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THE TORELLI GROUPS FOR GENUS 2 AND 3 SURFACES

GEOFFREY MESS

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1. INTRODUCTION

THE purpose of this paper is to show that the genus 2 Torelli group is free on infinitely many Dehn twists on separating curves. Moreover, the set of free generators can be identified with the set of splittings of the homology of a genus 2 surface into two subspaces mutually orthogonal and unimodular with respect to the intersection pairing. In addition, it is shown that the third integer homology of the genus 3 Torelli group naturally contains an infinitely generated free abelian group. This is a permutation module for the symplectic modular group. The method used is a study of the period mapping to Siegel space.

In Sections 2 and 3 we review some background material. Section 4 shows that the genus 2 Torelli group is free. Section 5 contains additional background material and the result on the third homology. Section 6 gives a homological application.

2. TEICHMÜLLER AND TORELLI SPACES

Let C be a closed Riemann surface of genus g. A *canonical homology basis* is a $2g$-tuple $(A_1, \ldots, A_g, B_1, \ldots B_g)$ of elements of $H_1(C, \mathbf{Z})$ such that $(A_i, A_j) = (B_i, B_j) = 0$, $(A_i, B_j) = \delta_{ij}$, where $(,)$ denotes the intersection pairing. We assume $g \geq 1$. The space of holomorphic differential forms on C has dimension g and a basis $(\phi_1, \ldots, \phi_g)$ such that $\int_{A_i} \phi_j = \delta_{ij}$. Let $\pi_{ij} = \int_{B_j} A_i$. Let $[C] \in H_2(C, \mathbf{Z})$ be the fundamental class. Then the diagonal $\Delta: H_2(C, \mathbf{Z}) \to H_2(C \times C, \mathbf{Z})$ is given by $\Delta[C] = i_{1*}[C] + i_{2*}[C] + \sum_{i=1}^{g} A_i \times B_i - B_i \times A_i$, where $i_1, i_2: C \to C \times C$ are the inclusions of the first and second factors and $\times$ is the exterior homology product. Riemann's first bilinear relation follows: for any two holomorphic 1-forms ϕ, ψ (or 1-forms representing any two cohomology classes with cupproduct equal to zero)

$$0 = \int_C \phi \wedge \psi = \sum_{i=1}^{g} \phi(A_i)\psi(B_i) - \psi(A_i)\phi(B_i)$$

and in particular $\pi_{ij} = \pi_{ji}$. If ϕ is a nonzero holomorphic 1-form, $i\phi \wedge \bar{\phi}$ is a nonnegative integrand so $i \int_C \phi \wedge \bar{\phi} > 0$. Riemann's second bilinear relation follows: the imaginary part of the *period matrix* $\{\pi_{ij}\}$ is positive definite. The *Siegel space* $\mathscr{X}_g$ is the space of all symmetric $g \times g$ complex matrices with positive definite imaginary part. An *abelian variety* is a complex torus which admits a projective embedding. Such an embedding $i: A \to \mathbf{P}^N$ determines a cohomology class $i^*(H) \in H^2(A, \mathbf{Z})$ where $H \in H^2(\mathbf{P}^N, \mathbf{Z})$ is the generator which satisfies $\langle H, \mathbf{P}^1 \rangle = 1$ where $\mathbf{P}^1$ has its orientation given by the complex structure. Let $g = \dim_{\mathbf{C}} A$; then the cup power $i^*(H)^g = d \cdot g![A]$ where $[A] \in H^{2g}(A, \mathbf{Z})$ is the fundamental class. d is a positive integer, and $d \cdot g!$ is the degree of $i(A)$. $H^{1,1}(A, \mathbf{C})$ is the subspace of $H^2(A, \mathbf{C})$ represented by forms which can be locally expressed as sums of terms of the form $f(z)dz_i \wedge d\bar{z}_j$. A *polarization* of A is a cohomology class x in $H^2(A, \mathbf{Z}) \cup H^{1,1}(A, \mathbf{C})$ which is *positive* and satisfies $x^g \neq 0$; then $x^g = d \cdot g![A]$ for some positive integer d. x is *positive* if its representing (1, 1)-form X satisfies $X(v, Jv) > 0$, where v is a tangent vector and J is the almost complex structure on A. x is a *principal polarization* if $d = 1$ or equivalently the symplectic form which x defines on $H_1(A, \mathbf{Z})$ is unimodular. Suppose (A, x) is a principally polarized abelian variety. Choose a homology basis $A_1, \ldots, A_g, B_1, \ldots, B_g$ for $H_1(A, \mathbf{Z})$ such that $x(A_i, A_j) = x(B_i, B_j) = 0$, $x(A_i, B_j) = \delta_{ij}$. Then there is a basis $(\phi_1, \ldots, \phi_g)$ of holomorphic differentials such that $\int_{A_i} \phi_j = \delta_{ij}$. Define the period matrix $\prod = \{\pi_{ij}\}$ by $\pi_{ij} = \int_{B_i} \phi_j$. Then $\prod \in \mathscr{X}_g$, and conversely for each $\prod \in \mathscr{X}_g$, there is a unique triple (up to biholomorphisms preserving the additional structures x, $(A_1, \ldots, B_g)$) such that (A, x) is a principal polarization of A and $(A_1, \ldots, B_g)$ is a homology basis such that $x(A_i, A_j) = x(B_i, B_j) = 0$, $x(A_i, B_j) = 1$. The symplectic group $Sp(g, \mathbf{Z}) \subset GL_{2g}\mathbf{Z}$ acts on $\mathscr{X}_g$ by $T \cdot (A, x, (A_1, \ldots, B_g)) = (A, x, (TA_1, \ldots, TB_g))$. Explicitly, if $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $T\Pi = (D\Pi - C)(-B\Pi + A)^{-1}$. (Cf. [46], p. 173. See pp. 174–175 of [46] for an explanation of the different formula given on p. 23 of [5].) Observe that the stabilizer of a point $p \in \mathscr{X}_g$ is a finite group containing $-1 \in Sp(g, \mathbf{Z})$ and this is the group of isomorphisms of the corresponding abelian variety which preserve the polarization and fix the identity $0 \in A(p)$ of the group structure. $\mathbf{\Omega}^1$ denotes the sheaf of holomorphic 1-differentials. Given C, the map $j\colon H_1(C, \mathbf{Z}) \to H^0(C, \Omega^1)^*$ defined by $j(A)(\phi) = \phi(A) = \int_{A'} \phi$ (where A' is any 1-manifold on C representing the class A) embeds $H_1(C, \mathbf{Z})$ as a discrete lattice. Using the dual basis to $\phi_1, \ldots, \phi_g$, $jH_1(C, \mathbf{Z})$ is generated by the $2g$ columns of the $g \times 2g$ matrix $(1_g, \Pi)$ where $\Pi = \pi_{ij}$. The Jacobian of C is the complex torus $H^0(C, \Omega^1)^*/jH_1(C, \mathbf{Z}) = J(C)$. Given any basepoint $P_0 \in C$, there is a natural map $\Psi\colon C \to J(C)$ given by $\Psi(P)(\phi) = \int_{P_0}^{P} \phi + jH_1(C, \mathbf{Z})$ where any path from P_0 to P may be chosen. Ψ defines an isomorphism $\Psi^*\colon H^2(J(C), \mathbf{Z})(\cong \wedge^2 H^1(J(C), \mathbf{Z})) \to \wedge^2 H^1(C, \mathbf{Z})$. The fundamental class $1_C \in H^2(M, \mathbf{Z})$ defines, by Poincaré duality, an element of $Hom(\wedge^2 H_1(C, \mathbf{Z}), \mathbf{Z})$ and so an element $x_C \in \wedge^2 H^1(C, \mathbf{Z})$. Note that the cup product can be regarded as a linear map $\bigcup\colon \wedge^2 H^1(C, \mathbf{Z}) \to H^2(C, \mathbf{Z})$, and $\bigcup(x_C) = g \cdot 1_C$. Ψ determines an element $[\Theta]$ of $H^2(J(C), \mathbf{Z})$ such that $\Psi^*[\Theta] = x_C$. This is a purely topological definition, but $[\Theta]$ has an analytic interpretation. Given $\Pi \in \mathscr{X}_g$, Riemann's theta function $\Theta\colon \mathbf{C}^g \to \mathbf{C}$ is defined by

$$\Theta(z, \Pi) = \sum_{\mathbf{m} \in \mathbf{Z}^g} \exp(\pi i \mathbf{m}^t \Pi \mathbf{m} + 2\pi i \mathbf{z}^t \mathbf{m})$$

