LINEAR REPRESENTATIONS OF THE MAPPING CLASS GROUP OF DIMENSION AT MOST $3g-3$

JULIAN KAUFMANN, NICK SALTER, ZHONG ZHANG, XIYAN ZHONG

ABSTRACT. We classify representations of the mapping class group of a surface of genus g (with at most one puncture or boundary component) up to dimension $3g-3$. Any such representation is the direct sum of a representation in dimension $2g$ or $2g+1$ (given as the action on the (co)homology of the surface or its unit tangent bundle) with a trivial representation. As a corollary, any linear system on the moduli space of Riemann surfaces of genus g in this range is of algebro-geometric origin.

1. INTRODUCTION

Let S be an oriented surface of genus g, either closed, with one puncture, or with one boundary component (notated as $\Sigma_g, \Sigma_{g,*}, \Sigma_{g,1}$, respectively). Assume $g \geqslant 3$ throughout. Let $\mathrm{Mod}(S)$ denote the mapping class group of S. This paper gives a classification of complex representations of $\mathrm{Mod}(S)$ in the dimension range $n \leqslant 3g-3$. In dimension $2g$, there is the the *symplectic representation*

$$\Psi : \mathrm{Mod}(S) \to \mathrm{GL}(H),$$

where $H = H_1(S;\mathbb{C})$ denotes the first homology of S, equipped with the intersection pairing $\langle \cdot, \cdot \rangle$. When S is not closed, there is a representation of dimension $2g+1$

$$\widetilde{\Psi} : \mathrm{Mod}(S) \to \mathrm{GL}(\widetilde{H}),$$

where $\widetilde{H} \cong \mathbb{C}^{2g+1}$ is a non-semisimple representation surjecting onto H given by the action on the homology of the unit tangent bundle of S; see Section 2.1. There is also the non-isomorphic dual representation $\widetilde{H}^*$.

Our main result shows that this gives a *complete* list of representations up to dimension $3g-3$.

Theorem A. *Let $g \geqslant 3$, and let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ be a nontrivial representation. Then for $n \leqslant 3g-3$, ρ is the direct sum of a trivial representation with one of the following:*

$$H, \widetilde{H}, \text{ or the dual } \widetilde{H}^*.$$

If S is closed, only H can appear.

This extends the work of Korkmaz [Kor23], who showed the uniqueness of the symplectic representation up to dimension $2g$, and Kasahara [Kas24], who classified representations of dimension $2g+1$.

Idea of proof. Our analysis is centered around the notion of a *bi-affine representation*. A group representation V is bi-affine if there is a filtration $V_1 \leqslant V_2 \leqslant V$ for which V_1 and V/V_2 are both trivial. Building off of the ideas developed by Korkmaz and Kasahara, we show inductively that every representation of $\mathrm{Mod}(S)$ up to dimension $3g-3$ is bi-affine. To do this, we develop a criterion for a $\mathrm{Mod}(S)$-representation to be bi-affine in Proposition 4.4. The other key results, Propositions 5.1, 5.6 and 6.7, establish conditions under which the hypotheses of Proposition 4.4 hold. Separately, we

Date: May 28, 2026.

show that any bi-affine representation of $\mathrm{Mod}(S)$ is in fact a direct sum of a trivial representation with a representation of dimension $2g+1$. This is a consequence of a homological calculation carried out in Proposition 3.8.

It is reasonable to ask why we use the language of bi-affine representations at all, if ultimately we are showing a stronger result. The answer is that we believe that this leads to the cleanest proofs of our results, with the least amount of fussing about cases and choosing coordinates. It is relatively painless to formulate a condition under which an extension of a bi-affine representation of $\mathrm{Mod}(S)$ remains bi-affine (cf. Lemma 3.3); the corresponding statement for an extension to be a direct sum with a trivial representation would be more elaborate, and would require us to carry around the data of the splitting for longer than necessary.

Interpretation in terms of local systems on moduli spaces. Local systems on the moduli space $\mathcal{M}$ of Riemann surfaces are determined by monodromy representations

$$\pi_1^{\mathrm{orb}}(\mathcal{M}) \to \mathrm{GL}(V),$$

where $\pi_1^{\mathrm{orb}}(\mathcal{M})$ is the mapping class group. A local system $\mathbb{V}$ on $\mathcal{M}$ is *of geometric origin* if there is a family $E \to \mathcal{M}$ of smooth projective varieties over $\mathcal{M}$ such that $\mathbb{V}$ is a subquotient of the local system of (co)homology associated with E. It is conjectured that all semi-simple representations of mapping class groups are of geometric origin [Lit24].

Local systems of geometric origin are necessarily semi-simple, while the local systems associated with $\widetilde{H}$ and $\widetilde{H}^*$ are not. For the purposes of this discussion, we will say that a local system *arises algebro-geometrically* if it is the monodromy of a family as above, where the fibers of the morphism are now only required to be quasiprojective, but are still required to be a topological fibration. Theorem A imposes strong constraints on the local systems that can appear on moduli spaces of closed surfaces or of surfaces with a puncture – up to rank $3g-3$, they must arise algebro-geometrically.

Corollary B. *For $g \geq 3$, any local system of rank at most $3g-3$ on $\mathcal{M}_{g,1}$ or $\mathcal{M}_g$ arises algebro-geometrically. Here are the families of algebraic varieties:*

- *for the symplectic representation H, it is the universal family of Riemann surfaces;*
- *for the representations $\widetilde{H}$ and $\widetilde{H}^*$, it is the relative tangent bundle of the universal family of curves with the zero section removed - the fiber of this bundle is homotopy equivalent to the unit tangent bundle of surfaces.*

Applications to rigidity. In [Far24], Farb proves that any nonconstant holomorphic map $f : \mathcal{M}_{g,n} \to \mathcal{A}_g$ must be the period mapping assigning a Riemann surface to its Jacobian (here, $\mathcal{A}_g$ denotes the moduli space of principally polarized Abelian varieties of dimension g). The first step in the argument is to appeal to Korkmaz's work classifying representations of the mapping class group up to dimension $2g$, as this governs the possibilities for the induced map on orbifold fundamental groups. The work of this paper opens the way to extending Farb's work to give a classification of holomorphic maps $\mathcal{M}_g \to \mathcal{A}_h$ in the range $2h \leqslant 3g-3$. We plan to revisit this topic in future work.

Beyond Theorem A. Here we offer some discussion and speculation about how the results of the paper may be extended. As a first comment, since this paper was first released, work of Brevidelli [Bre26] shows that any representation of $\mathrm{Mod}(S)$ of dimension at most $4g-4$ is non-faithful; Brevidelli moreover identifies an explicit subgroup of the Torelli group that must be in the kernel of any such representation.

There are at least three axes along which the results of the paper could potentially be extended:

(1) Improving the dimension range in which all representations $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ are classified. In particular, identifying the next dimension n in which an irreducible representation appears.
(2) Improving the result to handle general finite-type surfaces S with an arbitrary number of boundary components and/or punctures.
(3) Improving the result to incorporate representations of (some class of) finite-index subgroups of $\mathrm{Mod}(S)$.

As regards (1), the first question is the extent to which the range $n \leqslant 3g-3$ in Theorem A is sharp. We do not know of any new representations appearing in dimension $n = 3g-2$. The bottleneck imposing the bound $n \leqslant 3g-3$ of Theorem A is derived from Proposition 5.6, which shows that in this range, any representation has Dehn twists acting as *transvections*. Propositions 4.4 and 5.1 hold in the larger range $n \leqslant 4g-1$, resp. $n \leqslant 4g-3$ while the remaining cornerstone result, Proposition 6.7, holds in any dimension.

There is still a large gulf between a conjectural improved range $n \leqslant 4g-3$ in Theorem A and the dimensions of other known irreps of $\mathrm{Mod}(S)$. After the standard representation Ψ acting on $H_1(S;\mathbb{C})$, the next-smallest irreducible representation of $\mathrm{Sp}(2g;\mathbb{Z})$ is $\wedge^2 H_1(S;\mathbb{C})/\mathbb{C}$ of dimension $n = \binom{2g}{2} - 1$, whose dimension grows *quadratically* in g. To the authors' knowledge, this (or more precisely, its pullback to $\mathrm{Mod}(S)$) is the next known irrep.

Question 1.1. Does $\mathrm{Mod}(S)$ admit any irreducible representation of dimension $3g-2 \leqslant n < \binom{2g}{2}-1$?

