COHOMOLOGIES OF THE GROUP COS MOD 2

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Investigation of the cohomologies of the group COS in the sense of Artin was begun by V. I. Arnol'd [1]. In this article is given exhaustive information about the cohomologies of the group COS with coefficients in Z_2.

§1. THE PROBLEM POSED. DEFINITIONS. NOTATION

1.1. By $\widetilde{G}_n$ we denote the subspace of the space C^n consisting of points whose coordinates are pairwise different. The group S(n) of permutations of n things acts on $\widetilde{G}_n$ without fixed point. The factor space of G_n is obviously the space of unordered collections of n pairwise different complex numbers. As is known (see [1]) the fundamental group of the space G_n, in which as a marked point is taken the set $\{1, \ldots, n\}$, is none other than B(n), the group COS on n threads, but the homotopy groups $\pi_i(G_n)$ are trivial for $i \geq 2$. Accordingly, G_n is the classification space of the group B(n) and the cohomologies there, by definition are cohomologies of this group. The principal S(n)-fibering, $\widetilde{G}_n \to G_n$ is induced by the natural representation $B(n) \to S(n)$.

1.2. We have also the natural representation of the group S(n) in the group O(n): the transformation $R^n \to R^n$, inducing a permutation s: $\{1, \ldots, n\} \to \{1, \ldots, n\}$, conducts the k-th vector of the standard basis of the space R^n to the s(k)-th. The group O(n) can be considered as a subgroup of the group U(n). We obtain the desired homomorphisms

$$B(n) \dashrightarrow S(n) \to O(n) \to U(n),$$

inducing mappings of the corresponding classified spaces. There arise in particular the mappings $G_n \to BO(n)$, $G_n \to BU(n)$. These mappings induce over G_n, respectively, a real vector fibering ξ_n and a complex fibering ζ_n, the second of which is a complexification of the first. Obviously, for the mapping $\widetilde{G}_n \to G_n$ both these fiberings turn into the trivial fibering over $\widetilde{G}_n$. Hence the complete spaces of fiberings ξ_n, ζ_n are obtained from $\widetilde{G}_n \times R^n$, by factorizations with respect to the natural operations of the group S(n). Hence, it is clear that the complete space of fiberings ξ_n is the set of unordered sets of n pairs (z_i, x_i), where $z_1, \ldots, z_n$ are pairwise different complex numbers, $x_1, \ldots, x_n$ are any real numbers. The mapping of this space to G_n consists in discarding from each pair the numbers x_i. The fibering ζ_n is different from ξ_n only in the fact that the numbers x_i are complex.

1.3. Our purpose is to calculate the ring $H^*(B(n); Z_2)$, i.e., the ring of cohomologies of G_n with coefficients in Z_2. As will be shown, this ring is generated by the Stifel classes of fiberings ξ_n, i.e., the homomorphism $H^*(O(n); Z_2) \to H^*(B(n); Z_2)$ is an epimorphism. The fibering ζ_n also shows up as trivial, i.e., the homomorphism of the cohomologies of the group U(n) in the cohomologies of the group B(n) is trivial for any coefficients.

§2. HOMOMORPHISM OF GROUP COS IN THE UNITARY GROUP

We start with the simplest: with the study of the fibering ζ_n.

2.1. The fibering ζ_n is trivial.

<u>Proof.</u> It is obvious that the fibering ζ_n is isomorphic to the tangential fibering of the complex variety G_n. The latter is diffeomorphic to a region of space C^n and hence parallelized (the diffeomorphism of

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 4, No. 2, pp. 62-73, April-June, 1970. Original article submitted December 2, 1969.

the variety G_n in the completion of the "discriminant surface" in C^n is given by the formula $\{z_1, \ldots, z_n\} \to (\sigma_1(z_1, \ldots, z_n), \ldots, \sigma_n(z_1, \ldots, z_n))$, where $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric polynomials.

2.2. Remark. The homomorphism $H^*(U(n); Z_2) \to H^*(S(n); Z_2)$ here is by no means trivial. In fact, the $O(n)$-fibering over $BS(n)$, induced by the mapping $S(n) \to O(n)$, is obviously not oriented, i.e., its first Stifel class W_1 is different from zero. The first Chern class of the complexification of this fibering is compared, modulo 2, with the square of this class, since W_1^2 belongs to the transform of the homomorphism of interest. But the element W_1^2 is different from zero even in $H^*(S(1); Z_2) = H^*(Z_2; Z_2)$ and hence all the more so in $H^*(S(n); Z_2)$.

§3. CELLULAR SUBDIVISION

3.1. We denote by G_n^* the one-point compactification of the space G_n. In view of the presence of Poincaré isomorphism $H^*(G_n; Z_2) = H_*(G_n^*; Z_2)$ the problem of calculation of the cohomologies of the space G_n reduces to the problem of calculating the cohomologies of the space G_n^*. In this section we construct a cellular subdivision of the space G_n^*.