$\Theta\colon \mathbf{C}^g \to \mathbf{C}$ is holomorphic and the zero set $Z(\Theta)$ of Θ is invariant under translations by $(1_g, \Pi) \cdot \mathbf{Z}^{2g}$. So the divisor $Z(\Theta)$ determines a holomorphic line bundle on $\mathbf{C}^g/(1_g, \Pi) \cdot \mathbf{Z}^{2g}$, of which Θ is a section. From the equations

$$\Theta(\mathbf{z} + \mathbf{m}, \Pi) = \Theta(\mathbf{z}, \Pi)$$

$$\Theta(\mathbf{z} + \Pi\mathbf{m}, \Pi) = \exp(-\pi i \mathbf{m}^t \Pi \mathbf{m} - 2\pi i \mathbf{m}^t \mathbf{z})\Theta(\mathbf{z}, \Pi)$$

it follows that the Chern class of the line bundle is $\sum_{i=1}^{g} A_i' \cup B_i'$, where $A_i', B_i' (1 \leq i \leq g)$ are the elements of the dual basis to $(A_1, \ldots, B_g)$. So $[\Theta]$ is a principal polarization. We call $Z(\Theta)$ the theta divisor. Recall that Teichmüller space T_g is a complex manifold of dimension $3g - 3$ such that given a surface F of genus g, T_g is in bijection with the set of isotopy classes of complex structures. T_g is diffeomorphic to $\mathbf{R}^{6g-6}$. Each genus 2 surface is hyperelliptic, i.e. is a double branched cover of $\mathbf{P}^1$, with 6 branch points uniquely determined up to a fractional linear transformation. So for each point in $M = \{(\alpha, \beta, \gamma) \in (\mathbf{C} - \{0, 1\})^3 : \alpha \neq \beta \neq \gamma \neq \alpha\}$ there is a corresponding Riemann surface $y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\gamma)$.

Forgetting β, γ gives a fibering $M \to \mathbf{C} - \{0, 1\}$ with fiber M_1; M_1 fibers over a 4-times punctured sphere with fiber a 5-times punctured sphere. Thus the universal cover $\tilde{M}$ of M is a complex manifold diffeomorphic to $\mathbf{R}^6$. ("Fibering" is being used here in the C^∞ and not in the holomorphic sense.) In fact $\tilde{M} = T_2$.

The mapping class group Γ_g acts properly discontinuously on T_g [1]. By the Lefschetz fixed point formula, if $\phi: C \to C$ is biholomorphic and C has genus $g > 1$ then $2 - trace(H_1\phi) \geq 0$, so $H_1\phi$ is nontrivial. The kernel of the natural map $\Gamma_g \to Sp(g, \mathbf{Z})$ is called the *Torelli group* I_g. Since the stabilizer in Γ_g of a point $p \in T_g$ is the biholomorphism group of the corresponding complex structure F_p of F, I_g is the deck group of a covering $T_g \to T_g/I_g$; in particular I_g is torsion free. (As is well known Γ_g also has finite index torsion free subgroups, e.g. the kernel of the map $\Gamma_g \to Sp(g, \mathbf{Z}) \to Sp(g, \mathbf{Z}_n)$ for any $n > 2$.) The quotient space T_g/I_g is the Torelli space. The Teichmüller curve $V_g \xrightarrow{\pi} T_g$ is a proper submersion from V_g, a complex manifold of dimension $3g - 2$, to T_g such that the fiber F_p over $p \in T_g$ is a Riemann surface of genus g with the complex structure of F_p up to isotopy. (The Teichmüller curve is topologically trivial by Teichmüller's theorem [1] so the complex structure is well defined up to isotopy.) The quotient $U_g = V_g/I_g$ is the Torelli curve; it is the universal family of genus g surfaces with prescribed canonical homology basis. The *period* or *Torelli* map $T_g/I_g \xrightarrow{t} \mathscr{X}_g$ is defined by $t(C, (A_1, \ldots, B_g)) = \{\pi_{ij}\} \in \mathscr{X}_g$. t is holomorphic: tangent vectors to T_g can be represented by Beltrami differentials, and given a Beltrami differential μ on the fiber $\pi^{-1}q$ of $\pi: U_g \to T_g/I_g$, Rauch's variational formula [18] states that

$$d\pi_{ij}([\mu]) = \int_{\pi^{-1}q} \mu\phi_i\phi_j$$

where $[\mu]$ is the tangent vector determined by μ.

For more information on Teichmüller theory, theta functions, Jacobians etc., I have found the following references useful: [1, 2, 5, 20, 19, 21, 22, 23, 24, 38, 46, 47]. The Torelli group was previously investigated by Birman [11], Powell [48], Chillingworth [14], Wagoner [54], Birman and Craggs [13], Schiller [50], Johnson [30, 31, 32, 34, 33, 35, 36] and McCullough and Miller [43]. In particular, Birman showed that I_2 is the normal closure in Γ_2 of a Dehn twist on a separating curve, Johnson [33] showed that I_g is finitely generated if $g \geq 3$, and McCullough and Miller showed that I_2 is not finitely generated.

3. JACOBIANS OF GENUS TWO CURVES

Most of the material in this section is well known. Let J be a principally polarized abelian variety of dimension 2.

PROPOSITION 1. *Either the theta divisor C of J is a nonsingular curve C of genus 2 or $J = E_1 \times E_2$ for two elliptic curves E_1, E_2 as a product of polarized abelian varieties in which case $C = E_1 \times \{q\} + \{p\} \times E_2$ where p, q are the theta divisors of E_1, E_2 respectively.*

Proof. Suppose C is irreducible. Since J is a group variety, the canonical divisor K is trivial. By the genus formula (see e.g. I.15 in [8])

$$2g(C) - 2 = C \cdot C + C \cdot K$$

where $g(C) = H^1(C, \mathcal{O}_C)$. Since J is principally polarized, $g(C) = 2$. (We remark that if

C was known to be smooth, this would simplify to the argument that $C \cdot C = 2$ and $0 \to TC \to TJ|_C \to NC \to 0$ is exact, where TC and NC are the tangent and normal bundles of C, and $TJ|_C$ is the restriction of the tangent bundle of J to C. Since C and J are complex manifolds, this is an exact sequence of complex vector bundles, so $C \cdot C = 2$ implies $c_1(NC)([C]) = 2$ implies $c_1(TC)([C]) = -2$.) Following I.16 in [8], let $f\colon N \to C$ be the normalization of C and define a sheaf δ on C by the exact sequence

$$0 \to \mathcal{O}_C \to f_* \mathcal{O}_N \to \delta \to 0.$$

δ is supported at the singular points of C, so $H^1(C, \delta) = 0$. Because C is irreducible, $H^0(C, \mathcal{O}_C) = H^0(C, f_* \mathcal{O}_N) = \mathbf{C}$, and the induced map is an isomorphism. So the long exact cohomology sequence reduces to

$$0 \to H^0(C, \delta) \to H^1(C, \mathcal{O}_C) \to H^1(C, f_* \mathcal{O}_N) \to 0.$$

Now we will show that $H^1(C, f_* \mathcal{O}_N) \cong H^1(N, \mathcal{O})$. Let $\{U, V\}$ be a cover of C by two affine open subsets, and let $U' = f^{-1}U$, $V' = f^{-1}V$. Then we have exact Mayer–Vietoris sequences:

$$0 \to H^0(U \cup V, f_* \mathcal{O}_N) \to H^0(U, f_* \mathcal{O}_N) \oplus H^0(V, f_* \mathcal{O}_N)$$
$$\to H^0(U \cap V, f_* \mathcal{O}_N) \to H^1(U \cap V, f_* \mathcal{O}_N) \to 0$$
$$0 \to H^0(U' \cup V', \mathcal{O}_N) \to H^0(U', \mathcal{O}_N) \oplus H^0(V', \mathcal{O}_N) \to H^0(U' \cap V', \mathcal{O}_N) \to H^1(N, \mathcal{O}_N) \to 0.$$

By definition, $H^0(U, f_* \mathcal{O}_N) = H^0(U', \mathcal{O}_N)$ and similarly for V and $U \cap V$. So there is an isomorphism $H^1(C, f_* \mathcal{O}_N) \cong H^1(N, \mathcal{O}_N)$. So, writing $\dim \delta$ for $\dim H^0(C, \delta)$, $g(N) = 2 - \dim \delta$. (The argument that $H^1(C, f_* \mathcal{O}_N) \cong H^1(N, \mathcal{O})$ is a special case of exercise III.8.1 in [25]. I thank Donu Arapura for explaining this to me.) More generally if C is reducible we have $\chi(\mathcal{O}_C) = -1$, $\chi(\mathcal{O}_N) = \chi(\mathcal{O}_C) + \dim \delta$, and $g(N) = k - \dim \delta - \chi(\mathcal{O}_C) = k + 1 - \dim \delta$ where k is the number of components of N. Since J is an abelian variety, each component of C has genus at least one. So either C is an irreducible curve with $\dim \delta = 0$ and $g(N) = 2$, or $\dim \delta = 1$ and N is a union of curves of genus 1. δ is a skyscraper sheaf concentrated at the singular points of C. If $\delta = 0$ then C is normal and therefore smooth. Suppose N is a union of curves C_i of genus 1. A map $C_i \to J$ is a group homomorphism followed by a translation. So two elliptic curves in J are either translates of each other or else transverse. Each point of transverse intersection contributes 1 to $\dim \delta$, and $g(N) = k$. So $\dim \delta = 1$. So C contains 2 elliptic curves $E_1 E_2$ meeting transversely in 1 point. Any curve in J meets either E_1 or E_2, so $k = 2$ and $C = E_1 \times \{q\} + \{p\} \times E_2$.