The picture of irreducible representations of $\mathrm{Mod}(S)$ is even murkier when one restricts attention to those representations not factoring through the symplectic group. Constructions of such arise via the method of so-called *Prym representations*. The general idea here is to consider a finite covering $p : \widetilde{S} \to S$, to which there is associated a finite-index subgroup $\mathrm{Mod}(S)[p] \leqslant \mathrm{Mod}(S)$ consisting of mapping classes that lift to $\widetilde{S}$. There is an action of $\mathrm{Mod}(S)[p]$ on $H_1(\widetilde{S};\mathbb{Z})$ which can be inducted up to a representation of $\mathrm{Mod}(S)$ and subsequently decomposed into irreps. However, to the authors' knowledge, these constructions always have dimension *exponential* in the genus of S.

Question 1.2. Does $\mathrm{Mod}(S)$ admit any irreducible representation that does not factor through $\mathrm{Sp}(2g,\mathbb{Z})$ whose dimension is polynomial in g?

Regarding (2), there are some genuine novelties that arise when S has more boundary components/punctures, say p in total. Specifically, there is an action on $H_1(S;\mathbb{Z})$, which is free of rank $2g+p-1$. This is non-semisimple, and indeed co-affine (cf. Definition 3.1), as the action on the subspace spanned by peripheral classes is trivial. It is reasonable to conjecture that in a certain range, every representation of $\mathrm{Mod}(S)$ continues to arise from an action on the (co)homology of S or its unit tangent bundle; this seems like a reasonable target for future work.

Finally, with regards to (3), extending the classification to finite-index subgroups is in general a massive undertaking. For instance, the *Ivanov conjecture* posits that there is no finite-index subgroup of $\mathrm{Mod}(S)$ with nontrivial rational abelianization, i.e. that no finite-index subgroup admits a character with infinite image. While every finite-index subgroup contains some power of every Dehn twist, these will in general fragment into many distinct conjugacy classes, making their analysis vastly more complicated. Moreover special relations holding between individual such twists evaporate upon passing to powers, and so the specific techniques (e.g. in Section 6) used to obtain our results do not scale up.

It may be more reasonable to ask for rigidity results for specific, well-understood subgroups. For instance, returning to the setting of Prym representations, let $x \in H^1(S;\mathbb{Z}/2\mathbb{Z})$ be a nonzero vector, and define $\mathrm{Mod}(S)[x]$ as the stabilizer of x under the action of $\mathrm{Mod}(S)$ on $H^1(S;\mathbb{Z}/2\mathbb{Z})$.

Such x defines a $\mathbb{Z}/2\mathbb{Z}$-cover $\widetilde{S} \to S$ (with $\widetilde{S}$ consequently of genus $2g-1$), and there is a Prym representation $\rho : \mathrm{Mod}(S)[x] \to \mathrm{Aut}(H_1(\widetilde{S};\mathbb{Z})) = \mathrm{Sp}(4g-2;\mathbb{Z})$ obtained by acting on the homology of this cover (see [Loo97] for a detailed analysis of the image of such maps). This representation splits into two summands via the eigenspace decomposition of $H_1(\widetilde{S};\mathbb{Z})$ under the deck group action; the nontrivial eigenspace has rank $2g-2$, so there is an irrep of $\mathrm{Mod}(S)[x]$ of this dimension, in addition to the restriction of the symplectic rep. It was shown by Serván [Ser25] that this is the nontrivial representation of $\mathrm{Mod}(S)[x]$ of lowest dimension. How far can results of this nature be extended?

Acknowledgments. We would like to thank Benson Farb, Nate Harman, Yasushi Kasahara, Aaron Landesman, Daniel Litt, and Andy Putman for their input. We are also grateful to an anonymous referee for their close reading and many helpful suggestions that led to a substantial improvement in the exposition. The second-named author is supported by NSF grant DMS-2338485.

2. A recollection of prior results

2.1. **The work of Korkmaz and Kasahara.** The present work is deeply indebted to the papers [Kor23, Kas24] of Korkmaz and Kasahara. We recall their main results here.

Theorem 2.1 (Korkmaz, Theorem 1 of [Kor23])**.** *For $g \geqslant 3$ and $n \leqslant 2g-1$, any homomorphism $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ is trivial.*

Theorem 2.2 (Korkmaz, Theorem 2 of [Kor23])**.** *For $g \geqslant 3$, any homomorphism $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(2g,\mathbb{C})$ is either trivial or else is conjugate to the symplectic representation $\Psi : \mathrm{Mod}(S) \to \mathrm{GL}(H)$.*

We now turn to representations of dimension $2g+1$. Let UTS denote the unit tangent bundle of S. When S is non-closed, $UTS \cong S \times S^1$ splits as a product, and in particular, $H_1(UTS;\mathbb{C}) \cong \mathbb{C}^{2g+1}$. Let $\widetilde{H} = H_1(UTS;\mathbb{C})$. Any diffeomorphism of S induces a diffeomorphism of UTS; the *unit tangent representation*

$$\widetilde{\Psi} : \mathrm{Mod}(S) \to \mathrm{GL}(\widetilde{H})$$

is the induced action on homology. Note that $H \cong H_1(S;\mathbb{C})$ arises as a quotient of $\widetilde{H}$ via pushforward along the projection $\pi : UTS \to S$; the induced action on H is evidently Ψ. This representation was studied by Trapp [Tra92]. Kasahara classified representations of $\mathrm{Mod}(S)$ for S a surface of genus g with an arbitrary number of punctures and/or boundary components. In the setting we consider here (where S has at most one boundary component or puncture), his result specializes as follows.

Theorem 2.3 (Kasahara, cf. Theorem 1.1 of [Kas24])**.** *Let S be a surface of genus $g \geqslant 7$ which is either closed, has one boundary component, or one puncture. If S is closed, then every nontrivial representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(2g+1,\mathbb{C})$ is conjugate to $H \oplus \mathbb{C}$. If S is nonclosed, then any nontrivial ρ is conjugate to $H \oplus \mathbb{C}$, or to $\widetilde{H}$, or to $\widetilde{H}^*$.*

Remark 2.4. We explain here how to deduce this statement from [Kas24, Theorem 1.1]. The statement there asserts that isomorphism classes of nontrivial representations of dimension $2g+1$ up to dualizing are in bijection with $H^1(\mathrm{Mod}(S);H)/\mathbb{C}^\times$. When S is closed, Morita computed $H^1(\mathrm{Mod}(S);H) = 0$, and when S has one boundary component or puncture, $H^1(\mathrm{Mod}(S);H) \cong \mathbb{C}$ (see [Mor89], recalled here as Proposition 3.6). Thus up to dualizing, there is a unique representation not of the form $H \oplus \mathbb{C}$; it is not hard to show that $\widetilde{H}$ is such a representation. For further discussion of $\widetilde{H}$, see Section 3.2.

Remark 2.5. Our proof depends on Theorems 2.1 and 2.2 of Korkmaz, as well as various technical lemmas established therein. It is logically independent of Kasahara's main theorem (Theorem 2.3),

and indeed improves the range of his result from $g \geqslant 7$ down to $g \geqslant 4$, but we again make use of some of the internal technology.

2.2. **The Torelli group and the work of Johnson.** Recall that the *Torelli group* $\mathcal{I}(S) \leqslant \mathrm{Mod}(S)$ is the kernel of the symplectic representation Ψ. The structure of $\mathcal{I}(S)$ was greatly clarified in a series of papers of Dennis Johnson in the 1980's. We recall the relevant portions of his theory here.

We first recall the *Johnson homomorphism*, as developed in [Joh80]. For the sake of expediency we will not describe this in full detail. For our purposes it is sufficient to know that the Johnson homomorphism is a map

$$\tau : \mathcal{I}(S) \to A_S,$$

where A_S is a certain finitely generated torsion-free abelian group. Define the *Johnson kernel*

$$\mathcal{K}(S) = \ker(\tau)$$

as the kernel of τ.

A deep theorem of Johnson shows that $\mathcal{K}(S)$ is generated by just two types of elements. Let $c \subset S$ be a separating curve (i.e. one for which $S \setminus c$ is disconnected). In the case where S has a puncture or boundary component, the *genus* of S is the genus of the subsurface *not* containing this; if S is closed we define the genus as the smaller of the genera of the subsurfaces bounded by c. A *separating twist* is the Dehn twist T_c about a separating curve c; we define the genus of such T_c as the genus of c.

