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A SURVEY OF THE TORELLI GROUP

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In this survey I will discuss, in a very informal way, certain aspects of the mapping class group of a (forever compact and oriented) surface. Because of Thurston's work on it this subject has undergone some marked changes in the past few years and is now a fast moving target. The aspects which I aim at below are, however, not (yet) clearly related to Thurston's results and tend to be more algebraic in nature; but there is, I think, bound to be a great deal of interesting geometry lurking about here.

To begin with, here is the definition-notation of the mapping class group I will use:

a. F_g is a closed surface of genus g, $F_{g,*}$ the same but with distinguished base point $*$, and $F_{g,n}$ is a surface of genus g with n boundary circles.

b. The corresponding mapping class groups $\mathcal{M}_g$, $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,n}$ are the groups of orientation preserving diffeomorphisms of the surface which fix the distinguished point and/or boundary points, modulo isotopies which do the same. Actually, for surfaces with boundary I will only be using $F_{g,1}$. With this in mind then, $H_1(F,Z)$ is in every case free abelian of rank $2g$. The notation for this group will frequently be abbreviated to H_1. It has an attached nonsingular antisymmetric intersection pairing, that is, a "symplectic" inner product which we denote by $a \cdot b$ for $a,b \in H_1$. The action of the mapping class group on homology then gives a homomorphism $\mathcal{M} \to Sp(H_1)$ where the target is the group of symplectic (i.e., intersection preserving) automorphisms of H_1, and it was known classically that this map is onto.

It is the kernel of this map that I am going to discuss. The topologists had no name for it, but it has been known for a long time to the analysts, so I will use their label: the Torelli Group. This group,

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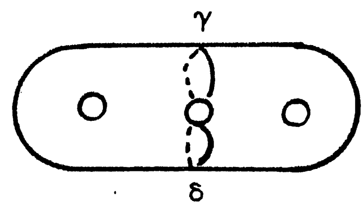
which we denote in its various forms by $\mathcal{I}_g$, $\mathcal{I}_{g,*}$ or $\mathcal{I}_{g,1}$ according to the nature of the surface F, is then that subgroup of $\mathcal{M}$ which acts trivially on the homology of F.

The Torelli Group has only become of interest to topologists in the past dozen years or so, and this interest was initiated principally through the work of Joan Birman. Some of the early interest in it stemmed from the fact that any homology 3-sphere can be created by cutting S^3 along an imbedded surface F and then regluing the two pieces by a map taken from the Torelli group of F. It was originaly felt that a careful study of $\mathcal{I}$ might thus lead to a deeper understanding of homology spheres. While this approach has not produced a great deal of fruit, it did indicate that $\mathcal{I}$ supports a lot of interesting algebraic topology. Most of the results to date have been proved with a mixture of algebraic and topological methods, with only a small amount of geometry and analysis. But some of the recent aspects of $\mathcal{I}$ studied recall something of its beginnings in Riemann surface theory: Moduli spaces, Jacobi varieties and period matrices, etc., and I think that the interesting features of this group are much more numerous than its original topological applications would have suggested.

GENERATORS FOR $\mathcal{I}$

The first problems of interest about $\mathcal{I}$ were, as might be expected, those concerning its generation and arise in Birman's paper [B]. This paper, one of the earliest on $\mathcal{I}$ by a topologist, is actually concerned with a different problem and is entirely algebraic in nature. It produced a finite presentation for the integral symplectic group $Sp(g,Z) \cong Sp(H_1F_g,Z))$, wherein the generators t_i are the images (via the map $\mathcal{M}_g \to Sp(H_1F_g))$ of Lickorisch's twist map generators T_i of $\mathcal{M}_g$. The relational words $r_k(t_i)$ of this presentation thus have distinguished lifts $r_k(T_i)$ in $\mathcal{M}_g$, and the latter words are then not only in $\mathcal{I}_g$ but are in fact normal generators for $\mathcal{I}_g$ in $\mathcal{M}_g$. This was the first (infinite) set of generators known for $\mathcal{I}$. Birman also raised in this paper the question of whether $\mathcal{I}$ is finitely generated.

The next result on generating $\mathcal{I}_g$ is by Powell in [P]. Beginning with Birman's normal generators, he reinterpreted them geometrically and showed that just three suffice to normally generate $\mathcal{I}_g$ when $g \geq 3$. These generators are quite natural and useful; in fact they are pervasive in much of the later work on the Torelli group. To describe them, we note first the following useful fact: If T_γ is a Dehn twist on the simple closed curve $\gamma \subset F_g$, then the action of T_γ on H_1F is completely determined by the homology class c of γ (the formula is $T_\gamma(x) = x + (c \cdot x)c$ for $x \in H_1$). Thus in particular, if γ is a <u>bounding</u> simple closed curve ("BSCC") then $T(x) = x$ for all x, i.e., $T_\gamma \in \mathcal{I}$ Similarly, suppose γ, δ are a pair of non-bounding, disjoint, homologous SCC's such as depicted below:



Then T_γ and T_δ act the same on H_1 and so $T_\gamma T_\delta^{-1} \in \mathcal{I}$ We shall call such a pair (γ, δ) a <u>bounding pair</u>, or <u>BP</u>. If we further define the <u>genus</u> of a BP to be the smaller of the genera of the two pieces into which $\gamma \cup \delta$ separates F (in the figure the genus is 1), and likewise for the genus of a BSCC, then Powell's result may be stated as follows:

<u>For</u> $g \geq 3$, <u>the set of all</u> BP <u>maps of genus</u> 1 <u>and all</u> BSCC <u>maps of genus</u> 1 <u>and</u> 2 <u>generate</u> $\mathcal{I}_g$. $\mathcal{I}_2$ <u>is generated by all twists on</u> BSCC's (<u>necessarily of genus</u> 1).

Alternately, we might state the result by saying that $\mathcal{I}_g$ $(g \geq 3)$ is normally generated by just three elements: a single BP map of genus 1 and single BSCC maps of genus 1 and 2.

The final result along these lines was my own: I showed in [J1] how to write any BSCC map as a product of genus 1 BP maps, and hence that:

<u>For</u> $g \geq 3$, $\mathcal{I}_g$ <u>is generated by all genus</u> 1 BP <u>maps</u>, <u>i.e.</u>, $\mathcal{I}_g$ <u>is normally generated in</u> $\mathcal{M}_g$ <u>by any genus</u> 1 BP <u>map</u>.

This result is the starting point for several others, and the fact that we no longer need consider the BSCC maps frequently simplifies the arguments. But before forgetting these maps, I should point out that they suggested a number of interesting problems themselves. Hence, we define $\mathcal{K}_g \subset \mathcal{I}_g$ to be the subgroup generated by all twists on bounding curves. Powell's theorem stimulated him and Birman to conjecture that perhaps $\mathcal{K}_g$ was actually equal to $\mathcal{I}_g$, or at least that $\mathcal{I}_g/\mathcal{K}_g$ was finite. Then, in [C], Chillingworth showed that a genus 1 BP map of $\mathcal{I}_g$ cannot be in $\mathcal{K}_g$, but this proof left open the possibility that their quotient is finite. The proof was via the construction of an obstruction to a map being in $\mathcal{K}_g$, and he raised the problem of whether this was the only such obstruction, thus attempting to characterize $\mathcal{K}_g$ by a computable invariant.

So now we have two questions concerning $\mathcal{K}_g$:

1. <u>Is $\mathcal{I}_g/\mathcal{K}_g$ finite</u>?
2. <u>Does Chillingworth's obstruction characterize</u> $\mathcal{K}_g$?

We will return to these problems below.

The only immediate problem remaining concerning the generation of $\mathcal{I}$, and one which by 1979 had been asked by several people, was whether it is finitely generated. Since $\mathcal{I}_g$ is normal and of infinite index in $\mathcal{M}_g$, the general feeling was that finite generation was very improbable. A lower bound existed (see [J2], Theorem 5) which stated that the number of generators is not less than $(4g^3 - g)/3$ and gave scant reinforcement to this feeling (compare this with the minimal number of generators of $\mathcal{M}_g$, which is ≤ 4). There is, however, a result of Magnus (see [MKS], p. 168-169) which made the finite generation of $\mathcal{I}$ seem more likely: it says that if A_n is the automorphism group of a free group π on n generators and $I_n \subset A_n$ is the (normal, infinite index) subgroup which acts trivially on π/π', then I_n <u>is</u> finitely generated. The relation of this result to the problem at hand lies in the fact that, e.g., $\mathcal{M}_{g,1}$ may be viewed as that subgroup of A_{2g} which leaves invariant a certain word in π (namely, the standard commutator relation for a closed surface) and $\mathcal{I}_{g,1}$ is the subgroup of $\mathcal{M}_{g,1}$ acting trivially on π/π'.

Nevertheless, it came as something of a surprise that $\mathcal{I}_g$ is indeed finitely generated when $g \geq 3$ (see [J5]). The proof (which leaves open the case $g = 2$) uses our previous result that $\mathcal{I}_{g\geq 3}$ is normally generated in $\mathcal{M}_g$ by any genus 1 BP map, and proceeds as follows: A certain set of BP maps of all genera from 1 to $g - 2$ is constructed and it is then shown that the subgroup of $\mathcal{I}_g$ which they generate is actually <u>normal in</u> $\mathcal{M}_g$, and so by the above result it must be all of $\mathcal{I}_g$. Although the number of generators constructed is large (exponential in g), the verification that the required conjugates remain in the generated subgroup boils down to a small number of topologically distinct cases. As for the gap between the previously mentioned lower bound $(4g^3 - g)/3$ and the exponential upper bound, I think one could probably use the methods of the proof a bit more carefully to get the number of generators down to polynomial order--in fact, I would guess polynomial of degree 3 or 4.