Here is an alternative proof in a more topological style. Given a singular curve C on a surface S, let $p \in C$ be a singular point. Blow up p. Let $\tilde{C}$, $\hat{C}$, E be the total transform of C, the strict transform, and the exceptional curve. Let k be the multiplicity of C at p. Let K and $\hat{K} = \pi^* K + E$ be canonical divisors on S and on the blown up surface $\hat{S}$ respectively. (For simplicity choose K disjoint from p.) Let $n = C \cdot C$. Then $n = \tilde{C} \cdot \tilde{C} = \hat{C} \cdot \hat{C} + 2k\hat{C} \cdot E - k^2$, so $\hat{C} \cdot \hat{C} + \hat{K} \cdot \hat{C} = n + K \cdot C - (k^2 - k)$. So after a finite sequence of blow ups at points where the strict transforms have multiplicities k_i which resolves the singularities of C, the strict transform C' of C satisfies

$$-\chi(C') = (C \cdot C + K \cdot C) - 2 \sum_i \binom{k_i}{2}$$

where $\chi(C')$ is the topological Euler characteristic, equal to $2\chi(\mathcal{O}_{C'})$ for a non-singular curve C'. It follows as before that either $g(C) = 2$ and C is nonsingular or C is a union of two elliptic curves meeting transversely in a point.

Proposition 2. *If J is a principally polarized Jacobian of dimension 2 and the Θ-divisor C is a nonsingular curve then $J = J(C)$ as a principally polarized abelian variety. In particular inclusion defines an isomorphism $H_1(C, \mathbf{Z}) \cong H_1(J, \mathbf{Z})$.*

Proof. By the universal property of the Jacobian there is a map $\pi: J(C) \to J$ such that $\pi \circ \psi = i$ where $\psi: C \to J(C)$ is the inclusion determined by a base point and $i: C \to J$ is the inclusion. Since C lies in no subvariety of J, π is a (necessarily abelian) covering. C lifts from J to $J(C)$ so any deck transformation g satisfies $g \cdot C \cap C = \varnothing$. But g is a translation, so this contradicts $C \cdot C = 2$.

Proposition 3. *The period map $t: T_2/I_2 \to \mathscr{X}_2$ is a holomorphic injection. Furthermore, the complement is a disjoint union of properly embedded copies of $U \times U$ where U is the upper half plane.*

Proof. t is holomorphic by Rauch's formula. Given $\{\pi_{ij}\} = p \in \mathscr{X}_2$, let C be the theta divisor of $J = \mathbf{C}^2/\left(\begin{smallmatrix} 1 & 0 & \pi_{11} & \pi_{12} \\ 0 & 1 & \pi_{21} & \pi_{22} \end{smallmatrix}\right) \cdot \mathbf{Z}^4$. First suppose that J is not a product. Identify $\pi_1 J$ with the free abelian group $H_1(C, \mathbf{Z})$ by Proposition 2, and also with the subgroup $\left(\begin{smallmatrix} 1 & 0 & \pi_{11} & \pi_{12} \\ 0 & 1 & \pi_{21} & \pi_{22} \end{smallmatrix}\right) \cdot \mathbf{Z}^4$ of $\mathbf{C}^2$. Let (A_1, A_2, B_1, B_2) be the homology basis of C such that $i_* A_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $i_* A_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $i_* B_1 = \left(\begin{smallmatrix} \pi_{11} \\ \pi_{12} \end{smallmatrix}\right)$, $i_* B_2 = \left(\begin{smallmatrix} \pi_{21} \\ \pi_{22} \end{smallmatrix}\right)$. This is in fact a canonical homology basis because (i) $[\Theta](A_i \times A_j) = [\Theta](B_i \times B_j) = 0$, $[\Theta](A_i \times B_j) = \delta_{ij}$ and (ii) the restriction of $[\Theta]$ to C is the fundamental class because $\cap [\Theta]: H^1(C, \mathbf{Z}) \to H_1(C, \mathbf{Z})$ defines an isomorphism by (i). We have exhibited an inverse to t. Now consider the case that J is a product: $J = E_1 \times E_2$ and the Θ-divisor is $E_1 \times q \cup p \times E_2$. Then as in Proposition 2, $i_*: H_1(E_1, \mathbf{Z}) \oplus H_1(E_2, \mathbf{Z}) \to H_1(J, \mathbf{Z})$ is an isomorphism, and $i_* H_1(E_1, \mathbf{Z})$, $i_* H_1(E_2, \mathbf{Z})$ are orthogonal subspaces with respect to the symplectic form determined by $[\Theta]$.

We introduce a definition.

Definition 1. Given a free abelian group L with a symplectic unimodular form $\langle , \rangle$ a homology splitting is an unordered pair $\{U, V\}$ of subgroups of L such that $L = U \oplus V$ and U and V are orthogonal with respect to $\langle , \rangle$.

Evidently the homology splittings of $H_1(J, \mathbf{Z})$ are in natural bijection with $Sp(2, \mathbf{Z})/\langle e \rangle \rtimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z})$ where $\rtimes$ denotes semidirect product, e is the order 2 matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and the two copies of $SL_2\mathbf{Z}$ are embedded as matrices such that the submatrix corresponding to the indices 1,3 (respectively 2,4) is the identity matrix and the ij entries are zero when i is 1 or 3 and j is 2 or 4 or vice versa. Let $X = \{g_i\}$ be coset representatives for $\langle e \rangle \rtimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z})$. Then there is a map $t': X \times U \times U \to \mathscr{X}_2$ defined by $(g_i, \tau_1, \tau_2) \mapsto g_i \cdot \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$. Given $\left(\begin{smallmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{smallmatrix}\right) = p$, if J is a product there is a unique $x \in X$ such that c takes the homology splitting $\{(A_1, B_1), (A_2, B_2)\}$ to $\{H_1 E_1, H_1 E_2\}$. Then $p = x \cdot \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)$ for some τ_1, τ_2 in U. So t' is a bijection onto the complement of the image $t(T_2/I_2)$ of t. In particular the image of t' is closed.

Corollary 1. *As a module over $Sp(2, \mathbf{Z})$, $H_1(I_2, \mathbf{Z})$ is isomorphic to $\mathbf{Z}[Sp(2, \mathbf{Z})/(\langle e \rangle \rtimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z}))]$.*

Proof. Observe that each embedding of $U \times U$ in $\mathscr{X}_2$ is proper. So Alexander duality applies.

We observe that the weaker result $I_2/[\Gamma_2, I_2] \cong \mathbf{Z}$ is implicit in earlier work. From Igusa's result [29] $\mathscr{M}_2 := T_2/I_2 = \mathbf{C}^3/(\mathbf{Z}/5\mathbf{Z})$ it follows that $H_2(\Gamma_2, \mathbf{Q}) = 0$. On the other hand $H_2(Sp(2, \mathbf{Z}); \mathbf{Z}) = \mathbf{Z}$. This follows from the presentations of $Sp(2, \mathbf{Z})$ obtained in [28] and [55]. By the five term exact sequence in group homology and the fact due to Birman [11] that I_2 is the normal closure in $\Gamma - 2$ of a single Dehn twist, $I_2/[\Gamma_2, I_2] = \mathbf{Z}$.

I point out that Propositions 1, 2 and 3 are well known to algebraic geometers. (I have not however been able to find a suitable reference.) Classically the complement of the image of the period map was known as the set of Humbert matrices. Here I have given a self-contained presentation of these results. The application to the Torelli group is new.

4. MORSE THEORY IN THE TORELLI SPACE

The argument of this section is a simple example of embedded Morse theory. Indeed, given a manifold, a submanifold, and a Morse function whose restriction to the submanifold is also a Morse function, we can obtain not only handlebody decompositions of the manifold and submanifold, but also a handlebody decomposition for the complement of the submanifold in the manifold. A general discussion is given in [51], pp. 65–71. In our case the Morse function has only one critical point on the manifold and only one critical point on each component of the submanifold.