Theorem 2.6 (Johnson, [Joh85])**.** *For $g \geqslant 3$, $\mathcal{K}(S)$ is generated by the set of separating twists of genus one and two.*

The Torelli group is itself generated by elements admitting a simple description. A *bounding pair* is a set of curves $a, b \subset S$ that are disjoint and such that $S \setminus \{a, b\}$ is disconnected. A *bounding pair map* is the product $T_a T_b^{-1}$ of Dehn twists; it is straightforward to show that these are elements of the Torelli group.

Theorem 2.7 (Johnson, [Joh79])**.** *For $g \geqslant 3$, $\mathcal{I}(S)$ is generated by bounding pair maps.*

It is worth remarking that both $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are in fact generated by a *finite* collection of such elements. This result for $\mathcal{I}(S)$ is due to Johnson [Joh83], while the result for $\mathcal{K}(S)$ is a much more recent result of Ershov-He [EH18]. Note however we will not use these finite generation results in our work.

3. Bi-affine representations of the mapping class group

Here we first define the notions of affine, co-affine, and bi-affine representations in general, then quickly specialize to the setting of the mapping class group, collecting some simple preliminary results. The main result of the section is Corollary 3.9, which gives a classification of bi-affine representations of $\mathrm{Mod}(S)$ with core H (as always, $H = H_1(S; \mathbb{C})$).

3.1. **Bi-affine representations.**

Definition 3.1 (Affine, co-affine, bi-affine)**.** A group representation $\rho : G \to \mathrm{GL}(V)$ is *affine* if there is an invariant subspace $W \leqslant V$ for which the quotient representation $\overline{\rho} : G \to \mathrm{GL}(V/W)$ is trivial. ρ is *co-affine* if there is an invariant subspace W for which $\rho|_W$ is trivial. ρ is *bi-affine* if there are invariant subspaces $V_1 \leqslant V_2 \leqslant V$ for which V_1 and V/V_2 are both trivial. In this setting, the quotient V_2/V_1 is called the *core* of the representation.

Remark 3.2. The dual of an affine representation is co-affine, and vice versa.

3.2. **The unit tangent representation as an affine representation.** Here we explain how to understand $\widetilde{H}$ and $\widetilde{H}^*$ as a co-affine and affine representation, respectively.

We first recall the construction of $\widetilde{H}$ and its dual $\widetilde{H}^*$. As above, UTS denotes the unit tangent bundle of S, and set $\widetilde{H} = H_1(UTS;\mathbb{C})$. The projection $\pi : UTS \to S$ induces a surjection $\pi_* : \widetilde{H} \to H$. The kernel is the trivial one-dimensional subrepresentation spanned by the class of the S^1 fiber. Thus there is a short exact sequence

$$0 \to \mathbb{C} \to \widetilde{H} \to H \to 0,$$

realizing $\widetilde{H}$ as a co-affine representation. Dually, $\widetilde{H}^* = H^1(UTS;\mathbb{C})$ is seen to be affine, with H embedded as a submodule of codimension one via pullback.

3.3. **Extensions.** The next lemma shows that any extension of a bi-affine representation of $\mathrm{Mod}(S)$ is again bi-affine, so long as the dimension does not grow too much.

Lemma 3.3. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a bi-affine representation with respect to the filtration $V_1 \leqslant V_2 \leqslant V$. Suppose that V embeds as a submodule of $\widetilde{V}$, and that $\dim(\widetilde{V}/V_2) = \dim(\widetilde{V}/V) + \dim(V/V_2) \leqslant 2g-1$. Then $\widetilde{V}$ likewise is bi-affine, with respect to the filtration $V_1 \leqslant V_2 \leqslant \widetilde{V}$.*

Dually, suppose $U \leqslant \widetilde{V}$ is a filtration of $\mathbb{C}[G]$-modules, with $V := \widetilde{V}/U$ bi-affine, filtered as $V_1 \leqslant V_2 \leqslant V$. Let $\widetilde{V}_1 \leqslant \widetilde{V}_2 \leqslant \widetilde{V}$ be the preimage of this filtration in $\widetilde{V}$. Suppose that $\dim(\widetilde{V}_1) = \dim(U) + \dim(V_1) \leqslant 2g-1$. Then $\widetilde{V}$ likewise is bi-affine, with respect to the filtration $\widetilde{V}_1 \leqslant \widetilde{V}_2 \leqslant \widetilde{V}$.

Proof. By hypothesis, $\dim(\widetilde{V}/V_2) \leqslant 2g-1$. By Theorem 2.1, $\widetilde{V}/V_2$ must be trivial; as V_1 is trivial by hypothesis, this realizes $\widetilde{V}$ as a bi-affine representation. The dual statement follows from this by Remark 3.2. □

3.4. **Classifying bi-affine representations of the mapping class group.** The objective of this subsection is Corollary 3.9, which shows that bi-affine representations of $\mathrm{Mod}(S)$ (with core H) are extremely simple. This is a homological calculation, built around a determination of certain Ext groups (recall that for R any ring and A, B any R-modules, $\mathrm{Ext}^1_R(A,B)$ classifies isomorphism classes of extensions $0 \to B \to E \to A \to 0$). We first recall some basic properties of Ext and its connection to group cohomology.

Proposition 3.4 (cf. [Bro94], III.2.2)**.** *Let G be a group and let M, N be $\mathbb{C}[G]$-modules. Then*

$$\mathrm{Ext}^*_{\mathbb{C}[G]}(M,N) \cong H^*(G;\mathrm{Hom}_{\mathbb{C}}(M,N)).$$

In general, $\mathrm{Ext}^*_R(A,B)$ carries commuting actions of $\mathrm{Aut}_R(A)$ and $\mathrm{Aut}_R(B)$. When A and/or B is a trivial R-module, this structure is extremely simple.

Lemma 3.5. *Let M be a $\mathbb{C}[G]$-module, and let $V \cong \mathbb{C}^n$ be a* trivial *$\mathbb{C}[G]$-module. Then as a $\mathrm{GL}(V)$-module,*

$$\mathrm{Ext}^*_{\mathbb{C}[G]}(V,M) \cong V^* \otimes \mathrm{Ext}^*_{\mathbb{C}[G]}(\mathbb{C},M) \qquad \textit{and} \qquad \mathrm{Ext}^*_{\mathbb{C}[G]}(M,V) \cong \mathrm{Ext}^*_{\mathbb{C}[G]}(M,\mathbb{C}) \otimes V.$$

Proof. Ext commutes with the formation of finite direct sums in either argument [Wei94, Proposition 3.3.4]. □

For simplicity of notation, in the remainder of the section, the ring R in Ext^*_R will always be $\mathbb{C}[\mathrm{Mod}(S)]$, and we will suppress it from the notation. Our determination of Ext groups in this setting is based on a fundamental result of Morita.

Proposition 3.6 (Morita, cf. Propositions 4.1 and 6.4 of [Mor89]). *For $S = \Sigma_{g,1}$ or $\Sigma_{g,*}$,*

$$H^1(\mathrm{Mod}(S); H) \cong \mathbb{C}.$$

For $S = \Sigma_g$,

$$H^1(\mathrm{Mod}(S); H) = 0.$$

Corollary 3.7. *Let V, W be trivial $\mathrm{Mod}(S)$-modules. Then*

$$\mathrm{Ext}^1(V, W) = 0.$$

If $S = \Sigma_{g,1}$ or $\Sigma_{g,*}$, then

$$\mathrm{Ext}^1(V, H) \cong V^* \qquad \text{and} \qquad \mathrm{Ext}^1(H, W) \cong W,$$

while if $S = \Sigma_g$, then

$$\mathrm{Ext}^1(V, H) = \mathrm{Ext}^1(H, W) = 0.$$

Proof. Using Proposition 3.4, these translate into assertions about $H^1(\mathrm{Mod}(S); \mathrm{Hom}_{\mathbb{C}}(A, B))$ for various A, B. The first assertion follows from the fact that $H^1(\mathrm{Mod}(S); V) = 0$ for all $g \geqslant 0$ and all trivial $\mathrm{Mod}(S)$-modules V. The remaining assertions follow by combining Lemma 3.5 and Proposition 3.6. □

Proposition 3.8. *Let V_1 and V_3 be trivial $\mathrm{Mod}(\Sigma_{g,1})$-modules, and let $[V_2] \in \mathrm{Ext}^1(H, V_1)$ be given. Suppose $[V_2] \neq 0$. Then $\mathrm{Ext}^1(V_3, V_2) = 0$.*

Proof. The short exact sequence $0 \to V_1 \to V_2 \to H \to 0$ induces the long exact sequence

$$\mathrm{Ext}^1(V_3, V_1) \to \mathrm{Ext}^1(V_3, V_2) \to \mathrm{Ext}^1(V_3, H) \xrightarrow{\delta^1} \mathrm{Ext}^2(V_3, V_1).$$

By Corollary 3.7 and Lemma 3.5, this simplifies to

$$0 \to \mathrm{Ext}^1(V_3, V_2) \to V_3^* \otimes \mathrm{Ext}^1(\mathbb{C}, H) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathbb{C}}(V_3, V_1) \otimes \mathrm{Ext}^2(\mathbb{C}, \mathbb{C}),$$

and so $\mathrm{Ext}^1(V_3, V_2) = 0$ if and only if δ^1 is injective. Moreover, it suffices to consider the case $V_1 = V_3 = \mathbb{C}$, and to take $V_2 = \widetilde{H}$.