AN ABELIAN QUOTIENT OF $\mathcal{I}$

In order to get more detailed information about $\mathcal{I}$, several things now suggested themselves. The most obvious thing would be to find some concrete representations of $\mathcal{I}$, and of these we might first look for abelian ones; in other words, we seek abelian quotients of $\mathcal{I}$.

The first abelian quotient of $\mathcal{I}$ is due to Sullivan [S], who showed how to get a map from $\mathcal{I}_g$ onto a free abelian group of rank $\binom{g}{3}$. The method here was to construct a 3-manifold with the same homology as the connected sum of g $S^1 \times S^2$'s by gluing together two copies of a genus g handlebody along their common boundary F, using a gluing map taken from the Torelli group of F, and then examining how the intersection ring of this 3-manifold differs from that of a true $g(S^1 \times S^2)$. Coming from a different direction (via a so-called Mangus representation of $\mathcal{I}$) I found in [J3] a homomorphism of $\mathcal{I}_{g,1}$ onto a free abelian group of rank $\binom{2g}{3}$; Sullivan's map is a quotient of this one. Explicitly, this homomorphism, denoted τ, maps $\mathcal{I}_{g,1}$ to the 3rd exterior power of the homology of F, i.e., $\tau : \mathcal{I}_{g,1} \to \Lambda^3 H_1(F_{g,1}, Z)$. It is natural in the sense that it commutes with the standard actions of $\mathcal{M}_{g,1}$ on the source and target: the action on the latter is the obvious one, and on the former $\mathcal{M}_{g,1}$ acts by conjugation. Because of the usefulness of this homomorphism, I will give several different definitions of it.

<u>First Definition</u>: Perhaps the easiest way to define τ and see most of its properties is via a certain nilpotent quotient of $\pi_1(F_{g,1})$: The latter is free on $2g$ generators and we put $E = \pi_1/[\pi_1, \pi_1']$. It stands to reason that, although $\mathcal{I}$ acts trivially on $H_1 = \pi_1/\pi_1'$, we may still learn something of use by examing how it acts on E. Note that H_1 is a quotient of E with kernel

$$N = E' = \frac{\pi_1'}{[\pi_1, \pi_1']} ,$$

and N is <u>central in</u> E. These facts imply that, if $f \in \mathcal{I}$ and we define $\delta f(e) = f(e)e^{-1}$ for $e \in E$, then $\delta f(e)$ is in N and $\delta f : E \to N$ is a homomorphism factoring through H_1, i.e., $\delta f : H_1 \to N$. Furthermore, it is not too hard to see that N is canonically isomorphic to $\Lambda^2 H_1$ (the map is given by sending $[e_1, e_2] \in N$ to $h_1 \wedge h_2 \in \Lambda^2 H_1$, where h_i is the image of e_i in H_1) so we may write $\delta f : H_1 \to \Lambda^2 H_1$.

In other words, $\delta f \in H_1^* \otimes \Lambda^2 H_1$, and a similar argument shows that $\delta : \mathcal{I}_{g,1} \to H_1^* \otimes \Lambda^2 H_1$ is a homomorphism. Now by applying the canonical isomorphism of H_1^* with H_1 given by the intersection pairing, we convert the above map into a map $\tau : \mathcal{I}_{g,1} \to H_1 \otimes \Lambda^2 H_1$. But $\Lambda^3 H_1$ sits inside $H_1 \otimes \Lambda^2 H_1$ in a very natural way: send $a \wedge b \wedge c$ to $a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$, and it turns out that $\operatorname{Im}\tau \subset \Lambda^3 H_1$. We may now summarize the immediate properties of τ as follows:

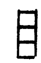
<u>THEOREM</u>: a. $\tau : \mathcal{I}_{g,1} \to \Lambda^3 H_1(F_{g,1}, Z)$ <u>is surjective</u>.

b. (<u>Naturality</u>): <u>For</u> $h \in \mathcal{M}_{g,1}$ <u>and</u> $f \in \mathcal{I}_{g,1}$ <u>we have</u> $\tau(hfh^{-1}) = h(\tau(f))$.

c. <u>If</u> $\mathcal{K}_{g,1} \subset \mathcal{I}_{g,1}$ <u>is the subgroup generated by all twists on bounding simple closed curves, then</u> $\tau(\mathcal{K}_{g,1}) = 0$, <u>i.e.</u>, $\mathcal{K}_{g,1} \subset \operatorname{Ker}\tau$.

I want to call particular attention to c). The facts that $\operatorname{Im}\tau = \Lambda^3 H_1$ is infinite and $\mathcal{I}_{g,1} \subset \operatorname{Ker}\tau$ immediately dispose of the first of the two problems concerning $\mathcal{K}$, and show that in fact $\mathcal{I}/\mathcal{K}$ is not finite (actually, this only establishes that $\mathcal{I}_{g,1}/\mathcal{K}_{g,1}$ is infinite, but a technical variation shows the same for $\mathcal{I}_g/\mathcal{K}_g$). As for the second problem, it can be seen ([J3]), Theorem 2) that Chillingworth's obstruction "factors through" the map τ, and also that there exists $f \in \mathcal{K}_{g>3}$ with zero Chillingworth obstruction but such that $\tau(f) \neq 0$, and hence that $f \notin \mathcal{K}$, thus answering the second problem in the negative also. The obvious conjecture to make now is that $\mathcal{K}_{g,1}$ is actually <u>equal</u> to $\operatorname{Ker}\tau$. Using the methods of the proof that $\mathcal{I}_g$ is finitely generated, I was able to prove this recently (see [J6]).