3.2. Let $m_1, \ldots, m_k$ be natural numbers whose sum is n. Denote by $e(m_1, \ldots, m_k)$the subset of the space G_n consisting of points $\{z_1, \ldots, z_n\} \in G_n$, such that the points $z_1, \ldots, z_n$ of the plane C lie on k pairwise distinct lines,where, moreover, m_1 points lie on the first line (reading from the left), m_2 on the second, m_3 on the third, etc. It is clear that the sets $e(m_1, \ldots, m_k)$ are pairwise nonintersecting and, in sum, make up all of G_n. We prescribe a cellular subdivision of the space G_n^* taking in the cells all sets $e(m_1, \ldots, m_k) \subset G_n \subset G_n^*$ and the point $\infty \in G_n^*$. The dimension of the cell $e(m_1, \ldots, m_k)$ is equal to $n + k$, and the characteristic mapping associated with it $f: I^{n+k} \to G_n^*$ is constructed as follows.

Let R^* be a line augmented by the points $-\infty$ and ∞, and let X_r be the space of rows $(x_1, \ldots, x_r)$, where $x_1, \ldots, x_r \in R^*$ and $x_1 \leq \ldots \leq x_r$. It is clear that the space X_r is homeomorphic to the r-dimensional cube. The mapping $f: I^{n+k} = X_k \times X_{m_1} \times \ldots \times X_{m_k} \to G_n^*$ we give by the formula $f[(x_1, \ldots, x_k), (x_{11}, \ldots, x_{1m_1}), \ldots, (x_{k1}, \ldots, x_{km_k})] = \{x_1 + ix_{11}, \ldots, x_1 + ix_{1m_1}, \ldots, x_k + ix_{k1}, \ldots, x_k + ix_{km_k}\}$, where,moreover, the set in braces is taken equal to $\infty \in G_n^*$, if at least two numbers coincide or at least one is equal to infinity. Clearly f homeomorphically maps $\mathrm{Int}(I^{n+k})$ on $e(m_1, \ldots, m_k)$. This mapping we take as characterizing the mapping corresponding to the cell $e(m_1, \ldots, m_k)$.

3.3. The cells of a cellular complex we identify with definite chains in the complex of cellular chains with coefficients in Z_2.

The boundary of the chain $e(m_1, \ldots, m_k) \in C_{n+k}(G_n^*; Z_2)$ is equal to

$$\sum_{i=1}^{k-1} C_{m_i+m_{i+1}}^{m_i} e(m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_k).$$

Proof. Under the mapping f constructed in Par. 3.2 the point $[(x_1, \ldots, x_k), (x_{11}, \ldots, x_{1m_1}), \ldots, (x_{k1}, \ldots, x_{km_k})] \in X_k \times X_{m_1} \times \ldots \times X_{m_k}$ goes over into a point of a $(n+k-1)$-dimensional cell of the space G_n^* if and only if for some i there holds $x_i = x_{i+1}$; the numbers $x_{i1}, \ldots, x_{im_i}, x_{i+1,1}, \ldots, x_{i+1,m_{i+1}}$ are all pairwise different and all numbers x_j are finite. Such a point goes over to a point of the cell $e(m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_k)$; the restriction of the mapping f to the appropriate boundary of the cube I^{n+k} at this point regularly at each point of the cell $e(m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_k)$ serves equally as the image of $C_{m_i+m_{i+1}}^{m_i}$ such points.

§4. ADDITIVE CONSTRUCTION OF COHOMOLOGIES

We begin by computing cohomologies of the complex $C_*(G_n^*; Z_2)$.

4.1. A chain of the complex $C_*(G_n^*; Z_2)$ is called symmetric if the cells $e(m_1, \ldots, m_k)$, $e(m_{s(1)}, \ldots, m_{s(k)})$, (where s is a permutation of k elements) enter into it with identical coefficients for any $m_1, \ldots, m_k$, s. The cell $e(m_1, \ldots, m_k)$ is called a 2-cell if all m_i are binary powers. The chain is called a 2-chain if it is a sum of 2-cells.

4.2. Every symmetric 2-chain is a cycle. Every cycle of the complex $C_*(G_n^*; Z_2)$ is homologous to a unique symmetric 2-chain. Thus, the graduated group $H_*(G_n^*; Z_2)$ is isomorphic to a subgroup of the graduated group of $C_*(G_n^*; Z_2)$, consisting of symmetric 2-chains.

This theorem, completely describing the additive structure of the cohomologies of the group B(n) with coefficients in Z_2, is covered by the proof below of the assertions 4.4-4.7, where moreover the basic difficulties lie in the proof of the last of them. We begin with an elementary arithmetic lemma.

4.3. Let $s_1, \ldots, s_k$ be pairwise different nonnegative integers. The number $C^{2^s}_{2^{s_1}+\ldots+2^{s_k}}$ is odd if and only if s is contained among the $s_1, \ldots, s_k$.

