part II: It suffices to look only at $C_4^{\pm 1} * [\beta 5 \ldots 2k+1]$ and in fact only at the case $k = g$ since smaller chains live in J_{g-1}. But $C_4^{\pm 1} * [\beta 5 \ldots 2g+1]$ is in J_g as well, again by Corollary 2.

Part IV. B normalizes J_g.

Proof. Since $C_5, C_6, \ldots$ all commute with B, the expression $B^{\pm 1} * [\beta i_1 \ldots]$ may be reduced to $B^{\pm 1} *$[consecutive β-chain] and again we need only consider $B^{\pm 1} * [\beta 5 \ldots 2g+1]$ which is, as before, in J_g. Let us look then at $B^{\pm 1}$ acting on a straight chain $(i_1 \ldots i_{2k})$; we may assume that $i_1 \leq 4$ and $i_{2k} \geq 5$. Let n be such that $i_n \leq 4$ and $i_{n+1} \geq 5$; C_1, C_2, and C_3 also normalize J_g and commute with B, and by using all the twists C_k, $k \neq 4$, we may reduce the chain to a consecutive straight chain whose n-th index is 4. For example, reduce the chain (1367) as follows:

$$(1367) \xrightarrow{C_3^{-1}} (1467) \xrightarrow{C_1^{-1}} (2467) \xrightarrow{C_2^{-1}} (3467) \xrightarrow{C_5} (3457) \xrightarrow{C_6} (3456).$$

Thus we need only consider chains beginning with 1, 2, 3 or 4, and among them only those whose last index is $2g$ or $2g+1$, since otherwise we may apply the induction hypothesis. But in this case, if the chain begins with 1 or 2 then the map in question is a maximal odd chain map and hence is in $K \subset J_g$. Also, if the chain begins with 4 then Corollary 2 tells us that $B^{\pm 1} * [45 \ldots 2g+1] \in J_g$. Thus only one case remains to be considered, namely $B^{\pm 1} * [34 \ldots 2g]$. To show that this is in J_g we need another geometric relation, again derived from the $M_{0,4}$ relation; we state this in the form of:

LEMMA 10.

$$[1234][1256 \ldots 2g] B * [345 \ldots 2g] = [56 \ldots 2g][123 \ldots 2g].$$

Proof. We start with the chain maps

$$[1234] = T_a T_{a'}^{-1}, \qquad [1256 \ldots 2g] = T_b T_{b'}^{-1}, \qquad [345 \ldots 2g] = T_c T_{c'}^{-1},$$
$$[56 \ldots 2g] = T_e T_{e'}^{-1}, \qquad [123 \ldots 2g] = T_f T_{f'}^{-1}$$

and $$B*[345\ldots 2g] = T_d T_{c'}^{-1}$$

where the curves are as shown in Figure 18.

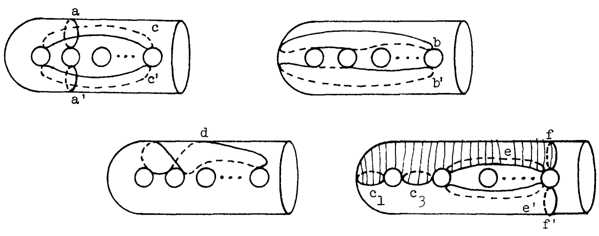


FIGURE 18

Note that c_1, c_3, e and f bound a surface S of type $M_{0,4}$ (shaded in the figure) and similarly c_1, c_3, e', f' bound S' in the lower part of the figure. The curves a, b, d all lie in S and are so placed there that we may apply the $M_{0,4}$ relation. To see this, note that the 3 curves α, β, γ of the $M_{0,4}$ relation, as pictured in Figure 16, are the respective boundaries of regular neighborhoods of $\varepsilon_1 \cup x \cup \varepsilon_2$, $\varepsilon_2 \cup y \cup \varepsilon_3$ and $\varepsilon_3 \cup z \cup \varepsilon_1$, where x, y, z are the arcs shown in Figure 19.