<u>Second Definition</u>: This is just a modification of the definition of Sullivan's map. Let $f \in \mathcal{I}_{g,1}$ and let W_f be the mapping torus of f, i.e., $F_{g,1} \times I$ with $x \times \{0\}$ glued to $f(x) \times \{1\}$. Then W_f is a homology $F_{g,1} \times S^1$, but it is not an <u>intersection</u> $F_{g,1} \times S^1$, that is, its intersection theory is different from that of $F_{g,1} \times S^1$. The triple intersection of three classes in $H_2(W_f, Z)$ gives a map $\Lambda^3 H_2 W_f \to Z$, i.e., an element $\tau(f)$ of $\Lambda^3 (H_2 W_f)^*$. Now $H_1 F_{g,1}$ is naturally contained in $H_1 W_f$, and an intersection argument in W_f give a natural isomorphism of this subspace $H_1 F \subset H_1 W_f$ with $(H_2 W_f)^*$, so we may think of $\tau(f)$ as contained in $\Lambda^3 H_1 F$. This definition is equivalent to the first.

In an attempt to see deeper into the structure of $\mathcal{I}$, we may generalize these two definitions as follows. In terms of the first definition, if we put $\mathcal{I} = \mathcal{I}^{(0)}$ and $\text{Ker}\ \tau = \mathcal{I}^{(1)}$ $(= \mathcal{K})$ then $\mathcal{I}^{(0)}$ is the group acting trivially on $H_1 = \pi/\pi'$ and $\mathcal{I}^{(1)}$ is the group acting trivially $E = \pi/[\pi, \pi'] = \pi/\pi^{[2]}$. By examining the action of $\mathcal{I}^{(1)}$ on $\pi/\pi^{[3]} = \pi/[\pi, [\pi, \pi']]$ we get a homomorphism defined on $\mathcal{I}^{(1)}$, let us call it τ_1, which is analogous to τ on $\mathcal{I}$, but whereas the target of τ is the "rank 3 tensor" $\Lambda^3 H_1$, the target of τ_1 is now a <u>rank 4</u> tensor space of the same type as the Riemann curvature tensor. (In terms of "Young diagrams", $\Lambda^3 H_1$ is represented by , and the target of τ_1 is represented by ⊞). The map τ_1 is no longer surjective, and the precise specification of its image is lacking; it can be shown, however, to be of finite index in the target--in other words, $(\text{Im}\,\tau_1) \otimes Q =$ ⊞ $H_1(F,Q)$.

Continuing in this way we get a sequence of groups $\mathcal{I}^{(n)}$ and homomorphisms τ_n defined on $\mathcal{I}^{(n)}$ with the following properties:

1. $\mathcal{I}^{(n)}$ is the group acting trivially on $\pi/\pi^{[n+1]}$, and is equal to $\text{Ker}\ \tau_{n-1}$.

2. τ_n is determined by the action of $\mathcal{I}^{(n)}$ on $\pi/\pi^{[n+2]}$, and the target of τ_n is (contained in) the $(n + 3)^{rd}$ tensor power of $H_1(F,Z)$.

3. $\bigcap_{n=0}^{\infty} \mathcal{I}^{(n)} = \{1\}$.

4. $\mathcal{I}^{(n)}$ contains the nth term $\mathcal{I}^{[n]}$ of the lower central series of $\mathcal{I}$ (where $\mathcal{I}^{[0]} = \mathcal{I}$, $\mathcal{I}^{[1]} = \mathcal{I}'$, etc.).

In terms of the second definition of τ (via the mapping torus) τ_n can be seen as Massey products. In fact, τ itself can be redefined in terms of the map $\Lambda^3 H^1(W_f,Z) \to Z$ given by $\alpha \wedge \beta \wedge \gamma \to \alpha \cup \beta \cup \gamma \in Z$ for $\alpha, \beta, \gamma \in H^1(W_f,Z)$. This map is an element $\tau(f)$ of $(\Lambda^3 H^1 W_f)^* = \Lambda^3 H_1 W_f$, and it actually lies in the subspace $\Lambda^3 H_1 F$ of $\Lambda^3 H_1 W_f$. This is just a cohomology version of the second definition. But now suppose that $\tau(f) = 0$, that is, $f \in \mathcal{I}^{(1)}$. A little Poincare duality shows that this implies $\alpha \cup \beta = 0$ for all $\alpha, \beta \in H^1 W_f$, and hence all triple Massey products $\langle \alpha, \beta, \gamma \rangle$ are defined and in $H^2(W_f,Z)$, so $\langle \alpha, \beta, \gamma \rangle \cup \delta \in H^3 W_f = Z$. This function of $\alpha, \beta, \gamma, \delta$ is quadrilinear on H^1 and has certain symmetries defining thereby an element of $\boxplus\ H_1F$. This definition is the same as the previous one.