Proof. From the obvious identity $(1+x^2)^q \equiv (1+x)^{2q} \bmod 2$, it follows that $C^{2r}_{2q} \equiv C^r_q \bmod 2$, $C^{2r-1}_{2q} \equiv 0 \bmod 2$. Suppose our assertion proved for $k < l$, and consider the case $k = l$. We shall suppose that $s_1 > \ldots > s_l$. There are two possibilities.

(1) $s \leq s_l$. Then $C^{2^s}_{2^{s_1}+\ldots+2^{s_l}} \equiv C^1_{2^{s_1-s}+\ldots+2^{s_l-s}} \bmod 2$, a $C^1_{2^{s_1-s}+\ldots+2^{s_l-s}} = 2^{s_1-s}+\ldots+2^{s_l-s}$ is odd only for $s = s_l$.

(2) $s > s_l$. Then $C^{2^s}_{2^{s_1}+\ldots+2^{s_l}} \equiv C^{2^{s-s_l}}_{2^{s_1-s_l}+\ldots+2^{s_{l-1}-s_l}+1} \bmod 2$, and $C^{2^{s-s_l}}_{2^{s_1-s_l}+\ldots+2^{s_{l-1}-s_l}+1} = C^{2^{s-s_l}}_{2^{s_1-s_l}+\ldots+2^{s_{l-1}-s_l}} + C^{2^{s-s_l}-1}_{2^{s_1-s_l}+\ldots+2^{s_{l-1}-s_l}} \equiv C^{2^{s-s_l}}_{2^{s_1-s_l}+\ldots+2^{s_{l-1}-s_l}} \bmod 2$.

Now the number $C^{2^{s-s_l}}_{2^{s_1-s_l}+\ldots+2^{s_{l-1}-s_l}}$ by the induction hypothesis is odd if and only if $s - s_l$ is contained among $s_1 - s_l, \ldots, s_{l-1} - s_l$, i.e., when s is contained among $s_1, \ldots, s_{l-1}$. The assertion is proved.

4.4. We return to the complex $C_*(G_n^*; Z_2)$.

Every symmetric chain is a cycle.

Proof. If c is a symmetric (n + k)-dimensional chain and the cell $e(m_1, \ldots, m_{k-1})$ enters into ∂c as a boundary cell $e(m_1, \ldots, m_{i-1}, m_i', m_i'', m_{i+1}, \ldots, m_{k-1})$, where $m_i' + m_i'' = m_i$, then $m_i' \neq m_i''$, since $C^m_{2m} = 2C^m_{2m-1} \equiv 0 \bmod 2$. Therefore the cell $e(m_1, \ldots, m_{k-1})$ enters into ∂c again as a boundary cell $e(m_1, \ldots, m_{i-1}, m_i'', m_i', m_{i+1}, \ldots, m_{k-1})$.

4.5. A chain, if it is a boundary, contains no 2-cells. In particular two chains which are 2-cells are homologous if and only if they coincide.

This follows from the fact that the number $C^i_{2^s}$ is even for $0 < i < 2^s$.

4.6. If a 2-chain is a cycle then it is symmetric.

Proof. If the cell $e(m_1, \ldots, m_{k-1})$ enters into the boundary of an (n + k)-dimensional 2-chain then every m_i except one is a binary power and one is the sum of two different binary powers (different since the number $C^{2^{s-1}}_{2^s}$ is always even), i.e., $e(m_1, \ldots, m_{k-1}) = e(2^{s_1}, \ldots, 2^{s_{i-1}}, 2^{s_i'} + 2^{s_i''}, 2^{s_{i+1}}, \ldots, 2^{s_{k-1}})$. Such a cell enters into the boundary of the two 2-cells: $e(2^{s_1}, \ldots, 2^{s_{i-1}}, 2^{s_i'}, 2^{s_i''}, 2^{s_{i+1}}, \ldots, 2^{s_{k-1}})$, $e(2^{s_1}, \ldots, 2^{s_{i-1}}, 2^{s_i''}, 2^{s_i'}, 2^{s_{i+1}}, \ldots, 2^{s_{k-1}})$ (the number $C^{2^{s_i'}}_{2^{s_i'}+2^{s_i''}}$ is odd in accord with 4.3). Hence, if a 2-chain is a cycle then the cells $e(2^{s_1}, \ldots, 2^{s_{i-1}}, 2^{s_i'}, 2^{s_i''}, 2^{s_{i+1}}, \ldots, 2^{s_{k-1}})$, $e(2^{s_1}, \ldots, 2^{s_{i-1}}, 2^{s_i''}, 2^{s_i'}, 2^{s_{i+1}}, \ldots, 2^{s_{k-1}})$ enter or do not enter it simultaneously, i.e., the 2-chain is symmetric.

4.7. Every cycle homologous to a cycle is a 2-chain.

Proof. We shall say that the cell $e(m_1, \ldots, m_k)$ has order larger than s if $m_1, \ldots, m_s$ are binary powers.

Let c be a cycle all of whose cells have degrees larger than $s \geq 0$. We show that it is homologous to a cycle all of whose cells have orders larger than s + 1. With this our assertion will be demonstrated.