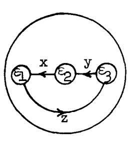


FIGURE 19

These arcs separate the $M_{0,4}$ into an annulus and a disc. If we give the boundary of the disc its standard orientation (i.e., so that the disc is on the left), then as we go around it, it traverses the arcs in the cyclic order z, y, x. In the relation $T_\gamma T_\beta T_\alpha = T_{\varepsilon_1} T_{\varepsilon_2} T_{\varepsilon_3} T_{\varepsilon_4}$ we see then that the twists $T_\gamma T_\beta T_\alpha$ must occur in the same cyclic order as their corresponding arcs z, y, x are traversed.

We apply this now to the surface S. Let x, y, z be the arcs shown in Figure 20.

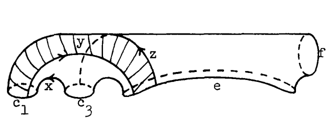


FIGURE 20

The orientation acquired from the disc component (shaded) is shown and the neighborhood boundaries of $c_3 \cup x \cup c_1$, $c_1 \cup y \cup e$, $e \cup z \cup c_3$ are a, b, d, respectively. The $M_{0,4}$ relation then reads: (S): $T_a T_b T_d = C_1 C_3 T_e T_f$.

Likewise, in S' we consider the arcs x', y', z' of Figure 21:

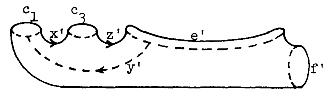


FIGURE 21

We see that they become oriented as shown and that the neighborhood boundaries of $c_1 \cup x' \cup c_3$, $e' \cup y' \cup c_1$, $c_3 \cup z' \cup e'$ are a', b', c', respectively. Thus the relation here reads: $T_{c'}T_{b'}T_{a'} = C_1C_3T_{e'}T_{f'}$, or, if we invert both sides: $T_{a'}^{-1}T_{b'}^{-1}T_{c'}^{-1} = C_1^{-1}C_3^{-1}T_{e'}^{-1}T_{f'}^{-1}$. Multiplying the latter relation with the relation (S) and noting that primed and unprimed maps commute, we get: $(T_aT_{a'}^{-1})(T_bT_{b'}^{-1})(T_dT_{c'}^{-1}) = (T_eT_{e'}^{-1})(T_fT_{f'}^{-1})$, i.e.,

$$[1234][1256\ldots 2g]B*[345\ldots 2g] = [56\ldots 2g][123\ldots 2g],$$

Q.E.D.

The above relation shows immediately that $B*[345\ldots 2g] \in J_g$. To see that $B^{-1}*[345\ldots 2g] \in J_g$, apply B^{-1} to the relation and get

$$B^{-1}*[1234]B^{-1}*[1256\ldots 2g][345\ldots 2g] = [56\ldots 2g]B^{-1}*[123\ldots 2g].$$

But $B^{-1}*[1234]$ is in J_{g-1} and $B^{-1}*[12\ldots 2g]$ is in K, so $B^{-1}*[1256\ldots 2g]$ is in J_g. We now conjugate $B^{-1}*[1\ 2\ 5\ 6\ldots 2g]$ successively by $C_2^{-1}, C_3^{-1}, C_1^{-1}, C_2^{-1}$, and, noting that these commute with B and normalize J_g, get successively that $B^{-1}*[1356\ldots 2g]$, $B^{-1}*[1456\ldots 2g]$, $B^{-1}*[2456\ldots 2g]$ and finally $B^{-1}*[3456\ldots 2g]$ are all in J_g. This finishes the proof of Part IV and Theorem 2. We remark that our proof leaves completely open the question of whether $\mathcal{J}_2$ is f.g.