Concerning the above properties of $\mathcal{I}^{(n)}$ and τ_n we have the following problems:

A) <u>What is the image of</u> τ_n <u>in</u> $H_1^{\otimes(n+3)}(F,Z)$, <u>or easier, what is</u> $(\mathrm{Im}\tau_n)\otimes Q \subset H^{\otimes(n+3)}(F,Q)$? (<u>These images are necessarily invariant under the action of the symplectic group</u> $Sp(H_1F)$.)

B) $\mathcal{I}' = \mathcal{I}^{[1]}$ is (as we will see below) of finite index in $\mathcal{I}^{(1)}$. <u>Is this true for all</u> n?

<u>MORE ON</u> τ; <u>THE HOMOLOGY OF</u> $\mathcal{I}$

The above problems were suggested by the first two definitions of τ. There is still another definition, and this one relates to some interesting homological problems about $\mathcal{I}$. For our third definition, recall that $F_{g,*}$ is a closed surface with distinguished base point $*$.

<u>Third Definition</u>: Let us define the <u>Jacobi variety</u> of $F_{g,*}$ to be the 2g-dimensional torus $J = H_1(F,R)/H_1(F,Z)$. Note that $H_1(J,Z)$ is canonically identifiable with $H_1(F,Z)$. Elementary $K(\pi,1)$ theory tells us that there is then a unique homotopy class of maps $j : (F,*) \to (J,0)$ such that $j_* : \pi_1(F) \to \pi_1(J) = H_1(J)$ is given by the quotient map $\pi_1(F) \to H_1(F) = H_1(J)$. Choose then a fixed such map j and consider the composite jf for $f \in \mathcal{I}_{g,*}$. Since $f_* = 1$ on H_1F, $(jf)_* = j_*$ and so jf is homotopic (rel $*$) to j. But $\mathrm{Im}(jf) = \mathrm{Im}j$ and thus the homotopy between jf and j gives rise to a 3-cycle in J. Its homology class does not depend on any of the choices we have made (e.g., j and the homotopy), so we have a well defined element $\tau(f)$ of $H_3(J,Z)$. Now H_k of a torus is well known to be the kth exterior power of its H_1, and using the identification of H_1J and H_1F, we have $\tau(f) \in H_3J = \Lambda^3H_1J = \Lambda^3H_1F$. We get thereby a homomorphism $\tau : \mathcal{I}_{g,*} \to \Lambda^3H_1(F_{g,*},Z)$. To relate this to our previous definitions of τ, which were defined on $\mathcal{I}_{g,1}$ rather than $\mathcal{I}_{g,*}$, let $\mathcal{I}_{g,1} \xrightarrow{p^*} \mathcal{I}_{g,*}$ be the obvious (surjective) homomorphism induced by the map $F_{g,1} \xrightarrow{p} F_{g,*}$ which collapses the boundary of $F_{g,1}$ to a point $*$. Then $\tau : \mathcal{I}_{g,1} \to \Lambda^3H_1(F_{g,1},Z)$ "factors through p", that is to say:

$$\begin{array}{ccc} \mathcal{I}_{g,1} & \xrightarrow{\ \tau\ } & \Lambda^3 H_1 F_{g,1} \\ p^* \downarrow & & \simeq \downarrow p^* \\ \mathcal{I}_{g,*} & \xrightarrow{\ \tau\ } & \Lambda^3 H_1 F_{g,*} \end{array} \qquad \text{commutes.}$$

There is a variation of this definition which will be useful to us. First of all, the complex analysts have a more refined definition of the map $j : (F,*) \to (J,0)$. Topologically this was defined only up to homotopy, but if we assume that F has a specific complex structure; i.e., is a Riemann surface, then the analysts give a corresponding natural complex structure to the Jacobi variety J and also produce a unique holomorphic map $j : (F,*) \to (J,0)$ in its homotopy class. This cleans things up a bit, but we can do even better: We globalize this phenomenon by removing its dependence on the particular complex structure given to F. For this purpose I need to introduce the Teichmuller space $\mathfrak{T}_{g,*}$ of $F_{g,*}$. The most convenient definition for us is the set of all equivalence classes of complex structures on $F_{g,*}$, where "equivalence" of two such structures means that there is an isotopy of F, rel $*$, which carries one structure into the other. This space is homeomorphic to R^{6g-4}. $\mathcal{M}_{g,*}$ acts on it properly discontinuously (the quotient is the "moduli space" $\mathfrak{M}_{g,*}$ of punctured Riemann surfaces) and it is a classical result, essentially due to Hurwitz, that $\mathcal{I}_{g,*}$ acts freely. Hence the quotient manifold $\mathfrak{T}_{g,*}/\mathcal{I}_{g,*} = \mathfrak{I}_{g,*}$, known as Torelli space, is a classifying space for $\mathcal{I}_{g,*} : \mathfrak{I}_{g,*} = B\mathcal{I}_{g,*} = K(\mathcal{I}_{g,*},1)$. We note the quotient group $\mathcal{M}_{g,*}/\mathcal{I}_{g,*} = Sp(H_1F)$ acts on $\mathfrak{I}_{g,*}$.