PROPOSITION 4. *The Torelli space T_2/I_2 has a handle decomposition with a single* 0*-handle and a set of* 1*-handles in one-to-one correspondence with the homology splittings. The Torelli group I_2 is a free group freely generated by a set of Dehn twists on separating curves which form a set of representatives for the homology splittings.*

Proof. Fix a point $p \in t(T_2/I_2) \subset \mathscr{X}_2$. We will consider $\mathscr{X}_2$ as a symmetric space. See e.g. [26]. Let $f(x) = d^2(x, p)$ where $d(,): \mathscr{X}_2 \times \mathscr{X}_2 \to [0, \infty)$ is the distance function. Then f is proper and strictly convex along every geodesic. See e.g. p. 27 of [44] or p. 4 of [7]. Now each component of $t'(X \times U \times U)$ is totally geodesic in $\mathscr{X}_2$ because it is the set of fixed points of some involution. To see this it suffices to consider the component $U \times U = \{(\begin{smallmatrix}\pi_1 & 0\\ 0 & \pi_2\end{smallmatrix}): Im\,\pi_1, Im\,\pi_2 > 0\}$ of $t'(X \times U \times U)$, which is fixed by the involution

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix} = j. \qquad j \cdot \begin{pmatrix} \pi_{11} & \pi_{12}\\ \pi_{21} & \pi_{22} \end{pmatrix} = \begin{pmatrix} \pi_{11} & -\pi_{12}\\ -\pi_{21} & \pi_{22} \end{pmatrix}.$$

So for each of the components $x \cdot (U \times U)$ of $\mathscr{X}_2 - t(T_2/I_2)$, $f|x \cdot (U \times U)$ is a proper Morse function with no critical points except for a unique minimum. Consider the balls $B_t = \{x : f(x) \le t\}$. If $[t_1, t_2]$ is an interval such that none of the critical values of $f|_{t'(X \times U \times U)}$ is in $[t_1, t_2]$, then the pairs $(B_{t_i}, B_{t_i} \cap t'(X \times U \times U))(i = 1, 2)$ are diffeomorphic, because the normal vector field to B_t can be homotoped in a neighborhood of $B_t \cap t'(X \times U \times U)$ to a vector field Y such that Y is nowhere zero and $Y|_{t'(X \times U \times U)}$ is the gradient of $f|_{t'(X \times U \times U)}$ and so is tangent to $t'(X \times U \times U)$. For simplicity we assume (using Baire category) that p is not equidistant to any pair of components of $t'(X \times U \times U)$. Suppose $t \ne 0$ is a critical value of f, attained at $q \in x \cdot U \times U$, $x \in X$. Then in a neighborhood of q it is possible to introduce coordinates f, x_1, x_2, x_3, x_4, x_5 such that $x \cdot U \times U$ is defined by the conditions $x_5 = 0, f = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Then $B^0_{t+\varepsilon} - t'(X \times U \times U)$ is diffeomorphic to $B^0_t - t'(X \times U \times U)$ union (a tubular neighborhood of the core $\{f = \varepsilon^2 - x_5^2, x_1 = x_2 = x_3 = x_4 = 0\}$ of a 1-handle). This establishes the first statement of the proposition. From the handlebody description it follows that I_2 is freely generated by a set of elements $e_x, x \in X$ such that e_x is conjugate in I_2 to the monodromy of the restriction $U_g|_{S^1_x}$ of the Torelli curve to a loop S^1_x bounding a disc D^2_x transverse to $t'(\{x\} \times U \times U)$. We may assume that the disc D^2_x is a complex submanifold. It remains to show that this is a Dehn twist. First I will give an indirect argument. By the uniformization theorem, the open punctured disc $\operatorname{int} D^2_x - \{p\}$ carries a complete hyperbolic metric compatible with the complex structure, which coincides with the Kobayashi metric. The conjugacy class of e_x can be represented by arbitrarily short geodesics in the hyperbolic metric on the punctured disc $\operatorname{int} D^2_x - \{p\}$ where $p = D^2_x \cap \{x\} \times U \times U$. Since the Teichmüller and Kobayashi metrics on T_2 are equal [49], [22] and all holomorphic maps are distance decreasing in the Kobayashi metric, the displacement function $f\colon T_2 \to \mathbf{R}, f(q) = d_T(q, e_x q)$ has infimum zero. (d_T denotes the Teichmüller metric.) Since e_x has infinite order the infimum is not attained. So e_x is parabolic in Bers's version [10] of the Nielsen–Thurston–Bers classification of mapping classes. Since e_x is parabolic and in I_2, e_x must be a product of Dehn twists on disjoint separating curves. Since $g = 2$, e_g is τ_x^n for some n and a Dehn twist τ_x. If $|n| > 1$, we would have a contradiction: $I_2/[\Gamma_2, I_2] = \mathbf{Z}$ with generator τ_x (by Birman's result [11] together with the Corollary to Proposition 3), but conjugates of e_x generate the subgroup $n\mathbf{Z}$ of $I_2/[\Gamma_2, I_2]$. Now let $h(x) \in X$ be the homology splitting defined by the curve upon which τ_x is a Dehn twist. Then $h\colon X \to X$ is $Sp(2, \mathbf{Z})$-equivariant. h must be the identity by the following lemma:

LEMMA 1. *Suppose $h\colon G/S \to G/S$ is a G-equivariant map on a transitive G-set G/S. Then if S is its own normalizer, h is the identity*

Proof. Let $h(S) = aS$. Then $h(sS) = saS = aS$, so $a^{-1}sa \in S$. So if S is its own normalizer $h(S) = S$ and then $h(gS) = gS$.

It is easy to see that $\langle e \rangle \rtimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z})$ is self-normalizing.

For a more geometric proof, consider the point $q = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \in \mathscr{X}_2$ and the transverse disc $D = \{q_\lambda = \begin{pmatrix} i & \lambda \\ \lambda & i \end{pmatrix}: |\lambda| < \varepsilon\}$ where ε is small enough that D meets $t'(X \times U \times U)$ only in q. Consider the family $D \times \mathbf{C}^2/\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \cdot \mathbf{Z}^4$ of Jacobians (here we identify λ with $q_\lambda \in D$) and the family $U_q = \{[(z_1, z_2, q_\lambda)]: \Theta(z_1, z_2, q_\lambda) = 0\}$, where $[a]$ denotes the equivalence class of a under the action of $\mathbf{Z}^4$. $\Theta(q_\lambda, z_1, z_2)$ is invariant under the reflection $j(z_1, z_2) = (1 + i + \lambda - z_1, 1 + i + \lambda_{z_2})$ (that is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is an even half period (cf. pp. 285–286 in [19])). Introduce $z_i' = z_i - (1 + i + \lambda)/2 (i = 1, 2)$. Then for λ sufficiently small, $\Theta(z_1, z_2, q_\lambda)$ has nonzero differential except at $(\lambda, z_1', z_2') = (0, 0, 0)$.

We will show that $\Theta(z_1, z_2, q_\lambda) = c_1\lambda + c_2 z_1' z_2' +$ (higher order terms), where $c_1, c_2 \neq 0$. It will follow that the monodromy of the punctured disc D_{q_0} is generated by a Dehn twist.

First,

$$\Theta(z_1, z_2, q_0) = \theta(i, z_1)\theta(i, z_2)$$

where $\theta(\tau, z) = \sum_{n \in \mathbf{Z}} \exp(\pi i n^2 \tau + 2\pi i n z)$ is Jacobi's theta function. Let $z = (1 + \tau)/2 + z'$.

$$\begin{aligned} \theta(\tau, z) &= \sum_{n \in \mathbf{Z}} \exp[\pi i(n^2\tau + n(1 + \tau) + 2nz')] \\ &= \sum_{n \in \mathbf{Z}} \exp[\pi i \tau(n + 1/2)^2] \exp[-\pi i \tau/4](-1)^n \exp(2n\pi i z) \end{aligned}$$

so

$$\frac{\partial \theta}{\partial z}\bigg|_{z'=0} = \sum_{n\in\mathbf{Z}} \exp[\pi i\tau(n+1/2)^2]\exp[-\pi i\tau/4](-1)^n 2\pi i n$$

$$= \sum_{n=0}^{\infty} \exp[\pi i\tau(n+1/2)^2]\exp[-\pi i\tau/4](-1)^n 2\pi i(2n+1)$$

$$\doteq 6.24799i, \quad \text{for } \tau = i.$$

So $\frac{\partial^2 \Theta}{\partial_{z_1}\partial_{z_2}}(z_1, z_2, q_0)|_{(z_1,z_2)=((1+i)/2,(1+i)/2)} = c_2 \doteq -39.0373 \neq 0$. Since

$$\frac{\partial \Theta}{\partial z_1'}\bigg|_{z_1'=0} = \frac{\partial \Theta}{\partial z_2'}\bigg|_{z_2'=0} = 0, \qquad \text{for } q = q_0,$$

$$\frac{\partial}{\partial \lambda}\Theta\left(\begin{pmatrix}\tau_1 & \lambda\\ \lambda & \tau_2\end{pmatrix}, ((1+\tau_1+\lambda)/2, (1+\tau_2+\lambda)/2)\right)\bigg|_{\lambda=0}$$

$$= \frac{\partial \Theta}{\partial \lambda}\left(\begin{pmatrix}\tau_1 & \lambda\\ \lambda & \tau_2\end{pmatrix}, ((1+\tau_1)/2, (1+\tau_2)/2)\right)\bigg|_{\lambda=0}$$

$$= \frac{\partial}{\partial \lambda}\sum_{n,m\in\mathbf{Z}} \{\exp[\pi i(\pi_1(n+1/2)^2 + \tau_2(m+1/2)^2)]\cdot$$

$$\exp[-(\pi i/4)(\tau_1+\tau_2)]\exp[\pi i(m+n)]\exp[2\pi i m n\lambda]\}|_{\lambda=0}$$

$$= \sum_{n,m\in\mathbf{Z}} 2\pi i\exp[-\pi(n^2+n+m^2+m)]\cdot(-1)^{m+n} mn$$

(setting $\tau_1 = \tau_2 = i$)

$$= \sum_{n,m\geq 0} 2\pi i\exp[-\pi(n^2+n+m^2+m)](-1)^{m+n}(2m+1)(2n+1)$$

$$= 2\pi i\left(\sum_{n\geq 0}\exp[-\pi(n^2+n)](-1)^n(2n+1)\right)^2 \doteq 6.21298i.$$