Choose from c terms of the form $e(2^{l_1}, \ldots, 2^{l_s}, j, m_{s+2}, \ldots, m_k)$, where j is not a binary power. If we take the sum of all such terms with fixed $j, m_{s+2}, \ldots, m_k$, entering in c, and in each of these terms we discard $j, m_{s+2}, \ldots, m_k$, then we get an (n + s)-dimensional 2-chain c'. As is easily verified from the fact that c is a cycle, it follows that c' is also a cycle. It therefore follows from the premise of 4.6 that the chain c is represented in the form of a sum of chains of the form $e([2^{l_1}, \ldots, 2^{l_s}], j, m_{s+2}, \ldots, m_k)$ (as the chains are denoted which arise from $e(2^{l_1}, \ldots, 2^{l_s}, j, m_{s+2}, \ldots, m_k)$ by symmetrization with respect to $2^{l_1}, \ldots, 2^{l_s}$) and the cell is of order greater than s + 1.

Clearly,

$$\partial e([2^{l_1}, \ldots, 2^{l_s}], j, m_{s+2}, \ldots, m_k)$$

$$= \sum C_{j+2^{l_i}}^{2^{l_i}} e([2^{l_1}, \ldots \hat{2}^{l_i} \ldots, 2^{l_s}], \; j + 2^{l_i}, m_{s+2}, \ldots, m_k) + \ldots, \tag{1}$$

where the summation extends over all different l_i, and "..." refers to a chain of order larger than s.

Suppose that $j = 2^{\lambda_1} + \ldots + 2^{\lambda_t}$. Then the sum on the right in Eq. (1) can be rewritten

$$\sum e([2^{l_1}, \ldots \hat{2}^{l_i} \ldots, 2^{l_s}], 2^{\lambda_1} + \ldots + 2^{\lambda_t} + 2^{l_i}, m_{s+2}, \ldots, m_k), \tag{2}$$

where the summation extends over all distinct l_i not contained among $\lambda_1, \ldots, \lambda_t$ (see (4.3). The cells entering into this sum may be reduced on ∂c only with cells entering in chains of the boundary $e([2^{l_1'}, \ldots, 2^{l_{s'}'}]$, $2^{\lambda_1'} + \ldots + 2^{\lambda_{t'}'}$, $m_{s+2}, \ldots, m_k)$, in which the set $\{l_1', \ldots, l_{s'}', \lambda_1', \ldots, \lambda_{t'}'\}$ coincides up to order with the set $\{l_1, \ldots, l_s, \lambda_1, \ldots, \lambda_t\}$. Denote the latter set by L and by c* the sum of the chains of the given form entering into c. Among the numbers of the set L some can be equal but p of them are different, where $t \leq p \leq t + s$. The different numbers of the set L we denote by $\mu_1, \ldots, \mu_p$. For $1 \leq i_1 < \ldots < i_t \leq p$ we set

$$E(i_1, \ldots, i_t) = e([2^{l_1}, \ldots, 2^{l_s}], 2^{\mu_{i_1}} + \ldots + 2^{\mu_{i_t}}, m_{s+2}, \ldots, m_k), \tag{3}$$

where $l_1, \ldots, l_s$ are numbers which remain in the set L if we eliminate from it t numbers $\mu_{i_1}, \ldots, \mu_{i_t}$. Identifying $E(i_1, \ldots, i_t)$ with the (t−1)-dimensional boundary of the (p−1)-dimensional simplex, we can represent c* as the (p−1)-dimensional ring of these simplexes with coefficients in Z_2. Using Eq. (2) we arrive at the conclusion that c* is the ring of (p−1)-dimensional simplexes. Since the simplex is a cyclic and since $t \geq 2$, it follows that this ring serves as the coboundary of some (t−2)-dimensional ring b, i.e. as a linear combination of the symbols $E(j_1, \ldots, j_{t-1})$. Each of these symbols, like their linear combination b, in relation to Eq. (3) denotes an (n + k−1)-dimensional chain of our complex. The boundary of the chain b, up to the order of the cells larger than s + 1, coincides with c*. Removing from c the boundary of the chain b and applying a suitable construction sufficiently often we get a cycle consisting of cycles of order larger than s, homologous to the cycle c.

The Assertion 4.7 is proved and with it Theorem 4.2.

4.8. Theorem 4.2 along with the Poincaré isomorphism $H^*(G_n; Z_2) = H_*(G_n^*; Z_2)$ permits us to handle things so as to include cohomologies of the group COS.

The rank of the group $H^k(B(n); Z_2)$ is equal to the number of ways in which the number n can be represented as the sum of n−k binary powers (among these binary powers there can be coincidences; representations differing only in the order of terms are considered identical).

Generators of the group $H^k(B(n); Z_2)$ we shall identify with partitions of the number n as a sum of n−k powers of two. In notation: $\langle 2^{l_1}, \ldots, 2^{l_{n-k}} \rangle \in H^k(B(n); Z_2)$. (Ordinarily we shall consider that $l_1 \geq \ldots \geq l_{n-k}$.)