There is an obvious bundle of Riemann surfaces over $\mathfrak{T}_{g,*}$ whose fiber over $\sigma \in \mathfrak{T}_{g,*}$ is just $F_{g,*}$ with structure on F given by σ (it is thus topologically trivial, though not holomorphically so). This bundle, which we denote $F\mathfrak{T}_{g,*}$, has a canonical cross section given by $*$ in each fiber. $\mathcal{I}_{g,*}$ acts on this bundle-with-section and gives thereby a bundle-with-section $F\mathfrak{I}_{g,*}$ of Riemann surfaces over $\mathfrak{I}_{g,*}$.

Likewise, over $\mathfrak{T}_{g,*}$ we have a bundle $J\mathfrak{T}_{g,*}$ of Jacobi varieties, where the fiber over $\sigma \in \mathfrak{T}_{g,*}$ is the (holomorphic) Jacobi variety of the Riemann surface σ. Again this bundle is topologically trivial and $\mathcal{I}_{g,*}$ acts on it, but note that here the action is trivial on the fiber, hence the quotient is a bundle $J\mathfrak{I}_{g,*}$ over $\mathfrak{I}_{g,*}$ which remains topologically trivial. We have then a smooth projection map from the total space to the fiber J.

The analyst's version of the map $j : (F,*) \to (J,0)$ now globalizes to a (holomorphic) map $j : (F\mathfrak{X}, *) \to (J\mathfrak{X},0)$; j commutes with the action of $\mathcal{I}_{g,*}$ and so we get a quotient map $j : F\mathfrak{T} \to J\mathfrak{T}$. Composing this with the projection map $J\mathfrak{T} \to J$ gives us finally a map $F\mathfrak{T} \xrightarrow{q} J$.

Now here is the punch line. If α is a k-cycle in the Torelli space $\mathfrak{T}$, its full inverse image (via the bundle projection) in $F\mathfrak{T}$ is a $(k+2)$-cycle, and we may project this to J by the map q. Thus we obtain a map from $H_k(\mathfrak{T}_{g,*}) = H_k(\mathcal{I}_{g,*})$ to $H_{k+2}(J) = \Lambda^{k+2}(H_1F)$. For $k = 1$, the composition of this map $H_1\mathcal{I}_{g,*} \to \Lambda^3 H_1F$ with the standard quotient $\mathcal{I}_{g,*} \to H_1\mathcal{I}_{g,*}$ gives still another definition of τ.

This was a lot of effort to go for just another definition of τ. The real point of it is that it shows the existence of a rather natural map from $H_k(\mathcal{I}_{g,*})$ to $\Lambda^{k+2}(H_1F_g)$ for all $1 \leq k \leq 2g-2$. As we shall see later, $\Lambda^3 H_1F$ embodies all rational abelian information about $\mathcal{I}_{g,*}$ -- in other words, $(\mathcal{I}/\mathcal{I}') \otimes Q = H_1(\mathcal{I},Q) \simeq \Lambda^3 H_1(F,Q)$. It is thus reasonable to ask if a similar result holds for all k, i.e.,:

C) <u>Is the map</u> $H_k(\mathcal{I}_{g,*},Q) \to \Lambda^{k+2}H_1(F,Q)$ <u>an isomorphism for all</u> k?

Related to this question and the fact that $\mathcal{I}_{g,*}$ is finitely generated are the following:

D) <u>Is</u> $\mathfrak{T}_{g,*}$ <u>the homotopy type of a finite</u> CW <u>complex</u>?

E) <u>Is</u> $\mathfrak{T}_{g,*}$ <u>the rational homotopy type of a finite</u> $2g-2$ <u>dimensional</u> CW <u>complex</u>?

Z_2 QUOTIENTS OF $\mathcal{I}$

I've now finished discussing τ and its ramifications, but I haven't yet exhausted the subject of abelian quotients of $\mathcal{I}$. One's first guess might be that all such quotients factor through $\Lambda^3 H_1$, i.e., that $\mathcal{I}/\mathcal{I}' = \Lambda^3 H_1$ -- but this is not the case. The undetected part of $\mathcal{I}/\mathcal{I}'$ remaining consists of 2-torsion and comes from a very different direction.