So in a neighborhood of $((1+i)/2, (1+i)/2, q_0)$ the family of Θ-divisors is topologically equivalent to the family of curves C_λ where C_λ is defined by $zw = \lambda$. The neighborhood may be taken to be $N = \{(z_1, z_2, q_\lambda): |\lambda|, |z_1'|, |z_2'| < \varepsilon\}$ where ε is sufficiently small that outside N projection of any C_λ along the fibers of a tubular neighborhood gives a diffeomorphism with $C_0 - (C_0 \cap N)$.

LEMMA 2. *In the 3-sphere* $L = \{|z| = 1, |w| \leq \varepsilon\} \cap \{|zw| = \varepsilon\} \cup \{|z| \leq \varepsilon, |w| = 1\} \subset N$, *the copy of* $T^2 \times I$ *given by* $|zw| = \varepsilon$ *is fibered by annuli* $A_\theta = \{zw = \varepsilon e^{i\theta}\}$, $0 \leq \theta < 2\pi$. *If* $\phi: A_0 \to A_0$ *is a representative diffeomorphism for the monodromy of this fibration which is the identity on* ∂A_0, ϕ *is a Dehn twist.*

Proof. Define an action of $\mathbf{R}$ on L by $F_\theta((z, w)) = (e^{ia(|z|)\theta}z, e^{i(1-a(|z|)\theta}w)$ where $\theta \in \mathbf{R}$, $a(|z|) = 0$ if $|z| = 1$, $a(|z|) = 0$ if $|z| \leq \varepsilon$, and a is a continuous increasing function of $|z|$. Then $F_\theta\{zw = \varepsilon\} = \{zw = \varepsilon e^{i\theta}\}$ and $F_{2\pi}\{zw = \varepsilon, |z| \in [\varepsilon, 1]\} = \{zw = \varepsilon, z = |z|e^{2\pi i a(|z|)}, |z| \in [\varepsilon, 1]\}$. So $\phi = F_{2\pi}$ is a Dehn twist.

Lemma 2 is well known. The homological monodromy was known to Picard. The proof of Proposition 4 is completed by the observation that the curve $|z| = |w|$ on the surface C_λ realizes the homology splitting of C_λ into the summands

$$\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}i\\ \lambda\end{pmatrix}\right\rangle \quad \text{and} \quad \left\langle \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\lambda\\ i\end{pmatrix}\right\rangle.$$

Without showing that $\partial\Theta/\partial\lambda \neq 0$ at the singular point, we could have concluded that the monodromy of a meridional loop was a power of a Dehn twist and deduced that the monodromy was a Dehn twist for homological reasons as in the argument using Bers's theorem.

The 1-skeleton which Proposition 4 yields is neither explicit nor canonical. There is no 1-dimensional complex embedded in $t(T_2/I_2)$ as a deformation retract which is equivariant with respect to $Sp(2, \mathbf{Z})$. Indeed, a 1-complex which lies in a manifold as a retract is necessarily locally finite. The preimage of the 1-complex in Teichmüller space would give a tree on which Γ_2 would act, with finite stabilizers by proper discontinuity. Such an action cannot exist, because Γ_2 has virtual cohomological dimension 3 and so is not the fundamental group of a graph of finite groups. It seems to be difficult to prescribe, on a genus 2 surface, a set of Dehn twists which freely generate I_2. That any such choice of generators must be complicated is suggested by the fact that Γ_2 is far from being a semidirect product of $Sp(2, \mathbf{Z})$ and I_2; in fact the virtual cohomological dimension of $Sp(2, \mathbf{Z})$ is 4 which is greater than 3, the virtual cohomological dimension Γ_2. Possibly the set of Dehn twists $\tau_{C(x)}$, $x \in X$ where $C(x)$ is the shortest curve (in a fixed hyperbolic metric) realizing the homology splitting x, is a set of free generators. Schiller [50] considered the subgroup $\mathcal{T}$ of I_2 centralizing an involution j which can be realized by a genus 2 surface which is the double branched cover of a genus 1 surface. He remarks without proof (pp. 112–113) that $\mathcal{T}$ is a free group. [50] contains other interesting results on the centralizer of j in Γ_2, using the embedding into $\mathcal{X}_2$.

5. THE THIRD HOMOLOGY OF THE GENUS THREE TORELLI GROUP

When $g = 3$, Teichmüller space T_3 has dimension $3g - 3$ and Siegel's space $\mathcal{X}_3$ has dimension $g(g + 1)/2 = 3$, while for $g > 3t(T_3/I_3)$ has positive codimension in $\mathcal{X}_3$ and is hard to describe topologically. (See [9] and references therein for recent progress in obtaining an analytic description of the closure of the image of the period map.) So one may hope to generalize the argument in Section 3 to the case $g = 3$. This was done by Johnson and Millson.

We will be using the sharp version of Torelli's theorem.

TORELLI'S THEOREM 1. (a) *Suppose the closed Riemann surfaces C, C' have period matrices $\{\pi_{ij}\}$, $\{\pi'_{ij}\}$ for some pair of choices of canonical homology basis, and $\{\pi_{ij}\}$, $\{\pi'_{ij}\}$ differ by the action of an element $a \in Sp(g, \mathbf{Z})$ where $g \geq 1$ is the genus of C and C'. Then C and C' are isomorphic Riemann surfaces.*

(b) *A Riemann surface is determined up to isomorphism by its polarized Jacobian $(J(C), [\Theta])$. Equivalently the map $\bar{t}$: $\mathcal{M}_g \to \mathcal{X}_g/Sp(g, \mathbf{Z})$ induced by the period map t (where $\mathcal{M}_g = T_g/\Gamma_g$) is injective.*

(c) *Suppose C, C' have the same polarized Jacobian $(J(C), [\Theta])$. Let $\Psi: C \to J(C)$, $\Psi': C' \to J(C)$ be the natural embeddings, which are determined up to translation in $J(C)$. Then $\Psi(C') = p + \Psi(C)$ or $\Psi(C') = p - \Psi(C)$ for some $p \in J(C)$. Equivalently every automorphism of $(J(C), [\Theta])$ fixing the identity $0 \in J(C)$ of the group $J(C)$ is of the form a or $-I \circ a$ where $-I: J(C) \to J(C)$ is the reflection $-I(x) = -x$, and a is the automorphism of $J(C)$ induced by some automorphism A of C.*

(d) *The map t: $T_g/I_g \to \mathcal{X}_g$ is a double branched cover of its image, and the branch locus $B \subset T_g/I_g$ is the set of points corresponding to hyperelliptic curves. $t: B \to t(B)$ is injective.*

(a) is the original statement of Torelli's theorem and (b) a reformulation which has the advantage of making sense over any algebraically closed field. (d) is a reformulation of (c).

(See e.g. [18].) (c) does not follow directly from (a), (b). From (a) or (b) it follows that if $q = t(p) \in t(T_g/I_g) \subset \mathscr{X}_g$ then $t^{-1}\{p\} = \{a \cdot q : a \in \mathrm{Stab}\, p \subset \mathscr{X}_g\}$. Observe that $-I \in \mathrm{Stab}\, p$ for any $b \in \mathscr{X}_g$. So the cardinality $\#\{t^{-1}\{p\}\} = \#\{\mathrm{Stab}\, p/\mathrm{Aut}\, a\}$ (where $\mathrm{Aut}\, q$ is identified with a subgroup of $Sp(g, \mathbf{Z})$).