4.9. We introduce an assertion of importance for the sequel.

The group $H^s(B(n); Z_2)$ is trivial* for $s \geq n$. The group $H^{n-1}(B(n); Z_2)$ is trivial if n is not a binary power and is generated uniquely by the generator $\alpha_n = \langle n \rangle$, if n is a binary power.

§5. HOMOMORPHISM OF THE GROUP COS IN AN ORTHOGONAL GROUP

The cellular subdivision of the space G_n^* constructed by us is suitable for an easy description of the Stifel class of the fibering ξ_n.

5.1. The Stifel class W_k of smooth n-dimensional vector fiberings over a smooth variety can be described thus: n−k cross sections of this fibering are constructed, lying in general position, and the set of all base points is taken on which the vectors of these cross sections linearly depend. This set of a cycle passes over for a Poincare isomorphism, to the k-th Stifel class.

The cross section of the fibering ξ_n is that function relating the unordered set $\{z_1, \ldots, z_n\}$ of pairwise distinct complex numbers of the unordered set $\{(z_1, x_1), \ldots, (z_n, x_n)\}$ with the same z_i and real x_i.

*This fact is contained in [1].

We form n−k cross sections of the fibering ξ_n putting

$$\sigma_i\{z_1, \ldots, z_n\} = \{(z_1, (\operatorname{Re} z_1)^{i-1}), \ldots, (z_n, (\operatorname{Re} z_n)^{i-1})\} \qquad (i = 1, \ldots, n-k).$$

It is easy to verify generality of position of these cross sections. These cross sections depend linearly on the points $\{z_1, \ldots, z_n\} \in G_n$ if and only if there exist real numbers $a_1, \ldots, a_{n-k}$ such that

$$a_1 (\operatorname{Re} z_j)^{n-k-1} + \ldots + a_{n-k-1} \operatorname{Re} z_j + a_{n-k} = 0$$

for $j = 1, \ldots, n$, i.e., if among the numbers $\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n$ not more than n−k are different, i.e., if the points $z_1, \ldots, z_n \in C$ lie on n−k vertical lines.

We arrive at that conclusion. Under the Poincaré W_k isomorphism there arises a cycle consisting of all (2n−k)-dimensional cells of the space G_n^*.

5.2. From the Assertion 4.5 and Theorem 4.7 it follows that every cycle of our complex is homologous to a sum of generators in its 2-cells. Therefore the demonstrated assertion shows that the class W_k passes over under the Poincaré isomorphism to a cycle equal to a sum of all (2n−k)-dimensional 2-cells. Applying Theorem 4.2 and Remark 4.8 we arrive at the indicated result.

The Stifel class $W_k \in H^k(B(n); Z_2)$ is equal to the sum of all elements of our chosen basis in $H^k(B(n); Z_2)$.

5.3. If $k \leq n-s$, where s is the number of units in the binary description of the number n, then $W_k \neq 0$; if $k > n-s$, then $W_k = 0$. In particular, $W_{n-1} \neq 0$ if and only if n is a binary power.

This follows from 5.2, 4.8, and 4.9.

§6. STABILIZATION

6.1. The natural imbedding $B(n) \to B(n+1)$ induces a homotopic class of mappings $G_n \to G_{n+1}$, one of which can be described thus: from the points $z_1, \ldots, z_n \in C$, we construct the point $\{z_1, \ldots, z_n\}$ of the space G_n, supplemented by the point $z + R + 1 \in C$, where z is the arithmetic mean of the numbers $z_1, \ldots, z_n$, and R is the maximum of the numbers $|z - z_i|$; there is produced a point of the space G_{n+1}. Under the construction of the imbedding φ: $G_n \to G_{n+1}$ a marked point goes into a marked point and the cell $e(m_1, \ldots, m_k)$ regularly maps into the cell $e(m_1, \ldots, m_k, 1)$, i.e., if $c \subset G_n$ is a compact cycle then the index of the intersection of the cycle $\varphi(c)$ with the cell $e(m_1, \ldots, m_{k+1})$ is equal to zero for $m_{k+1} > 1$ and is equal to the index of the intersection of the cycle c with $e(m_1, \ldots, m_k)$ for $m_{k+1} = 1$. From this observation and the definition of Poincaré isomorphism we get the following assertion.

The homomorphism φ: $H^*(B(n+1); Z_2) \to H^*(B(n); Z_2)$ carries $\langle 2^{l_1}, \ldots, 2^{l_s} \rangle$ to 0 for $l_s > 0$ and to $\langle 2^{l_1}, \ldots, 2^{l_{s-1}} \rangle$ for $l_s = 0$.

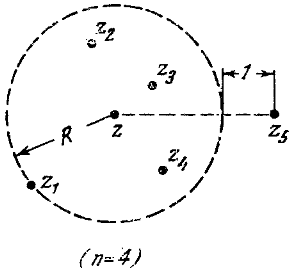
6.2. The homomorphism φ^* is an epimorphism.

This follows from the preceding assertion.