In [BC], Birman and Craggs produced a (finite) collection of homomorphisms from $\mathcal{I}_g$ onto Z_2. These homomorphisms may be defined as follows (compare [BC] and [J2]). Choose an imbedding $h : F_g \hookrightarrow S^3$ and identify F_g with its image. Given any f in the Torelli group of F we may split S^3 along F and reglue the two pieces by the map f. Since $f = 1$ on H_1F, the resulting 3-manifold $W(h,f)$ is a homology sphere, and its Rochlin invariant $\mu(h,f) \in Z_2$ is defined.* Birman and Craggs showed that for a fixed imbedding h and f ranging over $\mathcal{I}_g$, $\mu(h,f)$ defines a homomorphism ρ_h of $\mathcal{I}_g$ onto Z_2. They also showed that, although this homomorphism depends on the choice of the imbedding h, it is not very sensitive to it, and in fact that only a finite number of distinct homomorphisms can arise from different choices of h.

The precise dependency on h was determined in [J2]. Any surface $K \subset S^3$ has associated with it a bilinear Seifert linking form $L(\alpha,\beta)$ defined for $\alpha, \beta \in H_1(K,Z)$, and if we use Z_2 coefficients and restrict to $\alpha = \beta$ we get the "mod 2 self-linking form", which is a quadratic form defined on $H_1(K,Z_2)$. Given an imbedding $h : F \to S^3$, we identify F with its image, inducing thereby a self-linking form ω_h on $H_1(F,Z_2)$. Then we have

The Birman-Craggs homomorphisms of the imbeddings h_1, h_2 are equal iff h_1 and h_2 induce the same self-linking form ω on $H_1(F,Z_2)$.

Thus the homomorphism $\rho_h : \mathcal{I}_g \to Z_2$ depends only on the quadradic form ω induced by h, and we may replace the notation ρ_h by the notation ρ_ω to more aptly exhibit this fact.

An easy generalization found in the same paper gives a corresponding definition of surjective maps $\rho_\omega : \mathcal{I}_{g,1} \to Z_2$ for a surface with boundary, and it is shown there that these maps do not factor through $\Lambda^3 H_1 F$, so we have here some new data about $\mathcal{I}/\mathcal{I}'$. Some further questions concerning these maps were obvious ones to ask:

a) How many distinct Birman-Craggs homomorphisms are there and are they linearly independent? If not, then:

b) What is the dimension of the space of homomorphisms they span?

c) Give a "natural" representation of this space of homomorphisms as a module over $Sp(H_1 F)$. Here are the results (again from [J2]):

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*The definition of the Rochlin invariant of a homology sphere W is: Let $W = \partial X$ where X is a simply connected parallelizable 4-manifold; such an X always exists, and the signature of X is always divisible by 8. Furthermore, $(\frac{\text{signature}}{8} \bmod 2)$ is independent of the choice of X, so it is actually an invariant of W; this is the Rochlin invariant.

a) The number of Birman-Craggs homomorphisms $\mathcal{I}_{g,1} \to Z_2$ is 2^{2g}.

b) They are not linearly independent, but span a Z_2 vector space of dimension $\binom{2g}{3} + \binom{2g}{2} + \binom{2g}{1} + \binom{2g}{0}$.

To describe the answer to c), I must construct from $H_1(F,Z_2)$ a certain Z_2-algebra. This algebra B is commutative with unit 1 and has a generator for each non-zero element a of $H_1(F,Z_2)$. This generator being denoted by $\bar{a}$, we require the following relations to hold in B:

1. $\bar{a}^2 = \bar{a}$ for all $a \neq 0$ in $H_1(F,Z_2)$.

2. $\overline{(a+b)} = \bar{a} + \bar{b} + a \cdot b$, where $a \cdot b \in Z_2 \subset B$ is the algebraic intersection of a and b. (This relation establishes the contact between B and the set of quadratic forms on $H_1(F,Z_2)$, which are by definition functions $\omega : H_1(F,Z_2) \to Z_2$ satisfying $\omega(a+b) = \omega(a) + \omega(b) + a \cdot b$).

Because of the form of the relations, the degree of an element of B (thought of as a polynomial in the generators) is well defined, and we put B_k equal to the vector space of all elements of B of degree $\leq k$. Note that $Sp(H_1F)$, and hence also $\mathcal{M}_{g,1}$, acts naturally on B as algebra isomorphisms and on B_k as linear isomorphisms. The principal theorem of [J2] is to show how all the Birman-Craggs homomorphisms can be assembled simultaneously into a single surjective homomorphism $\sigma : \mathcal{I}_{g,1} \to B_3$ (the cubics of B). The properties of σ may then be stated thus:

a. (Naturality): For $h \in \mathcal{M}_{g,1}$ and $f \in \mathcal{I}_{g,1}$ we have $\sigma(hfh^{-1}) = h(\sigma(f))$.

b. Each Birman-Craggs homomorphism ρ_ω factors through σ via a corresponding linear map $\lambda_\omega : B_3 \to Z_2$.

c. The Birman-Craggs homomorphisms generate thereby $B_3^* = \mathrm{Hom}(B_3, Z_2)$.