To proceed, we move to the setting of group cohomology, recasting $\delta^1 : \mathrm{Ext}^1(\mathbb{C}, H) \to \mathrm{Ext}^2(\mathbb{C}, \mathbb{C})$ as the connecting map $\delta^1 : H^1(\mathrm{Mod}(\Sigma_{g,1}); H) \to H^2(\mathrm{Mod}(\Sigma_{g,1}); \mathbb{C})$ in the cohomology long exact sequence associated to the extension

$$0 \to \mathbb{C} \to \widetilde{H} \to H \to 0. \tag{1}$$

From here, we follow a calculation of Kawazumi-Soulié.[1] In [KS24, Proposition 3.7], they show that for all $g \geqslant 0$, the connecting map

$$\delta^1 : H^1(\mathrm{Mod}(\Sigma_{g,1}); H) \to H^2(\mathrm{Mod}(\Sigma_{g,1}); \mathbb{C})$$

is given by the (presently mysterious) formula

$$\delta^1(v) = \mu_*(m_{1,1} \cup v).$$

The terms $m_{1,1}$ and μ_* require explanation. The class $m_{1,1} \in H^1(\mathrm{Mod}(\Sigma_{g,1}); H)$ is originally constructed in [KM01];[2] as shown in [KS24, Eq. (2.5)], it coincides with the extension class of $\widetilde{H} \in \mathrm{Ext}^1(H, \mathbb{C}) \cong H^1(\mathrm{Mod}(\Sigma_{g,1}); H)$ for any $g \geqslant 2$. In particular, by Proposition 3.6, $m_{1,1}$ spans $H^1(\mathrm{Mod}(\Sigma_{g,1}); H)$. The map $\mu_* : H \otimes H \to \mathbb{C}$ is the contraction map induced by the

[1] Their argument employs coefficients in $\mathbb{Z}$, not $\mathbb{C}$, but this is immaterial.

[2] We follow Kawazumi-Soulié in referencing the unpublished expanded version of the original research announcement [KM96].

intersection pairing on H. Injectivity of δ^1 now follows from the *contraction formula* established in [KM01, Theorem 6.2]:

$$\mu_*(m_{1,1} \cup m_{1,1}) = -e_1 \neq 0,$$

where $e_1 \in H^2(\mathrm{Mod}(\Sigma_{g,1}); \mathbb{C})$ is the first MMM class. □

Corollary 3.9. *Let S be a surface of genus $g \geqslant 3$ which is either closed, has one boundary component, or one puncture. Let V be a bi-affine representation of $\mathrm{Mod}(S)$ of arbitrary dimension with core $V_2/V_1 \cong H$. Then $V \cong W \oplus \mathbb{C}^n$, where W is the symplectic representation H, the unit tangent representation $\widetilde{H}$, or the dual $\widetilde{H}^*$. If S is closed, the latter two possibilities cannot arise.*

Proof. Note first that any representation of $\mathrm{Mod}(\Sigma_{g,*})$ is *a fortiori* a representation of $\mathrm{Mod}(\Sigma_{g,1})$. Therefore it suffices to assume that S is either closed or has one boundary component.

Let $V_1 \leqslant V_2 \leqslant V$ be the bi-affine filtration on V, determining associated classes $[V_2] \in \mathrm{Ext}^1(H, V_1)$ and $[V] \in \mathrm{Ext}^1(V/V_2, V_2)$. First suppose $[V_2] = 0$, so that $V_2 \cong H \oplus V_1$. Then, by Lemma 3.5 and Corollary 3.7, in the case $S = \Sigma_{g,1}$,

$$\mathrm{Ext}^1(V/V_2, V_2) \cong \mathrm{Ext}^1(V/V_2, H) \oplus \mathrm{Ext}^1(V/V_2, V_1) \cong (V/V_2)^*,$$

while if $S = \Sigma_g$, then by Corollary 3.7, $\mathrm{Ext}^1(V/V_2, H) = 0$, so that $\mathrm{Ext}^1(V/V_2, V_2) = 0$. As $\mathrm{GL}((V/V_2)^*)$ acts transitively on the nonzero elements of $(V/V_2)^*$, it follows that in the case of $S = \Sigma_{g,1}$ there is a *unique* isomorphism class of $\mathrm{Mod}(S)$-module forming a nontrivial extension of V/V_2 by $V_2 \cong H \oplus V_1$. This is represented explicitly by $\widetilde{H}^* \oplus V_1 \oplus U$, where $U \leqslant V/V_2$ is a subspace of codimension one, and is of the claimed form. If instead also $[V] = 0$ in $\mathrm{Ext}^1(V/V_1, V_1)$, then $V \cong H \oplus V_1 \oplus V/V_2$. This completes the proof in the case where $[V_2] = 0$.

Now suppose $[V_2] \neq 0$. By Proposition 3.8, $\mathrm{Ext}^1(V/V_2, V_2) = 0$. The argument now proceeds as before: $V \cong V/V_2 \oplus V_2$, and as $\mathrm{Ext}^1(H, V_1) \cong V_1$ by Corollary 3.7, we again find (by exploiting the action of $\mathrm{GL}(V_1)$ to construct an explicit representative) that $V_2 \cong \widetilde{H} \oplus U$, where again $U \leqslant V_1$ is codimension one. □

In later arguments, we will make use of the following structural corollary of this result.

Corollary 3.10. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a bi-affine representation with core H. Let c be a nonseparating simple closed curve, and T_c the associated Dehn twist. Then $\rho(T_c)$ is unipotent and the 1-eigenspace $E_1^c \leqslant V$ is of codimension one.*

Proof. By Corollary 3.9, $V = W \oplus \mathbb{C}^n$, where W is one of $H, \widetilde{H}$, or $\widetilde{H}^*$. It therefore suffices to prove the claim for $W = H$ and $W = \widetilde{H}^*$. If $W = H$ (and hence $\rho = \Psi$), then $\Psi(T_c)(x) = x + \langle x, c \rangle\, [c]$, which is unipotent and has 1-eigenspace $[c]^\perp \leqslant H$ of codimension one as claimed.

In the case $W = \widetilde{H}^*$, the action of a Dehn twist is given as

$$\widetilde{\Psi}^*(T_c) = \left(\begin{array}{c|c} \Psi(T_c) & \phi(T_c) \\ \hline 0 & 1 \end{array}\right),$$

where $\phi : \mathrm{Mod}(S) \to H$ is a crossed homomorphism (obtained as a representative of the extension class for $[\widetilde{H}^*] \in \mathrm{Ext}^1(\mathbb{C}, H) \cong H^1(\mathrm{Mod}(S); H)$). According to [Kas24, Theorem 4.2], there is a constant $\lambda \in \mathbb{C}$ (depending on ϕ, c, and a choice of orientation of c) such that $\phi(T_c) = \lambda[c]$. As we saw in the previous paragraph, $\Psi(T_c) - I$ likewise has image contained in the span of $[c]$, and annihilates $[c]$, so that $\widetilde{\Psi}^*(T_c)$ has 1-eigenspace of codimension one and is unipotent, as claimed. □

4. A criterion to be bi-affine

The goal of this section is to prove Proposition 4.4, which gives a sufficient condition under which a representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ is bi-affine with core H. **To streamline the discussion, we impose the standing convention that if left unspecified, the core of a bi-affine representation is H.** We first recall an extremely useful result formulated by Korkmaz (readily derived from the vanishing of $H^1(\mathrm{Mod}(S);\mathbb{C})$) which we will use throughout the rest of the paper, as well as two preparatory results about representations of the symplectic group.