6.3. If n is even then the homomorphism φ^*: $H^*(B(n+1); Z_2) \to H^*(B(n); Z_2)$ is an isomorphism.†

Proof. In the decomposition of an odd number (here n + 1) as a sum of binary powers there explicitly enters $1 = 2^0$.

6.4. If $k \leq n/2$, then the homomorphism φ^*: $H_k(B(n+1); Z_2) \to H^k(B(n); Z_2)$ is an isomorphism.‡



(n=4)

Fig. 1

Proof. In the decomposition of the number n + 1 as a sum of n + 1 binary powers there explicitly enters $1 = 2^0$.

6.5. On the strength of Assertion 6.4 the groups $H^k(B(n); Z_2)$ for fixed k and for $n \geq 2k$ are canonically isomorphic (independently of n). Put $H^*(B(\infty); Z_2 = H^*(B(2k); Z_2)$. The graduated group $H^*(B(\infty); Z_2)$ is representable as a group of cohomologies of the space G_∞ defined as the direct limit of of the sequence

†This assertion is proved in [1].
‡This assertion is proved in [1].

$$G_1 \to G_2 \to G_3 \to \dots .$$

We recall that $H^*(B(\infty);\ Z_2)$ is a graduated ring. For each n there holds a ring of epimorphisms $H^*(B(\infty);\ Z_2) \to H^*(B(n);\ Z_2)$, since knowing multiplication in $H^*(B(\infty);\ Z_2)$, we can compute it in $H^*(B(n);\ Z_2)$.

The additive basis in $H^*(B(\infty);\ Z_2)$ makes up the collection $\langle 2^{l_1}, \dots, 2^{l_s} \rangle$, where $s \geq 0$, $l_1 \geq \dots \geq l_s > 0$.

Here $\dim \langle 2^{l_1}, \dots, 2^{l_s} \rangle = 2^{l_1} + \dots + 2^{l_s} - s$, and for the epimorphism $H^*(B(\infty);\ Z_2) \to H^*(B(n);\ Z_2)$ the element $\langle 2^{l_1}, \dots, 2^{l_s} \rangle$ passes over into 0 if and only if $2^{l_1} + \dots + 2^{l_s} > n$.

§7. HOPF ALGEBRA

7.1. The natural imbedding $B(m) \times B(n) \to B(m+n)$ induces a homotopical class of mappings $G_m \times G_n \to G_{m+n}$, one of which can be expressed as follows. The point $(\{z_1, \dots, z_m\}, \{w_1, \dots, w_n\} \in G_m \times G_n$ passes over into the point $\{z_1, \dots, z_m, w_1 + a, \dots, w_n + a\}$, where $a = z - w + R_1 + R_2 + 1$, and where in its turn z is the arithmetic mean of $z_1, \dots, z_m$; w is the arithmetic mean of the numbers $w_1, \dots, w_n$; R_1 is the largest of the numbers $|z - z_i|$; R_2 is the largest of the numbers $|w - w_i|$. Under the construction of the imbedding ψ: $G_m \times G_n \to G_{m+n}$ a marked point goes over to a marked point, and the product $e(m_1, \dots, m_k) \times e(m_1', \dots, m_l')$ regularly maps into the cell $e(m_1, \dots, m_k, m_1', \dots, m_l')$. It is obvious that the diagram

$$\begin{array}{ccccc} G_m \times G_n & \longrightarrow & G_{m+n} & \searrow & \\ \downarrow & & & & G_{m+n+1} \\ G_{m+1} \times G_{n+1} & \longrightarrow & G_{m+n+2} & \swarrow & \end{array}$$

is homotopically commutative since the homomorphism ψ^*: $H^*(B(m+n);\ Z_2) \to H^*(B(m);\ Z_2 \otimes H^*(B(n);\ Z_2)$ with increasing m and n is stabilized and converted into the homomorphism Δ: $H^*(B(\infty);\ Z_2) \to H^*(B(\infty);\ Z_2 \otimes H^*(B(\infty);\ Z_2$.

7.2. Simple calculations with the indices of the intersection, analogous to those conducted in Par. 6.1, lead us to the formula

$$\Delta \langle 2^{l_1}, \dots, 2^{l_s} \rangle = 1 \otimes \langle 2^{l_1}, \dots, 2^{l_s} \rangle$$
$$+ \sum (\langle 2^{l_{i_1}}, \dots, 2^{l_{i_u}} \rangle \otimes \langle 2^{l_{j_1}}, \dots, 2^{l_{j_v}} \rangle) + \langle 2^{l_1}, \dots, 2^{l_s} \rangle \otimes 1, \tag{4}$$

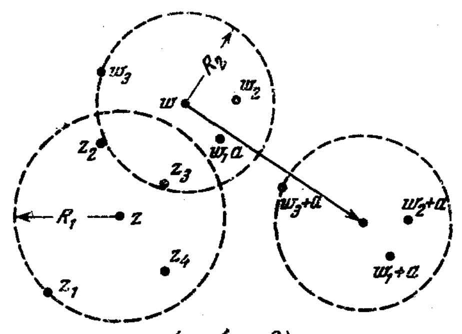
where the summation extends to all different partitions of the set $l_1, \dots, l_s$ into two nonempty sets. (For example $\Delta \langle 2,2 \rangle = 1 \otimes \langle 2,2 \rangle + \langle 2 \rangle \langle 2,2 \rangle \otimes 1$.)