Let I_g^* be the preimage in Γ_g of $\{\pm Id\} \subseteq Sp(g, \mathbf{Z})$; then (a), (b) show that the period map $t^*: T_g/I_g^* \to \mathscr{X}_g$ is injective on $Q = \{q \in T_g/I_g^* : \mathrm{Stab}\, t(q) = \{Id, -Id\}\}$. In fact Q is open and dense in T_g/I_g^*. First $t(T_g/I_g^*)$ lies in no proper geodesic subspace $\mathscr{L} = \mathrm{Fix}\, a \subseteq \mathscr{X}_g (a \in Sp(g, \mathbf{Z}))$, because then $\mathscr{L}$ would be invariant under the image $Sp(g, \mathbf{Z})/\{\pm Id\}$ of Γ_g and therefore also invariant under the real Zariski closure $Sp(g, \mathbf{R})$ of $Sp(g, \mathbf{Z})$. But $\mathscr{X}_g$ contains no proper invariant subspaces. Second, the union of the subspaces $\mathrm{Fix}\, a (a \neq Id \in Sp(g, \mathbf{Z}))$ is closed in $\mathscr{X}_g$. Now if $g = 3$, $t^*(T_g/I_g^*)$ is an open subset of $\mathbf{C}^6$. Since t^* is holomorphic with finite fibers, t^* is open with positive local degree at each point of T_g/I_g^*, so t^* is injective everywhere; thus we recover the sharp forms (c), (d) of Torelli's theorem.

Torelli's original proof [53] actually yields (c). Matsusaka [42] gave a more rigorous version of Torelli's proof, valid in abstract algebraic geometry. The second proof of Torelli's theorem was by Comessatti [16] and also gives the sharp version (c). Ciliberto [15] gives a modern and rigorous presentation of Comessatti's proof. A third proof is due to Andreotti, first over $\mathbf{C}$ [3] and then in general in papers of Weil [56] and Andreotti [4]. A fourth proof is due to Martens [40]. It is a short proof of the sharp version. For more recent proofs see pp. 261–269 in [5]. Torelli's theorem is often (e.g. in [45, 24, 5]) stated in the weaker form (a), (b). Andreotti's proof only yields (a), (b) in general. However when $g = 3$, Andreotti's proof (given in [24, 5]) actually identifies C as the branch locus of the Gauss map $\gamma: \Theta \to \mathbf{P}^2$ which maps $p \in \Theta$ to the tangent plane at p translated to the origin of $J(C)$, and therefore gives the strong versions (c), (d). (If C is hyperelliptic then C is an irreducible component of the branch locus.)

Recall that a *stable curve* C is a complex analytic space such that (i) C is compact and connected (ii) each point $p \in C$ has a neighborhood biholomorphic to $\{z \in C : |z| < \varepsilon\}$ or to $\{(z, w) \in \mathbf{C}^2 : zw = 0, |z|, |w| < \varepsilon\}$ and in the latter case p is called a node, (iii) $C - \{\text{nodes of } C\}$ is a union of Riemann surfaces of negative Euler characteristic. The genus $g(C)$ of C is the number such that $2 - 2g(C) = \chi(C) - d$ where $\chi(C)$ is the topological Euler characteristic of C and d is the number of nodes. Given a stable curve C construct a graph $G(C)$ with one vertex i for each component C_i of $C - \{\text{nodes}(C)\}$ and, for each node p, an edge from i to j where C_i, C_j are the two components which meet a neighborhood of p. Possibly $i = j$. We call C a *finite* stable curve if $G(C)$ is a tree. Hoyt [27] showed that the closure in $\mathscr{X}_g$ of the image $t(T_g/I_g)$ consists of the Jacobians of finite stable curves, where the Jacobian $J(C)$ of a finite stable curve is the products of the Jacobians of the components $\hat{C}_i$ of the normalization of C. $\hat{C}_i$ is obtained from C_i by filling each puncture.

D. Johnson and J. Millson [37] generalized Proposition 3 as follows.

PROPOSITION 5. *There is a free abelian subgroup $A \subset H_3(I_3, \mathbf{Z})$ of infinite rank. I_3^*/I_3 acts trivially on A, and given $x \in H_3^{inv}(I_3, \mathbf{Z}) = (y \in H_3(I_3, \mathbf{Z}) : T_* y = y\}$ where $I_3^*/I_3 = \{1, T\}$, $2x \in A$. As a $Sp(3, \mathbf{Z})$-module, $A \cong \mathbf{Z}[Sp(3, \mathbf{Z})/Sp(1, \mathbf{Z}) \times Sp(2, \mathbf{Z})]$.*

Proof. By Hoyt's theorem, the frontier in $\mathscr{X}_3$ of $t(T_3/I_3)$ consists of the period matrices of products $E_1 \times J$ where E_1 is a genus 1 curve and J is the Jacobian of a finite stable genus 2 curve (so $J = J(C)$ for a genus 2 curve C or else $J = E_2 \times E_3$ for two elliptic curves E_2, E_3). Explicitly the frontier is the union $\bigcup_{y \in Y} y \cdot (U \times \mathscr{X}_2)$ where Y is a set of coset representatives for $Sp(3, \mathbf{Z})/Sp(1, \mathbf{Z}) \times Sp(2, \mathbf{Z})$ and

$$U \times \mathscr{X}_2 \text{ is the subspace } \left\{ \begin{pmatrix} \tau & 0 & 0 \\ 0 & \pi_{22} & \pi_{23} \\ 0 & \pi_{32} & \pi_{33} \end{pmatrix} : \tau \in U, \Pi = \begin{pmatrix} \pi_{22} & \pi_{23} \\ \pi_{32} & \pi_{33} \end{pmatrix} \in \mathscr{X}_2 \right\}$$

of $\mathscr{X}_3$. By Theorem 2, p. 419 of [27], the image in $\mathscr{X}_3/Sp(3, \mathbf{Z})$ of $t(T_3/I_3) \cup Y \cdot (U \times \mathscr{X}_2)$ is Zariski closed in the quasiprojective variety $\mathscr{X}_3/Sp(3, \mathbf{Z})$. So $t(T_3/I_3) = \mathscr{X}_3 - Y \cdot (U \times \mathscr{X}_2)$. (Quasiprojectivity was proven by Bailey [6].) Since t: $T_3/I_3 \to \mathscr{X}_3$ is open, $Y \cdot (U \times \mathscr{X}_2)$ is closed. If $y_1 \cdot (U \times \mathscr{X}_2)$ meets $y_2 \cdot (U \times \mathscr{X}_2)$, then the intersection is of the form $g \cdot (U \times U \times U)$ for some $g \in Sp(3, \mathbf{Z})$ and a third subspace $y_3 \cdot (U \times \mathscr{X}_2)$ satisfies $y_3 \cdot (U \times \mathscr{X}_2) \cap y_1 \cdot (U \times Z_2) = y_1 \cdot (U \times \mathscr{X}_2) \cap y_2 \cdot (U \times \mathscr{X}_2) = y_2 \cdot (U \times \mathscr{X}_2) \cap y_3 \cdot (U \times \mathscr{X}_2)$. The map i_*: $H_c^*(U \times U \times U) \to H_c^*(U \times \mathscr{X}_2)$ defined by extension by 0 from a tubular neighborhood of $U \times U \times U$ in $U \times \mathscr{X}_2$ is 0, so using Mayer–Vietoris we obtain

$$H^k_{cpt}(Y \cdot (U \times \mathscr{X}_2), \mathbf{Z}) = 0 \quad \text{if} \quad k \neq 8, \; H^8_{cpt}(Y \cdot (U \times \mathscr{X}_2), \mathbf{Z}) = \mathbf{Z}Y,$$

the free abelian group on Y. By Alexander–Lefschetz duality, $H_3(\mathscr{X}_3 - Y \cdot (U \times \mathscr{X}_2), \mathbf{Z}) \cong H^8_{cpt}(Y \cdot (U \times \mathscr{X}_2), \mathbf{Z}) = \mathbf{Z}Y$, and $H_i(\mathscr{X}_3 - Y \cdot (U \times \mathscr{X}_2), \mathbf{Z}) = 0$ for $i \neq 0, 3$. Let p: $H_i(\mathscr{X}_3 - Y \cdot (U \times \mathscr{X}_2), \mathbf{Z}) \to H_i(T_3/I_3, \mathbf{Z})$ satisfy $p_* p^* x = 2x$, $p^* p_* x = x + T_* x$ where T: $T_3/I_3 \to T_3/I_3$ is the covering involution.

Given $x \in H_3(I_3, \mathbf{Z})$, $2x = (x - T_* x) + (x + T_* x) = (x - T_* x) + p^* p_* x$ so if $x = T_* x$, $2x \in A$ where A is the image of p^*.

$H_3(t(T_3/I_3), \mathbf{Z})$ is generated by spherical classes. $H_3(I_3, \mathbf{Z})$ contains no nonzero spherical classes; we will find representatives for the elements of A.