Lemma 4.1 (Flag triviality criterion (Korkmaz), Lemma 7.1 of [Kor23])**.** *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ be a representation. If there is a flag*

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = \mathbb{C}^n$$

of $\mathrm{Mod}(S)$-invariant subspaces such that each quotient W_i/W_{i-1} is a trivial $\mathrm{Mod}(S)$-representation, then the image of ρ is trivial. In particular this holds if $\dim(W_i/W_{i-1}) \leqslant 2g-1$ for each $i = 1,2,\dots,k$.

Invariant flags turn out to be ubiquitous, in large part because of the following principle. Suppose $R \subset S$ is a subsurface, and let $c \subset S \setminus R$ be a simple closed curve. Then $\mathrm{Mod}(R)$ commutes with T_c, and since commuting linear transformations preserve (generalized) eigenspaces, any flag of generalized eigenspaces for $\rho(T_c)$ is a $\mathrm{Mod}(R)$-invariant flag. We will often use this principle without further comment.

Lemma 4.2. *For $g \geqslant 3$, the irreducible representations of $\mathrm{Sp}(2g,\mathbb{C})$ of smallest dimension are the trivial representation $\mathbb{C}$ of dimension 1, the standard representation H of dimension $2g$, and $\wedge^2 H/\mathbb{C}$ of dimension $\binom{2g}{2}-1$ (for $g=3$ there is a second representation of dimension $\binom{2g}{2}-1=14$).*

Proof. The low-dimensional representations of semi-simple Lie groups are tabulated in [AVÈ67, Table 1]. □

Recall that a *transvection* on H is a transformation $T_x(y) = y + \langle x,y\rangle\, x$, where $x,y \in H$.

Lemma 4.3. *For $g \geqslant 3$, let V be a $\mathrm{Sp}(2g,\mathbb{Z})$-module of dimension at most $4g-1$. Suppose that every transvection acts unipotently on V. Then either V is trivial, or else $V = H \oplus \mathbb{C}^d$; in particular, V is bi-affine.*

Proof. A result of Harman [Har19, Corollary 2.3] asserts that for $n > 2$, any representation $\rho : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{GL}_N(\mathbb{C})$ for which the elementary matrices act unipotently extends to an algebraic representation $\widetilde{\rho} : \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$. The proof of this uses only superrigidity for $\mathrm{SL}_n(\mathbb{Z})$ and the congruence subgroup property for $\mathrm{SL}_n(\mathbb{Z})$, both of which hold for $\mathrm{Sp}(2g,\mathbb{Z})$ for $g \geqslant 2$. Thus ρ extends to a representation $\rho : \mathrm{Sp}(2g,\mathbb{C}) \to GL(V)$. As $\mathrm{Sp}(2g,\mathbb{C})$ is semi-simple, V decomposes as a direct sum of irreducibles $V = \bigoplus_{i=1}^k V_i$. As $\dim(V) = 4g-1 < \binom{2g}{2}-1$, by Lemma 4.2, at most one of the summands V_i can be nontrivial, in which case it must be isomorphic to H. □

Proposition 4.4. *Let $g \geqslant 3$, and let $S = \Sigma_{g,1}$ be a surface of genus g with one boundary component. Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ be a representation, endowing $V = \mathbb{C}^n$ with the structure of a $\mathrm{Mod}(S)$-module. Suppose that the following conditions hold:*

(A) The restriction of ρ to $\mathcal{K}(S)$ is trivial,
(B) For any nonseparating curve $c \subset S$, $\rho(T_c)$ is unipotent.

Then for $n \leqslant 4g-1$, either V is trivial, or else V is bi-affine (with core H).

Proof. We proceed by induction on $m = n - 2g$. Korkmaz's Theorem 2.2 provides the base case $m = 0$. Before proceeding any further, observe that if V satisfies hypotheses (A) and (B), then so does any submodule $W \leqslant V$, and likewise for any quotient module V/W.

We consider the restriction of ρ to the Torelli group $\mathcal{I}(S)$. By hypothesis (A), $\rho|_{\mathcal{I}(S)}$ factors through $\mathcal{I}(S)/\mathcal{K}(S)$. Recalling the discussion of Section 2.2, $\rho|_{\mathcal{I}(S)}$ factors through the Johnson homomorphism $\tau : \mathcal{I}(S) \to A_S$; in particular, the image is abelian. By Theorem 2.7, $\mathcal{I}(S)$ is generated by bounding pair maps $T_a T_b^{-1}$, where a and b are disjoint and homologous. By hypothesis (B), it follows that every bounding pair map acts unipotently. In summary, the action of $\mathcal{I}(S)$ on V is both unipotent and abelian.

In light of this, the fixed space $V^{\mathcal{I}(S)}$ is nonempty. This is a $\mathrm{Mod}(S)$-submodule, and moreover since $\mathcal{I}(S)$ acts on it trivially, the action of $\mathrm{Mod}(S)$ on $V^{\mathcal{I}(S)}$ factors through the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. If $V^{\mathcal{I}(S)} = V$, then the result follows by Lemma 4.3. Otherwise, $V^{\mathcal{I}(S)}$ is a proper submodule of V. By induction, either $V^{\mathcal{I}(S)}$ is bi-affine, or else $V^{\mathcal{I}(S)}$ is trivial.

In the former case, as $\dim(V) \leqslant 4g - 1$ and V contains a copy of H of dimension $2g$, the dimension bound of Lemma 3.3 is seen to hold, and so by that result, V is bi-affine. If $V^{\mathcal{I}(S)}$ is trivial, then by Lemma 4.1, either V itself is trivial, or else the complement $V/V^{\mathcal{I}(S)}$ must be a nontrivial $\mathrm{Mod}(S)$-module. By induction, this must be bi-affine, and again since $\dim(V) \leqslant 4g - 1$, the dimension bound of Lemma 3.3 holds and we conclude that V is bi-affine. □

5. Unipotence

In this section we establish two key results regarding the structure of Dehn twists in representations up to a certain range. In Proposition 5.1, we show that Dehn twists act unipotently in every $\mathrm{Mod}(S)$-representation up to dimension $4g-3$. It is worth noting that a well-known folklore result shows that Dehn twists act *quasi-unipotently* (i.e. all eigenvalues are roots of unity) in *any* linear representation (see [But19, Corollary 3.5] and/or [AS16, Proposition 2.4]). The second key result is Proposition 5.6, which shows that under stricter hypotheses, moreover the genuine 1-eigenspace has codimension 1.

Proposition 5.1. *Let $g \geqslant 4$, and let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ be a representation. Let $c \subset S$ be a nonseparating simple closed curve. Then for $0 \leqslant n \leqslant 4g - 3$, $\rho(T_c)$ is unipotent.*

Remark 5.2. Proposition 5.1 holds more generally for $S = \Sigma_{g,n}^b$ a surface of genus $g \geqslant 4$ with any number of punctures and/or boundary components, since the results of Korkmaz on which it relies hold in this setting.

The proof uses the following lemma, a generalization of [Kor23, Lemma 4.3] with the same proof. Here and throughout, for a simple closed curve $c \subset S$, we write

$$E_{\lambda,k}^c = \ker((\rho(T_c) - \lambda I)^k)$$

for the degree-k generalized λ-eigenspace of $\rho(T_c)$. In the case $k = 1$ of the genuine eigenspace, we will write simply E_λ^c.

Lemma 5.3 (Korkmaz, cf. Lemma 4.3 of [Kor23])**.** *Let $b, c \subset S$ be nonseparating simple closed curves satisfying $i(b, c) = 1$. Let ρ be a linear representation of $\mathrm{Mod}(S)$ and let λ be an eigenvalue of $\rho(T_b)$ (and hence of $\rho(T_c)$ also). If $E_{\lambda,k}^b = E_{\lambda,k}^c$ for some $k \geqslant 1$, then $E_{\lambda,k}^b$ is $\mathrm{Mod}(S)$-invariant.*

We now give the proof of Proposition 5.1:

Proof. We proceed by induction on n. The base cases $n \leqslant 2g$ follow from Korkmaz (Theorem 2.2). For $n > 2g$, suppose for contradiction that $\rho(T_c)$ has an eigenvalue $\lambda \neq 1$. Let $\lambda_\sharp$ denote the multiplicity of λ as an eigenvalue.