7.3. As is readily verified, the ring $H^*(B(\infty);\ Z_2)$ is equipped with a homomorphism Δ constituting a Hopf algebra over the field Z_2. Essential for us in the sequel will be only the multiplicativeness of the homomorphism Δ: it is used to compute products in $H^*(B(\infty);\ Z_2)$.

§8. A GEOMETRIC LEMMA

The assertion to be demonstrated in this section will be used in the proof of Theorem 9.1.

8.1. An element $\alpha_n \in H^{2^n - 1}(B(2^n);\ Z_2)$ is not representable as a product of elements of positive dimensions.



($m=4,\ n=3$)

Fig. 2

Proof. Suppose $N = 2^n$. We shall begin with the fact that the imbedding in G_N of a smooth compact $(N-1)$-dimensional variety M in which α_n carves out an $(N-1)$dimensional class of cohomologies modulo 2 different from zero.

Draw in the plane **C** a circle of radius 1 with center at the point 0 and mark on it two diametrically opposite points A_1 and A_2. Further draw circles of a small radius ε with centers at points A_1 and A_2 and mark on each of them a pair of diametrically opposite points: on the first, A_{11} and A_{12}, and on the second, A_{21} and A_{22}. Then draw circles of radius ε^2 and center at each of these four points and in each of these mark a pair of diametrically opposite points $A_{111}, \dots, A_{222}$. Producing the construction further, analogously, we obtain 2^n pairwise distinct points $A_{i_1 \ldots i_n}$ (i_s = 1 or 2), i.e., points of a space G_N. The set of all points of the space G_N which can be obtained in this way we denote by $M_n = M$. Clearly M_n is an $(N-1)$-dimensional smooth variety. It is obtained from a product $M_{n-1} \times M_{n-1} \times S^1$ by identifying $(x, y, z) = (y, x, -z)$, i.e., it is represented as a space of smooth fibering with basis S^1 and fibers $M_{n-1} \times M_{n-1}$. When $n > 1$ the variety M_n is nonoriented.

The variety M_n intersects the set $e(2^n) \subset G_N$ in one point (compiled by N pure imaginary complex numbers). This intersection is transversal since the element $\alpha_n = \langle 2^n \rangle \in H^{N-1}(G_N;\ Z_2)$ passes over under the homomorphism indicated by the imbedding $M \to G_N$, to the generator of the group $H^{N-1}(M;\ Z_2) = Z_2$.

It will be enough to prove that the product of arbitrary classes of cohomologies of the variety M with complements of positive dimensions, generated in the form of the homomorphism $H^*(G_N;\ Z_2) \to H^*(M;\ Z_2)$, is equal to zero.

The homologies of the variety M are readily determined from the cellular decomposition or from the chain of spectral sequences. The basis of these homologies is describable as follows. Let F be any subset of the sets consisting of the symbols $\alpha, \alpha_1, \alpha_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{\underbrace{1\ldots1}_{n-1}}, \ldots, \alpha_{\underbrace{2\ldots2}_{n-1}}$. Denote by c(F) the subset of varieties M consisting of points in whose construction there was fulfilled the condition: if $\alpha_{i_1 \ldots i_k} \in F$, then the points $A_{i_1 \ldots i_k 1}$, $A_{i_1 \ldots i_k 2}$ lie on the line parallel to the line $x = y$. It is easy to check that each of the sets c(F) is a cycle (mod 2) of the variety M. Codimensionality of the cycle c(F) in M is equal to the number of elements of the set F. Some of the cycles c(F) coincide: if $\alpha_{i_1 \ldots i_k} \notin F$, then c(F) does not change if in F we replace $\alpha_{i_1 \ldots i_k 1 j_1 \ldots j_l}$ by $\alpha_{i_1 \ldots i_k 2 j_1 \ldots j_l}$ for all $j_1 \ldots j_l$, and conversely. Cycles not passing over into each other under that identification constitute a basis of homologies of M.

The index of the intersection of cycles $c(F_1)$ and $c(F_2)$ is equal to 1 if and only if $c(F_2) = c(F_1^*)$, where F_1^* is the complement of the set F_1. We show that if the sets F and F^* are nonempty, then either $c(F) \sim 0$, or $c(F^*) \sim 0$ in G_N, which will prove our assertion.

The sets c(F) are not all varieties but they can be considered as a kind of variety under smooth mappings. Self intersections can arise just as in the situation where the segment $A_{i_1 \ldots i_k 1} A_{i_1 \ldots i_k 2}$ is parallel to the line $x = y$ although $\alpha_{i_1 \ldots i_k} \notin F$.