5. Minimal generating sets

The set of generators we have given is by no means minimal; in fact, relations we have already developed would allow us to eliminate certain generators immediately. Consider $g = 3$ for example. By Lemma 5, $W_7 = [123456]$ is a word in $W_1 = [234567]$, $W_2 = [134567], \ldots, W_6 = [123457]$ and $B*W_1$. But since $B*[2345] \in J_2$ and the latter is generated by straight 3-chains (see Lemma 6), we see that the relation of Lemma 9 may be written in the form

$$B*W_1 = W_1 \cdot [2345]^{-1}[4567]^{-1}[\beta 567] \cdot (\text{a word in straight 3-chains}).$$

Hence W_7 may be expressed in terms of W_1 through W_6 and the 3-chain generators, and we may eliminate W_7 from our generating set. In higher genera this type of argument would allow us to eliminate a number of generators.

Returning to genus 3, we note that the number of straight 3-chain generators is $\binom{2g+1}{3+1} = \binom{7}{4} = 35$; using these and the 7 generators for $\mathrm{Ker}(\mathcal{I}_{3,1} \to \mathcal{I}_{3,0})$ allows us (by Lemma 9 again) to generate $[\beta 5 6 7]$, and hence all of $\mathcal{I}_{3,1}$. This gives 42 generators for $\mathcal{I}_{3,1}$ consisting of 35 straight 3-chain maps and 7 5-chain maps; the latter die in $\mathcal{I}_{3,0}$ and thus the 35 3-chain maps alone generate it. Now in [J3] it is shown that lower bounds for the number of generators of $\mathcal{I}_{g,0}$ and $\mathcal{I}_{g,1}$ are $\binom{2g}{3} + \binom{2g}{2}$ and $\sum_{i=0}^{3} \binom{2g}{i}$, respectively. For $g = 3$ this gives 35 and 42, respectively, and thus we have proved:

Theorem 3. *$\mathcal{I}_{3,1}$ and $\mathcal{I}_{3,0}$ are generated by 42 and 35 elements, respectively, and these numbers are the smallest possible.*

We might ask if this result can be extended to higher genera. To begin with, Lemma 5 gives us $2g + 1 = \binom{2g}{1} + \binom{2g}{0}$ generators of $\mathrm{Ker}(\mathcal{I}_{g,1} \to \mathcal{I}_{g,0})$, so the problem reduces to the following:

Question. Can $\mathcal{I}_{g,0}$ be generated by $\binom{2g}{3} + \binom{2g}{2}$ elements for $g \geq 4$?

University of California, Los Angeles

References

[B] J. Birman, *Braids, Links and Mapping Class Groups*, Annals of Math. Studies, No. 82, Princeton Univ. Press, Princeton, 1975.

[G] A. Gramain, Sur le type d'homotopie du groupe de difféomorphisme d'une surface compacte, Ann. Scient. Ec. Norm. Sup. **6** (1973), 53–66.

[H] S. Humphries, Generators for the mapping class group, in *Topology of Low-Dimensional Manifolds*, Lecture Notes in Math. 722, Springer, Berlin (1979), 44–47.

[J1] D. Johnson, An abelian quotient of the mapping class group $\mathcal{I}_g$, Math. Ann. **249** (1980), 225–242.

[J2] ———, Homeomorphisms of a surface which act trivially on homology, Proc. AMS **75** (1979), 119–125.

[J3] ———, Quadratic forms and the Birman-Craggs homomorphisms, Trans. AMS **261** (1980), 235–254.

[K] R. Kirby, Problems in low-dimensional manifold theory, Proc. Symp. Pure Math. **32** (1975), 273–312.

[L] W. B. R. Lickorisch, A representation of orientable combinatorial 3-manifolds, Ann. of Math. **76** (1962), 531–540.

[P] J. Powell, Two theorems on the mapping class group of surfaces, Proc. AMS **68** (1978), 347–350.

(Received February 16, 1982)