Suppose $P \in \mathscr{X}_3 - t(T_3/I_3)$ and suppose the corresponding abelian variety is not a product (as a polarized variety) of 3 elliptic curves. Then P is the Siegel point of the Jacobian of a stable curve $E_1 \cup C$ where C has genus 2 and $E_1 \cap C = u \in C$. Moreover for each $v \in C$ there is stable curve (obtained by joining E_1 to C at v) with the same Jacobian. The nodes of these stable curves can be replaced by annuli, and we obtain a map of the unit tangent bundle of C into the Torelli space T_3/I_3. First we give a hyperbolic description. Consider the family of once punctured genus 2 curves π: $C \times C - \Delta \to C$, $\pi(x, y) = y$, where $\Delta = \{(v, v): v \in C\}$ is the diagonal. There is a smooth metric on $C \times C - \Delta$ which restricts to a conformal hyperbolic metric on each fiber $p^{-1}\{v\} = C_{\{v\}}$. On each $C - \{v\}$ there is a horocircle H_v of length ε about the cusp v. From the family $C \times C - \Delta$ remove the outside of each horocircle H_v, obtaining a family $\bigcup_{v \in V} C'_v$ of hyperbolic surfaces of genus 2 with boundary. Now take a fixed oriented hyperbolic surface E' of genus 1 with a single boundary geodesic of length ε. Fix a point $e \in \partial E'$. The subset $\bigcup_{v \in C} H_v$ can be identified with the unit tangent bundle $UT(C)$ of C. For each point $w \in \bigcup_{v \in C} H_v$, form a Riemann surface by isometrically identifying $\partial E'$ with H_v, identifying e with w. This gives a family $C'' \to UT(c)$ of Riemann surfaces such that the monodromy acts trivially on the homology of the fiber.

Alternatively we could choose H_v so that in the hyperbolic metric in which H_v was geodesic, H_v would have length ε; then the gluing would give a nonsingular hyperbolic metric.

To discuss the period map we need a holomorphic description following [41] and [21]. The family will be homotopy equivalent but not identical to the previously described family. Given $p \in C$ let U be a neighborhood of p and for each $q \in U$ let z_q: U let z_q: $U \to D_2 = \{z \in \mathbf{C}: |\mathbf{z}| < 2\}$ be a chart such that $z_q(r) = z_p(r) - z_p(q)$ and for each q $z_q(U)$ contains $D = \{z \in \mathbf{C}: |\mathbf{z}| < 1\}$. Choose Beltrami differentials v_1, v_2, v_3 in C and v_4 in E_1 with supports disjoint from U, V such that v_1, v_2, v_3, v_4 generate the tangent space to the

Teichmüller space $T_2(C) \times T_1$ of $C \coprod E_1$. On the elliptic curve E_1 fix a chart $w\colon V \to D$. Now for $(t, q) \in D \times U$ form a Riemann surface $X_{t,q}$ by removing the discs $\{0 \le |z_q| \le |t|\}$ and $\{0 \le |w| \le |t|\}$ from U and E and identify $s_1 \in U$ with $s_2 \in V$ provided $z_q(s_1)w_q(s_2) = t$. (If $t = 0$, $X_{0,q}$ is a stable curve rather than a Riemann surface.) Now given $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$ in a sufficiently small neighborhood of $0 \in \mathbf{C}^3$, let $\mu(\tau) = \sum_{i=1}^{4} \tau_i v_i$ and let $X_{q,t,\tau}$ be the Riemann surface such that there is a quasiconformal homeomorphism $w^{\mu(\tau)}\colon X_{q,t} \to X_{q,t,\tau}$ with Beltrami coefficient $w_{\bar{z}}^{\mu(\tau)}/w_z^{\mu(\tau)} = \mu(\tau)$. Then by [41] there is a complex manifold X of dimension 7 and a proper holomorphic submersion $\pi\colon X \to U \times D \times W$ such that the fiber over q, t, τ is $X_{q,t,\tau}$. As q varies over the surface C, the families patch together to form a family $\pi\colon X \to N \times W$ where N is a neighborhood of the diagonal in $C \times C$. To see this, consider the complex surface $zw = t$ in C^3; let $y = (z - w)/2$, $x = (z + w)/2$ so the surface is given by $y^2 = x^2 - t$. So the annulus given by $z_q w = t \subset U \cap V$ (where $U - \{0 \le |z_q| \le t\}$ has been identified with $V - \{0 \le |w| \le t\}$) is the double branched cover of the disc U, with branching over $z_q^{-1}(\pm t^{1/2})$.

Now [21] shows that on $\pi^{-1}(N \times \{w\})$ there are 3 holomorphic 2-forms U_1, U_2, U_3 such that the residue $v_{i,q,t,w}$ of U_i on $X_{q,t,w}$ is a holomorphic 1-form with periods δ_{ij} on A_j where $(A_1, A_2, A_3, B_1, B_2, B_3)$ is the canonical homology basis such that P has period matrix

$$\begin{pmatrix} \tau & 0 & 0 \\ 0 & \pi_{22} & \pi_{23} \\ 0 & \pi_{32} & \pi_{33} \end{pmatrix}$$

with respect to $(A_1, A_2, A_3, B_1, B_2, B_3)$. Let $\pi_{ij}(q, t, w) = \int_{B_j} v_{i,q,t,w}$. Furthermore [57], p. 129 shows that (if $i = 1, 2 \le j \le 3$)

$$\pi_{ij}(q, t, 0) = \pi_{ij}(q, 0, 0) - tv_i(0)v_j(q) + O(t^2)$$

where v_1, v_2, v_3 is a basis for the holomorphic differentials on $E_1 \amalg C$ such that $\int_{A_i} v_j = \delta_{ij}$, and $v_i(0)$, $v_i(q)$ are the values of the holomorphic functions such that $v_1 = v_1(z)dz$, $v_j = v_j(w)dw$ in the regions U, V respectively; while for all other (i, j) with $i \le j$, $\pi_{ij}(q, t, 0) = \pi_{ij}(q, 0, 0) + O(t^2)$. (This formula was earlier derived by Fay [21] but with an incorrect multiplicative constant. Yamada [57] also obtains all the terms of higher order in t. Cf. [52].) Together with $\pi_{ij}(q, 0, \tau) = \pi_{ij}(q, 0, 0) + \int_{C \amalg E} \tau v_i v_j + O(|t|^2)$ this shows that the period map $P\colon N \times W \to \mathscr{X}_3$ defined by $t(q, t, \tau) = [\pi_{ij}(q, t, \tau)]$ is holomorphic. (Alternatively, Masur shows the existence of holomorphic 2-forms A_1, A_2, A_3 on X.) Write $T_P\mathscr{X}_3 = \mathbf{C}^4 \oplus \mathbf{C}^2$, where the summands correspond to the variables $(\tau_{11}, \tau_{22}, \tau_{23}, \tau_{33})$ and (τ_{12}, τ_{13}). Then $dP\colon T_{(q,0,0)}N \to T_P\mathscr{X}_3$ is onto the first factor. So X contains a holomorphic subfamily $\pi\colon X \to N_0$ such that N_0 is biholomorphic to a neighborhood of the diagonal ΔC in $C \times C$, and (thereby identifying C with a Riemann surface in N_0) $\pi_1(C) = P$. Furthermore, the map $(t, q) \to (t/4)(v_1(q), v_2(q))$ has rank 2 unless q is a Weierstrass point of C. So $P'\colon N_0 \to \mathbf{C}^2$ defined by $t''(q, t, \tau) = (\pi_{12}(X_{q,t,\tau}), \pi_{13}(X_{q,t,\tau}))$ is holomorphic and open. Blow up the origin of $\mathbf{C}^2$. We may assume that $v_i(0) = 1$, and then the Yamada–Fay formula 1 shows that P' lifts to $P''\colon N_0 \to \hat{\mathbf{C}}^2$ where $\hat{\mathbf{C}}^2$ is $\mathbf{C}^2$ with blown up origin. Furthermore, $P''\colon N_0 \to \mathbf{P}^1 \subset \hat{\mathbf{C}}^2$ is the canonical map, so it is a double branched cover branched over 6 points. For any q the derivative of P restricted to a fiber $\{q\} \times D \times \{w\}$ is nonzero. So $t'\colon F \to \mathbf{C}^2$ has nonzero derivative where F is any curve in N_0 transverse to C. So $P\colon N_0 \to \mathbf{C}^2$ has degree 2 and is branched over the union of 6 smooth curves. So if $S^3(\varepsilon) \subset \hat{\mathbf{C}}^2$ is a sufficiently small sphere, $P\colon t^{-1}S^3(\varepsilon) \to S^3(\varepsilon)$ expresses the unit tangent bundle of C as a double branched cover of $S^3(\varepsilon)$; the branch locus $B_\varepsilon \subset S^3(\varepsilon)$ is 6 distinct curves of the Hopf fibration up to isotopy; and, because t is differentiable, as $\varepsilon \to 0$, $\varepsilon^{-1}B_\varepsilon$ converges to 6 Hopf circles $H_1, \ldots, H_6$ which are preimages by the Hopf map of the 6 branch points of the canonical image of the genus 2 curve C. Note that the derivative defines a lift of the hyperelliptic involution to a free involution on the unit tangent bundle; the composite of this free involution and a rotation of π on each fiber is an involution which fixes 6 circles and has the 3-sphere as quotient space.