Firstly, suppose $\lambda_\sharp = n$. If $\rho(T_c) = \lambda I_n$, applying ρ to the lantern relation results in $\lambda^4 = \lambda^3$ and hence $\lambda = 1$, a contradiction. Thus

$$1 \leqslant \dim E^c_\lambda \leqslant n-1 \leqslant 4g-4.$$

In particular, either E^c_λ or $\mathbb{C}^n/E^c_\lambda$ has dimension between 1 and $2g-2$. Both of these are invariant under $\mathrm{Mod}(S \setminus \{c\})$, where $S \setminus \{c\}$ is a surface of genus $g-1$ obtained by cutting c open in S. Since $g-1 \geqslant 3$, any representation of $\mathrm{Mod}(S \setminus \{c\})$ with dimension between 1 and $2g-2$ is known by Korkmaz (Theorem 2.2), and in particular Dehn twists act unipotently, so $\lambda = 1$, a contradiction.

Thus $\lambda_\sharp < n$. Let $E^c_{\lambda,gen}$ be the generalized λ-eigenspace. As above, either $E^c_{\lambda,gen}$ or $\mathbb{C}^n/E^c_{\lambda,gen}$ has dimension between 1 and $2g-2$. If $1 \leqslant \dim E^c_{\lambda,gen} \leqslant 2g-2$, by a similar argument as above, we get $\lambda = 1$, a contradiction. Thus $1 \leqslant \dim \mathbb{C}^n/E^c_{\lambda,gen} \leqslant 2g-2$, and $\mathrm{Mod}(S \setminus \{c\})$ acts on this space either trivially or via the symplectic representation; in either case, Dehn twists act unipotently. Therefore any nonseparating simple closed curve on $S \setminus \{c\}$ has the same generalized λ-eigenspace as c. In particular, taking two nonseparating simple closed curves d, d' on $S \setminus \{c\}$ with $i(d,d') = 1$, then by Lemma 5.3, $E^c_{\lambda,gen}$ is $\mathrm{Mod}(S)$-invariant, of dimension $< n$, so by induction, $\lambda = 1$, concluding the proof. □

This also leads to the following corollary:

Corollary 5.4. *Any homomorphism from* $\mathrm{Mod}(S)$ *to a compact Lie group of dimension at most* $4g-3$ *is trivial.*

Proof. Given $f : \mathrm{Mod}(S) \to G$ such a homomorphism, the adjoint representation furnishes a linear representation of $\mathrm{Mod}(S)$ of dimension $\leqslant 4g-3$, which sends Dehn twists to unipotent elements by Proposition 5.1. But the only unipotent element in a compact Lie group is the identity matrix [Bor91]. Thus $\mathrm{Im}(f)$ is contained in $Z(G)$, an abelian group, and since $g \geqslant 3$ by standing assumption, it follows that $\mathrm{Im}(f)$ is trivial. □

For the results of the next section, it is also necessary to specify the dimension of the 1-eigenspace. We make use of the following piece of linear algebra that governs the structure of a generalized eigenspace.

Lemma 5.5 (Jordan inequalities, cf. Section 2.2 of [CS23])**.** *Let* $A \in \mathrm{End}(\mathbb{C}^n)$ *be a linear transformation, and let* λ *be an eigenvalue of* A*. Consider the filtration*

$$0 = E_{\lambda,0} \leqslant E_{\lambda,1} \leqslant E_{\lambda,2} \leqslant \ldots \leqslant E_{\lambda,d} = E_{\lambda,gen}$$

of the generalized eigenspace $E_{\lambda,gen}$*. Then the dimensions of the associated graded quotients form a non-increasing sequence:*

$$\dim(E_{\lambda,j}/E_{\lambda,j-1}) \geqslant \dim(E_{\lambda,j+1}/E_{\lambda,j})$$

for $1 \leqslant j \leqslant d-1$*.*

Proposition 5.6. *Let* $\rho : \mathrm{Mod}_{g,1} \to \mathrm{GL}(2g+m, \mathbb{C})$ *be a representation with* $0 < m \leqslant g-3$*. Let* $c \subset S$ *be a nonseparating simple closed curve. Suppose that for* $h = g$ *or* $g-1$*, every nontrivial representation* $\rho' : \mathrm{Mod}_{h,1} \to \mathrm{GL}(2h+(m-1), \mathbb{C})$ *is bi-affine with core* H*. Then the* 1*-eigenspace* E^c_1 *of* $\rho(T_c)$ *has codimension at most 1.*

Proof. By Proposition 5.1, T_c acts unipotently on $\mathbb{C}^{2g+m}$. We consider the flag of generalized eigenspaces

$$0 \leqslant E^c_{1,1} \leqslant E^c_{1,2} \leqslant \ldots \leqslant E^c_{1,d} = \mathbb{C}^{2g+m};$$

in this notation, $E^c_{1,1} = E^c_1$ is the genuine eigenspace. This is invariant under the action of $\mathrm{Mod}(S \setminus \{c\})$. By the Jordan inequalities (Lemma 5.5), the sequence of dimensions $\dim(E^c_{1,j}/E^c_{1,j+1})$

is nonincreasing, so that $\dim(E_1^c)$ is an upper bound on the dimension of any such quotient. By the flag triviality criterion (Lemma 4.1), it follows that $\dim(E_1^c) \geqslant 2g-2$.

Let $S'_c \subset S \setminus \{c\}$ be a subsurface homeomorphic to $\Sigma_{g-1,1}$. If $\dim(E_1^c) \leqslant 2(g-1)+(m-1)$, then by hypothesis, the restriction of $\rho(\mathrm{Mod}(S'_c))$ to E_1^c must be bi-affine or else trivial. By Lemma 3.3, it follows that $\rho(\mathrm{Mod}(S'_c))$ itself is either bi-affine or trivial, and then the claim follows from Corollary 3.10.

Otherwise $\mathrm{codim}(E_1^c) \leqslant 2$, so it remains only to rule out the case $\mathrm{codim}(E_1^c) = 2$. Let $b \subset S \setminus S'_c$ be a simple closed curve b with $i(b,c) = 1$. Then by Lemma 5.3, either E_1^c is invariant under $\rho(\mathrm{Mod}(S))$, or else $E_1^b \cap E_1^c$ is a $\mathrm{Mod}(S'_c)$-invariant subspace of dimension at most $2(g-1)+(m-1)$. In the former case, by hypothesis, the action of $\mathrm{Mod}(S)$ on E_1^c is either bi-affine or trivial, and we conclude as in the preceding paragraph. In the latter case, we again conclude by hypothesis that the representation of $\mathrm{Mod}(S'_c)$ on $E_1^b \cap E_1^c$ is bi-affine or trivial, and the argument finishes along the same lines once again. □

6. Transvective representations

The work of the previous section shows that in a certain range, Dehn twists act unipotently, and with genuine eigenspace of codimension 1. Here we study representations of this type (which we call *transvective representations*) in greater detail. The culminating result is Proposition 6.7, which shows that transvective representations necessarily annihilate the Johnson kernel $\mathcal{K}(S)$.

Definition 6.1 (Transvective representation)**.** A representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ is *transvective* if for all nonseparating simple closed curves $c \subset S$, $\rho(T_c)$ is unipotent and $\mathrm{codim}(E_1^c) = 1$.

It turns out that transvective representations have an extremely rigid structure on Dehn twists. In Lemma 6.2-6.6, we establish this theory.