In other words, we may identify the space of Birman-Craggs homomorphisms with B_3^*. This is the method used to calculate the dimension of the space of Birman-Craggs homomorphisms: For $\dim B_3^* = \dim B_3$, and the fact that the latter is $\sum_{j=0}^{3} \binom{2g}{j}$ follows from the defining relations 1 and 2 for B.

We now have two abelian quotients of $\mathcal{I}_{g,1}$, namely:

$$\tau : \mathcal{I}_{g,1} \to \Lambda^3 H_1(F_{g,1}, Z) : \text{rank} = \binom{2g}{3} \qquad \text{and}$$

$$\sigma : \mathcal{I}_{g,1} \to B_3 \qquad : \text{dimension} = \sum_{j=0}^{3} \binom{2g}{j} \ .$$

As the common term $\binom{2g}{3}$ in the ranks of the targets suggests, these two maps are not independent. They can, however, be assembled via a fibered product construction to get an abelian quotient embodying them both. That this is the final word on abelian quotients of $\mathcal{I}$ is the result of [J7], namely:

<u>THEOREM</u>: a. $\mathcal{I}_{g,1}/\mathcal{I}'_{g,1}$ <u>is isomorphic to the above fibered product</u>. <u>In particular</u>, $\mathcal{I}'_{g,1} = \operatorname{Ker} \tau \cap \operatorname{Ker} \sigma$.

b. $\sigma : H_1(\mathcal{I}_{g,1}, Z_2) \to B_3$ <u>is an isomorphism</u>.

c. <u>The Birman-Craggs homomorphisms</u> $\{\rho_\omega\}$ <u>generate</u> $\operatorname{Hom}(\mathcal{I}_{g,1}, Z_2)$.

There is one final aspect of the Birman-Craggs homomorphisms on which I wish to harp. It is that their definition is rather mysterious. In fact, if we take the point of view of Magnus and the combinatorial group theorists, $\mathcal{M}_{g,1}$ could be defined (by a theorem of Nielsen) as the automorphisms of a free group which fix a certain word, and $\mathcal{I}_{g,1}$ as the subgroup which acts trivially on the abelianization of this free group. This definition apparently removes all topology from the subject, and yet we have no definition of the homomorphisms $\rho_\omega : \mathcal{I}_{g,1} \to Z_2$ which does not involve the (at least implicit) construction of a 4-manifold. It would be interesting and perhaps enlightening to see the ρ_ω's defined in a more direct and fundamental way. By studying certain relations in $\mathcal{I}_{g,1}$, I was able to produce a group-theoretic result of this kind (see [J4]). It suffices to describe the (index 2) kernel of ρ_ω. Define $\mathcal{O}_\omega$ to be the subgroup of $\mathcal{M}_{g,1}$ consisting of those diffeomorphisms which act on $H_1(F_{g,1}, Z_2)$ so as to preserve the quadratic form ω. Since $\mathcal{I}_{g,1}$ is normal in $\mathcal{M}_{g,1}$, the commutator group $[\mathcal{O}_\omega, \mathcal{I}_{g,1}]$ is contained in $\mathcal{I}_{g,1}$.

<u>THEOREM</u>: $\operatorname{Ker} \rho_\omega = [\mathcal{O}_\omega, \mathcal{I}_{g,1}]$.

This is suggestive, but I still find it curiously unsatisfying, because in spite of the fact that $[\mathcal{O}_\omega, \mathcal{I}]$ is of index only two in $\mathcal{I}$, the process of deciding whether a given $f \in \mathcal{I}$ is in $[\mathcal{O}_\omega, \mathcal{I}]$ or not remains very devious and indirect. It would be more interesting to know whether $\rho_\omega(f)$ can be calculated directly from, say, the action of f on $\pi_1(F)$. (The word "directly" is vague but essential here; in principal at least, <u>everything</u> can be computed from the action of f on π_1.) This desire is not completely arbitrary. For example, it might shed light on the question of whether the Rochlin invariant of a homology sphere is predictable from its fundamental group (this is not known even in the simply connected case!). A more precisely stated problem is:

F) Can $\rho_\omega(f)$ be computed from the action of f on some nilpotent quotient of $\pi_1(F)$? If so, it should suffice to look at $\pi/\pi^{[4]}$ or $\pi/\pi^{[5]}$. I have shown by an unenlightening brute force calculation, however, that one cannot quite find $\rho_\omega(f)$ from the action of f on $\pi/\pi^{[3]}$ ($\pi^{[3]} = [\pi, [\pi,\pi']]$). The meaning of "not quite" is more precisely this: The cubic "polynomial" $\sigma(f) \in B_3$ can be computed modulo its constant term by means of the action of f on $\pi/\pi^{[3]}$. This constant term is not, however, predictable from the action, and unfortunately knowledge of the constant term is necessary to compute $\rho_\omega(f)$.

Since $\tau(f)$ can also be computed from the action of f on $\pi/\pi^{[3]}$ (in fact, from its action on $E = \pi/\pi^{[2]}$) we see that this action gives all abelian information about $\mathcal{I}_{g,1}$ except for a single missing Z_2.

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