To clarify the geometric reason for the branching, we observe that the elliptic curve E_1 has an involution i_1 inducing $-Id$ on $H_1(E, \mathbf{Z})$ and fixing $0 \in E$. Take the neighborhood V of $0 \in E_1$ to be i_1-invariant and choose the chart $w: V \to D$ so $w(i_1 s) = -w(s)$. Suppose $q \in C$ is a Weierstrass point, and let i_2 be the hyperelliptic involution on C. Then i_1 and i_2 together define a hyperelliptic involution i on the curve $X_{q,t}$. So the corresponding point in Torelli space is fixed by I_g^*/I_g. Finally we show that $(N - \Delta C) \times W$ is canonically isomorphic to an open subset of T_3/I_3. For any $p \in N \times W$, the inclusion $H_1(\pi^{-1}p, \mathbf{Z}) \to H_1(X, \mathbf{Z})$ defines a canonical isomorphism with $H_1(E \cup C, \mathbf{Z})$. So there is a basis $(A_1, \ldots, B_3)$ on each $H_1(\pi^{-1}p, \mathbf{Z})$ and therefore a natural map, the Kodaira–Spencer map $KS: (N - \Delta C) \times W \to T_3/I_3$, $p \mapsto (\pi^{-1}p, (A_1, \ldots, B_3))$. KS is holomorphic (see pp. 373–388 in [47].) Since $P: N \times W \mapsto \mathscr{X}_3$ has degree 2 in a neighborhood of ΔC and $P = t \circ KS$ where $t: T_3/I_3 \to \mathscr{X}_3$ is the Torelli map, and t has degree 2, KS is a local biholomorphism. We summarize the discussion:

Proposition 6. *For each sphere $S^3(\varepsilon)$ representing a generator of $H_3(t(T_3/I_3), \mathbf{Z})$, $p^*([S^3(\varepsilon)])$ is represented by an embedding of the unit tangent bundle $UT(C)$ of a genus 2 surface in T_3/I_3. If the radius ε of the sphere $S^3(\varepsilon)$ (lying in a complex plane perpendicular to a component of $Y \cdot (U \times \mathscr{X}_2)$), then $S^3(\varepsilon)$ meets the set of period matrices of hyperelliptic curves in B_ε, a link of 6 circles, and $UT(C)$ is the branched cover of S^3 over B_ε. As $\varepsilon \to 0$, $\varepsilon^{-1} B_\varepsilon \subseteq S^3$ approaches a union of 6 Hopf circles, say $\phi^{-1}\{z_1, \ldots, z_6\}$ (where $\phi: S^3 \to \mathbf{P}^1$ is the Hopf map and the center of the sphere $S^3(\varepsilon)$ represents the stable Jacobian $E \times J(C)$ where C is the genus 2 Riemann surface branched over $\{z_1, \ldots, z_6\}$.*

6. A HOMOLOGICAL APPLICATION

We will now give a homological application of Proposition 5 and Corollary 1 of Proposition 3, and a result on the cohomological dimension of quotient groups. Consider the Lyndon–Hochschild–Serre spectral sequence for the group extension $I_2 \hookrightarrow \Gamma_2 \to Sp(2, \mathbf{Z})$. Since I_2 is free there are only two nonzero lines in the E_2 term and d_2 is the only differential. Furthermore $H_k(Sp(2, \mathbf{Z}), H_1(I_2, \mathbf{Z})) \cong H_k(Sp(2, \mathbf{Z}), \mathbf{Z}[Sp(2, \mathbf{Z})/(\langle e \rangle \rtimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z}))]) \cong H_k(\langle e \rangle \rtimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z}), \mathbf{Z})$, where the second isomorphism is by Shapiro's lemma. So the spectral sequence degenerates to an exact sequence $\cdots \to H_k(\Gamma_2, \mathbf{Z}) \to H_k(Sp(2, \mathbf{Z}), \mathbf{Z}) \to H_{k-1}(\langle e \rangle \rtimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z}), \mathbf{Z}) \to H_{k-1}(\Gamma_2, \mathbf{Z}) \to \cdots H_2(Sp(2, \mathbf{Z}), \mathbf{Z}) \to \mathbf{Z} \to H_1(Sp(2, \mathbf{Z}), \mathbf{Z})$. In particular $H_k(\Gamma_2, \mathbf{Q}) = 0$ except for $k = 0$(by Igusa's theorem) yielding $H_k(Sp(2, \mathbf{Z}), \mathbf{Q}) = 0$ for $k > 0$. This exact sequence could be used to calculate $H_*(Sp(2, \mathbf{Z}), \mathbf{Z})$ using the complete calculation of $H_*(\Gamma_2, \mathbf{Z})$ by F. R. Cohen [17]. Now consider the extension $I_3^* \to \Gamma_3 \to PSp(3, \mathbf{Z})$ where $PSp(3, \mathbf{Z}) = Sp(3, \mathbf{Z})/\langle \pm I \rangle$. By Proposition 5, $H_3(I_3^*, \mathbf{Z}[1/2]) \cong \mathbf{Z}[Sp(3, \mathbf{Z})/Sp(1, \mathbf{Z}) \times Sp(2, \mathbf{Z})]$, and $H_k(I_3^*, \mathbf{Z}[1/2]) = 0$ for $k \neq 0, 3$. $\mathbf{Z}[1/2]$ may be replaced by any ring in which 2 is invertible. Again there are only 2 lines in the E_2 term of the spectral sequence, and d_4 is the only nontrivial differential. In particular, with rational coefficients we have:

Proposition 7. *The spectral sequence in rational homology for the extension $I_3^* \to \Gamma_3 \to PSp(3, \mathbf{Z})$ degenerates to give* (a) $H_k(PSp(3, \mathbf{Z}), \mathbf{Q}) \cong 0$ *for* $k > 7$, (b) $H_7(\Gamma_3, \mathbf{Q}) \cong$

$H_7(PSp(3, \mathbf{Z}), \mathbf{Q})$ (c) *There are exact sequences*

$$\text{(i)} \qquad 0 \to H_6(\Gamma_3, \mathbf{Q}) \to H_6(PSp(3, \mathbf{Z}), \mathbf{Q}) \xrightarrow{d_4} \mathbf{Q} \to H_5(PSp(3, \mathbf{Z}), \mathbf{Q}) \to 0.$$

and

$$\text{(ii)} \quad 0 \to H_4(\Gamma_3, \mathbf{Q}) \to H_4(PSp(3, \mathbf{Z}), \mathbf{Q}) \xrightarrow{d_4} \mathbf{Q} \to H_3(\Gamma_3, \mathbf{Q}) \to H_3(PSp(3, \mathbf{Z}), \mathbf{Q}) \to 0.$$

Proof. For (a) use $H_k(Sp(2, \mathbf{Z}), \mathbf{Q}) = 0$ except for $k = 0, 2$. In (c) (i), d_4 takes values in $H_2(PSp(3, \mathbf{Z}), H_3(I_3^*, \mathbf{Q})) \cong H_2((Sp(1, \mathbf{Z}) \times Sp(2, \mathbf{Z}))/\langle \pm I \rangle, \mathbf{Q})$ in (i) and, in (c) (ii), in $H_0(PSp(3, \mathbf{Z}), H_3(I_3^*, \mathbf{Q})) \cong H_0(Sp(1, \mathbf{Z}) \times Sp(2, \mathbf{Z})/\langle \pm I \rangle, \mathbf{Q})$.

Similar results hold in cohomology. Suppose given a group extension $F \to G \to Q$ where the kernel F is free on Q/A for some subgroup A of Q. One example is that Q is a torsion-free 1-relator group, and $A = 1$; another is $G = \Gamma_2, F = I_2, A = \mathbf{Z}/2 \ltimes (SL_2\mathbf{Z} \times SL_2\mathbf{Z})$. We may ask for a bound for the cohomological dimension of Q.

Proposition 8. *With F, G, Q, A as above $cd_R Q \leq \max(cd_R G, cd_R A + 2)$ where cd_R is cohomological dimension over R.*

Proof. Let M be a Q-module. Then M can be considered as a G-module. Consider the E_2 term of the spectral sequence of the group extension: $E^{0k} = H^k(Q, M)$,

$$E^{1k} = H^k(Q, H^1(F, M)) \cong H^k(Q, \mathrm{Hom}_R(RQ/A, M))$$
$$= H^k(Q, \mathrm{Coind}_A^Q M) \cong H^k(A, M).$$

So if $H^k(Q, M) \neq 0$ and $k > cd_R A + 2$, $H^k(G, M) \cong H^k(Q, M) \neq 0$. To apply Proposition 7 to $Sp(2, \mathbf{Z})$, use $R = \mathbf{Q}$ or else pass to a finite index subgroup and generalize to the case that F is free on the union of finitely many sets Q/A_i, to obtain $cd_{\mathbf{Q}} Sp(2, \mathbf{Z}) \leq 5$, $vcd\, Sp(2, \mathbf{Z}) \leq 5$. (In fact $vcd\, Sp(2, \mathbf{Z}) = 4$.) It would be interesting to have more examples to which Proposition 8 is applied.

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Department of Mathematics
University of California at Los Angeles
Los Angeles, California 90024,
U.S.A.