Lemma 6.2. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Then for any nonseparating simple closed curve $c \subset S$, there is $\alpha_c \in V^*$ and $v_c \in V$ for which*

$$\rho(T_c)(x) = x + \alpha_c(x)v_c$$

for all $x \in V$, and $\alpha_c(v_c) = 0$

Proof. Since $\mathrm{codim}(E_1^c) = 1$, there is $\alpha_c \in V^*$ and $v_c \in V$ for which

$$\rho(T_c) - I = \alpha_c v_c.$$

Since $\rho(T_c)$ is unipotent, $\rho(T_c) - I$ is nilpotent, and so necessarily $\alpha_c(v_c) = 0$. □

Lemma 6.3. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a, b \subset S$ be nonseparating simple closed curves. Suppose that either $i(a,b) = 1$, or else that $i(a,b) = 0$ but that a and b are not homologous. Let $\alpha_a, \alpha_b, v_a, v_b$ be as in Lemma 6.2. Then α_a and α_b are linearly independent, and the same is true of v_a and v_b.*

Proof. This is a variant of the method of proof of Lemma 5.3; we recall the argument. Suppose for contradiction that α_a, α_b or v_a, v_b are linearly dependent. For convenience, we will notate the span of α_a by $\mathbb{C}\alpha_a$, and similarly for other objects. Let $c \subset S$ be an additional simple closed curve satisfying the same topological constraints as b (i.e. in the first case, $i(a,c) = 1$ and similarly in the second case). By the change-of-coordinates principle, there is $f \in \mathrm{Mod}(S)$ such that $f(a) = a$ and $f(b) = c$. Then $\rho(f)$ commutes with $\rho(T_a)$ and so preserves $\mathbb{C}\alpha_a$ and $\mathbb{C}v_a$. On the other hand, $\rho(f)$ conjugates $\rho(T_b)$ to $\rho(T_c)$, and so takes $\mathbb{C}\alpha_b$ to $\mathbb{C}\alpha_c$, and $\mathbb{C}v_b$ to $\mathbb{C}v_c$. If $\mathbb{C}\alpha_a = \mathbb{C}\alpha_b$, this shows that likewise $\mathbb{C}\alpha_a = \mathbb{C}\alpha_c$, and similarly for $\mathbb{C}v_a, \mathbb{C}v_b, \mathbb{C}v_c$.

Let $\mathcal{C}$ be the graph with vertices in correspondence with isotopy classes of nonseparating simple closed curves on S, and with edges connecting a, b if $i(a,b) = 1$; let $\mathcal{C}'$ be the graph on the same vertex set, with a joined to b if $i(a,b) = 0$ and a, b are non-homologous. Both $\mathcal{C}$ and $\mathcal{C}'$ are connected for $g \geqslant 3$ (see [FM12, Chapter 4]). Thus if $\mathbb{C}\alpha_a = \mathbb{C}\alpha_b$ for some a, b adjacent in $\mathcal{C}$ or $\mathcal{C}'$, and if c is an arbitrary vertex of $\mathcal{C}^{(')}$, there is a path $c_0 = a, c_1, \dots, c_k = c$ in $\mathcal{C}^{(')}$. The argument of the above paragraph shows that $\mathbb{C}\alpha_a = \mathbb{C}\alpha_{c_1}$, and successively $\mathbb{C}\alpha_{c_i} = \mathbb{C}\alpha_{c_{i+1}}$, ultimately showing $\mathbb{C}\alpha_a = \mathbb{C}\alpha_c$. The same argument of course works with vectors in place of covectors.

We show that in either situation, this leads to a contradiction. If $\mathbb{C}\alpha_a = \mathbb{C}\alpha_b$ for all pairs of nonseparating simple closed curves on S, then this shows that the codimension-1 subspace $E_1^a = \ker(\alpha_a)$ is ρ-invariant and trivial; by the flag triviality criterion (Lemma 4.1), ρ itself is trivial. If $\mathbb{C}v_a = \mathbb{C}v_b$ for all pairs of curves a, b, we reduce to the previous situation by considering the dual representation. □

Lemma 6.4. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a, b \subset S$ be nonseparating simple closed curves satisfying $i(a,b) = 1$. Let $\alpha_a, \alpha_b, v_a, v_b$ be as in Lemma 6.2. Then*

$$\alpha_a(v_b)\alpha_b(v_a) = -1.$$

Proof. Since $i(a,b) = 1$, the twists T_a, T_b satisfy the braid relation: $T_aT_bT_a = T_bT_aT_b$. A direct computation with the formula of Lemma 6.2 shows that

$$\rho(T_aT_bT_a)(x) = x + (2\alpha_a(x) + \alpha_b(x)\alpha_a(v_b) + \alpha_a(x)\alpha_b(v_a)\alpha_a(v_b))v_a + (\alpha_b(x) + \alpha_a(x)\alpha_b(v_a))v_b$$

and, by reversing the roles of a and b,

$$\rho(T_bT_aT_b)(x) = x + (2\alpha_b(x) + \alpha_a(x)\alpha_b(v_a) + \alpha_b(x)\alpha_a(v_b)\alpha_b(v_a))v_b + (\alpha_a(x) + \alpha_b(x)\alpha_a(v_b))v_a.$$

By Lemma 6.3, v_a and v_b are linearly independent. Comparing coefficients on v_a (and using the fact that α_a is not identically zero, also by Lemma 6.3), we obtain the equation

$$1 + \alpha_a(v_b)\alpha_b(v_a) = 0$$

as desired. □

Lemma 6.5. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a, c \subset S$ be nonseparating simple closed curves that satisfy $i(a,c) = 0$ but are not homologous. Let $\alpha_a, \alpha_c, v_a, v_c$ be as in Lemma 6.2. Then*

$$\alpha_a(v_c) = \alpha_c(v_a) = 0.$$

Proof. We compute

$$\rho(T_cT_a) = x + \alpha_a(x)v_a + (\alpha_c(x) + \alpha_a(x)\alpha_c(v_a))v_c$$

and

$$\rho(T_aT_c) = x + \alpha_c(x)v_c + (\alpha_a(x) + \alpha_c(x)\alpha_a(v_c))v_a.$$

By Lemma 6.3, v_a, v_c are linearly independent. Comparing coefficients on v_a and v_c then yields the desired identities. □

Recall that a *chain* $a_1, \dots, a_k$ of simple closed curves in S is a set of curves for which $i(a_i, a_{i+1}) = 1$ for $i = 1, \dots, k-1$ and for which $i(a_i, a_j) = 0$ for $|i-j| > 1$. We say that a chain is *standardly embedded* if the classes $[a_1], \dots, [a_k]$ are linearly independent in $H_1(S; \mathbb{Z})$.

Lemma 6.6. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a_1, \dots, a_{2k}$ be a standardly-embedded chain; let $v_{a_1}, \dots, v_{a_{2k}}$ be the vectors associated to $\rho(T_{a_1}), \dots, \rho(T_{a_{2k}})$ as in Lemma 6.2. Then $[v_{a_1}], \dots, [v_{a_{2k}}]$ are linearly independent in V, and can be normalized so that $\alpha_{a_i}(v_{a_{i+1}}) = -\alpha_{a_{i+1}}(v_{a_i}) = 1$ for $i = 1, \dots, 2k-1$, and $\alpha_{a_i}(v_j) = 0$ for $|i-j| > 1$.*

Proof. To see that $\{[v_{a_i}]\}$ is linearly independent, suppose

$$w = c_1 v_{a_1} + \cdots + c_{2k} v_{a_{2k}} = 0.$$

Applying T_{a_1} and applying Lemma 6.4 and Lemma 6.5,

$$0 = T_{a_1}(w) = w + \alpha_{a_1}(w) v_a = c_2 \alpha_{a_1}(v_{a_2}) v_a,$$

showing that $c_2 = 0$. Successively applying $T_{a_3}, \ldots, T_{a_{2k-1}}$ then shows that all coefficients $c_{2i} = 0$. Working from the other end, applying $T_{a_{2k}}$ shows that $c_{2k} = 0$; working backwards applying $T_{a_{2k-2}}, \ldots, T_{a_2}$ then shows that all remaining coefficients $c_{2i+1} = 0$ as well.

The desired normalization can be defined by setting

$$c_i = \prod_{j=1}^{i-1} \alpha_{a_j}(v_{j+1})$$

and then defining

$$\alpha'_{a_i} = c_i \alpha_{a_i} \qquad \text{and} \qquad v'_{a_i} = \frac{v_{a_i}}{c_i}.$$

It is then a routine calculation to verify that $\alpha'_i(v'_{a_{i+1}}) = 1$ and the other claimed relations. □

We come to the main result of the section.

Proposition 6.7. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Then the restriction of ρ to $\mathcal{K}(S)$ is trivial.*

Proof. Let a, b, c, d be a standardly-embedded chain of length 4. Following Lemma 6.6, we can extend v_a, v_b, v_c, v_d to a basis of V. In such a basis, after normalizing *à la* Lemma 6.6, the elements $\rho(T_a), \ldots, \rho(T_d)$ have the following matrix expressions:

$$\rho(T_a) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \alpha_a \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right) \qquad \rho(T_b) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \alpha_b \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right)$$

$$\rho(T_c) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \alpha_c \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right) \qquad \rho(T_d) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & \alpha_d \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right).$$

Here, it should be emphasized that the last column is actually a block, recording what happens to basis vectors beyond $v_a, \ldots, v_d$. The entries $\alpha_a, \ldots, \alpha_d$ are in fact covectors (row vectors), expressing these functionals in the chosen coordinates.

By Theorem 2.6, $\mathcal{K}(S)$ is generated by separating twists of genus 1 and 2, and so it suffices to examine these two mapping classes. Let $a, b, c, d \subset S$ be a standardly-embedded chain. Then a regular neighborhood of $a \cup b$ is a subsurface of genus 1, bounded by a separating curve Δ_1. There is an alternate form of the chain relation [FM12, Section 4.4.1] which asserts

$$T_{\Delta_1} = (T_a^2 T_b)^4.$$

A computation shows that

$$\rho(T_a^2T_b) = \left(\begin{array}{cccc|c} -1 & 2 & 0 & 0 & 2(\alpha_a+\alpha_b) \\ -1 & 1 & 1 & 0 & \alpha_b \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right) \tag{2}$$

and hence

$$\rho(T_a^2T_b)^2 = \left(\begin{array}{cccc|c} -1 & 0 & 2 & 0 & 2\alpha_b \\ 0 & -1 & 2 & 0 & -2\alpha_a \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right);$$

a final squaring shows that $\rho(T_{\Delta_1}) = \rho(T_a^2T_b)^4 = I$ as required.

The computation for a separating twist of genus 2 proceeds along the same lines. We use the alternate formulation of the chain relation

$$T_{\Delta_2} = (T_a^2T_bT_cT_d)^8,$$

where Δ_2 is the boundary of the surface of genus 2 containing a, b, c, d. Picking up from (2),

$$\rho(T_a^2T_bT_cT_d) = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 2 & 2(\alpha_a+\alpha_b+\alpha_c+\alpha_d) \\ -1 & 0 & 0 & 1 & \alpha_b+\alpha_c+\alpha_d \\ 0 & -1 & 0 & 1 & \alpha_c+\alpha_d \\ 0 & 0 & -1 & 1 & \alpha_d \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right).$$

Taking successive powers yields

$$\rho(T_a^2T_bT_cT_d)^2 = \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2\alpha_d \\ 1 & 0 & -1 & -1 & -2\alpha_a-\alpha_b-\alpha_c \\ 1 & 0 & -1 & 0 & -\alpha_b+\alpha_d \\ 0 & 1 & -1 & 0 & -\alpha_c+\alpha_d \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right)$$

and

$$\rho(T_a^2T_bT_cT_d)^4 = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 2(\alpha_b+\alpha_d) \\ 0 & -1 & 0 & 0 & -2\alpha_a \\ 0 & 0 & -1 & 0 & 2\alpha_d \\ 0 & 0 & 0 & -1 & -2(\alpha_a+\alpha_c) \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right),$$

and a final squaring yields $\rho(T_a^2T_bT_cT_d)^8 = I$. □

7. Proof of Theorem A

Proof of Theorem A. We will show that any representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(2g+n, \mathbb{C})$ for $g \geqslant 3$ and $0 \leqslant n \leqslant g-3$ is either trivial or else bi-affine with core H. The classification given in Theorem A then follows by Corollary 3.9.

This will be established by induction on the pairs (g, n) (again, with $g \geqslant 3$ and $0 \leqslant n \leqslant 3g-3$), endowed with the lexicographic ordering. The base case $(3, 0)$ follows from Korkmaz (Theorem 2.2). Now let (g, n) be given (in particular with $g \geqslant 4$), and assume that the result holds for any pair (g', n') with $g' < g$ or else with $g' = g$ and $n' < n$. By Proposition 5.1, $\rho(T_c)$ is unipotent for any

nonseparating simple closed curve $c \subset S$. Invoking the inductive hypothesis, the hypotheses of Proposition 5.6 are satisfied, and so either ρ is trivial, or else the 1-eigenspace E_1^c of $\rho(T_c)$ has codimension 1. In this case, ρ is transvective (Definition 6.1), and so by Proposition 6.7, the restriction of ρ to $\mathcal{K}(S)$ is trivial. Then by Proposition 4.4, it follows that ρ is bi-affine with core H. □

References

[AS16] J. Aramayona and J. Souto. Rigidity phenomena in the mapping class group. In *Handbook of Teichmüller theory. Vol. VI*, volume 27 of *IRMA Lect. Math. Theor. Phys.*, pages 131–165. Eur. Math. Soc., Zürich, 2016.

[AVÈ67] E. M. Andreev, E.B. Vinberg, and A. G. Èlašvili. Orbits of highest dimension of semisimple linear Lie groups. *Funkcional. Anal. i Priložen.*, 1(4):3–7, 1967.

[Bor91] Armand Borel. *Linear Algebraic Groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2 edition, 1991.

[Bre26] T. Brevidelli. Lower bounds for faithful linear representations of subgroups of the mapping class group. arXiv preprint, https://arxiv.org/abs/2510.12591, 2026.

[Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[But19] J. O. Button. Aspects of non positive curvature for linear groups with no infinite order unipotents. *Groups Geom. Dyn.*, 13(1):277–292, 2019.

[CS23] N. Caplinger and N. Salter. Totally symmetric sets in the general linear group. *Michigan Math. J.*, page To appear, 2023.

[EH18] M. Ershov and S. He. On finiteness properties of the Johnson filtrations. *Duke Math. J.*, 167(9):1713–1759, 2018.

[Far24] Benson Farb. Global rigidity of the period mapping. *J. Algebraic Geom.*, 33(2):199–212, 2024.

[FM12] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Har19] N. Harman. Effective and infinite-rank superrigidity in the context of representation stability. arXiv preprint, https://arxiv.org/abs/1902.05603, 2019.

[Joh79] D. Johnson. Homeomorphisms of a surface which act trivially on homology. *Proc. Amer. Math. Soc.*, 75(1):119–125, 1979.

[Joh80] D. Johnson. An abelian quotient of the mapping class group $\mathcal{I}_g$. *Math. Ann.*, 249(3):225–242, 1980.

[Joh83] D. Johnson. The structure of the Torelli group. I. A finite set of generators for $\mathcal{I}$. *Ann. of Math. (2)*, 118(3):423–442, 1983.

[Joh85] D. Johnson. The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves. *Topology*, 24(2):113–126, 1985.

[Kas24] Y. Kasahara. Crossed homomorphisms and low dimensional representations of mapping class groups of surfaces. *Trans. Amer. Math. Soc.*, 377(2):1183–1218, 2024.

[KM96] N. Kawazumi and S. Morita. The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes. *Math. Res. Lett.*, 3(5):629–641, 1996.

[KM01] N. Kawazumi and S. Morita. The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes. Pre-print, 2001.

[Kor23] M. Korkmaz. Low-dimensional linear representations of mapping class groups. *J. Topol.*, 16(3):899–935, 2023.

[KS24] N. Kawazumi and A. Soulié. Stable twisted cohomology of the mapping class groups in the unit tangent bundle homology. *Bull. Lond. Math. Soc.*, 56(11):3358–3381, 2024.

[Lit24] Daniel Litt. Motives, mapping class groups, and monodromy. arXiv preprint, https://arxiv.org/abs/2409.02234, 2024.

[Loo97] E. Looijenga. Prym representations of mapping class groups. *Geom. Dedicata*, 64(1):69–83, 1997.

[Mor89] S. Morita. Families of Jacobian manifolds and characteristic classes of surface bundles. I. *Ann. Inst. Fourier (Grenoble)*, 39(3):777–810, 1989.

[Ser25] C. Serván. On the uniqueness of the Prym map. *J. Differential Geom.*, 131(3):801–830, 2025.

[Tra92] Rolland Trapp. A linear representation of the mapping class group M and the theory of winding numbers. *Topology Appl.*, 43(1):47–64, 1992.

[Wei94] C. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

JK: Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720
Email address: jkaufma2@alumni.nd.edu

NS: Department of Mathematics, University of Notre Dame, 255 Hurley Building, Notre Dame, IN 46556
Email address: nsalter@nd.edu

ZZ: Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637
Email address: zhongz@uchicago.edu

XZ: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
Email address: xiyanmath@gmail.com