In each cycle c(F) there operate two transformations of period 2: the first, for $\alpha_{i_1 \ldots i_k} \notin F$, reflects all points $A_{i_1 \ldots i_k j_1 \ldots j_{n-k}}$ in the horizontal line drawn through the point $A_{i_1 \ldots i_k}$, and the second for $\alpha_{i_1 \ldots i_k} \notin F$ interchanges $A_{i_1 \ldots i_k 1 j_1 \ldots j_{n-k-1}}$ with $A_{i_1 \ldots i_k 2 j_1 \ldots j_{n-k-1}}$. We stress that this transformation can "disperse" the self-intersections of the cycles c(F). The intersections of the set c(F) with the cells of the space G_N are invariant with respect to the first transformation, and those of them which do not move during the first transformation are invariant with respect to the second. Hence, if in the cycle c(F) there are no points passing over into themselves under both transformations then the index of the intersection of the cycle c(F) with any of the cycles chosen by us in G_N is equal to zero, which is to say such cycles are homologous to zero in G_N. We call a set F symmetric if from $\alpha_{i_1 \ldots i_k} \notin F$ follows that $\alpha_{i_1 \ldots i_k 1 j_1 \ldots j_l}$, $\alpha_{i_1 \ldots i_k 2 j_1 \ldots j_l}$ for any $j_1, \ldots, j_l$ either all belong or all do not belong to the set F. It is easy to deduce from the above that if $c(F) \nsim 0$ in G_N then $c(F) = c(F')$, where the set F' is symmetrized. Hence it follows that if $c(F) \nsim 0$ and $c(F^*) \nsim 0$, then the set F is constructed in the following way: if $\alpha_{i_1 \ldots i_k} \in F$, then all α with k indices belong to F. It remains for us to show that if F is a nonempty set satisfying this condition and $\alpha \notin F$, then $c(F) \sim 0$ in G_N.

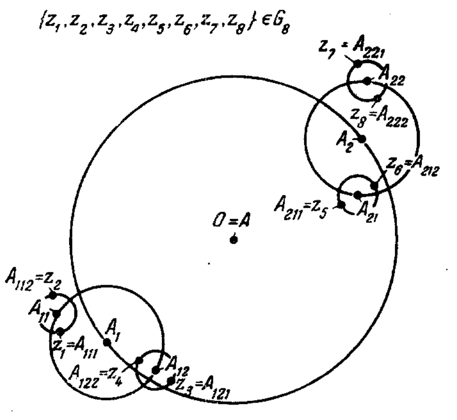


Fig. 3

We first remark that $c(\{\alpha_1, \alpha_2\}) \sim 0$ in G_4. Indeed, this is the one dimensional cycle defining in $\pi_1(G_4) = B_4$ a cross represented in Fig. 4 (we transfer markings from point to point $\left\{\frac{(1+\varepsilon)(1+i)}{\sqrt{2}}, \frac{(1-\varepsilon)(1+i)}{\sqrt{2}}, \frac{(-1+\varepsilon)(1+i)}{\sqrt{2}}, \frac{(-1-\varepsilon)(1+i)}{\sqrt{2}}\right\}$); in $H_1(G_4;\ Z) = Z$ it represents the element 4 and in $H_1(G_4;\ Z_2)$, zero. We fix the chain $b \subset G_4$, to which the cycle $c(\{\alpha_1, \alpha_2\})$ is limited modulo 2.

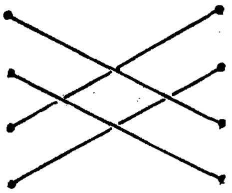


Fig. 4.

If $F=\{\alpha_{\underbrace{1\dots1}_{k-1}},\dots,\alpha_{\underbrace{2\dots2}_{k-1}},\dots\}$ is a set of the considered form then we construct the chain B as follows. The points $A_{i_1\dots i_k}$ are constructed just as in the construction of arbitrary points of the variety M. Four points $A_{i_1\dots i_k11}$, $A_{i_1\dots i_k12}$, $A_{i_1\dots i_k21}$, $A_{i_1\dots i_k22}$ are chosen from which for the lowest of the points $A_{i_1\dots i_k}$ similarly, four points, comprising the chain b (if there is no "lowest" then we take one close to them all), and for the remaining points $A_{i_1\dots i_k}$ four points $A_{i_1\dots i_k11}$, $A_{i_1\dots i_k12}$, $A_{i_2\dots i_k21}$, $A_{i_1\dots i_k22}$ are chosen as in the construction of the cycle c(F). The subsequent construction is conducted as in the construction of the cycle c(F). The set obtained $B\subset G_N$, and it is a chain whose boundary (mod 2) is c(F).

§ 9. MULTIPLICATIVE CONSTRUCTION OF COHOMOLOGIES OF THE GROUP COS

9.1. The following formula holds:

$$\langle\underbrace{2^m,\dots,2^m}_{k_m},\dots,\underbrace{2,\dots,2}_{k_1}\rangle\langle\underbrace{2^m,\dots,2^m}_{l_m},\dots,\underbrace{2,\dots,2}_{l_1}\rangle = C_{k_m+l_m}^{k_m}\dots C_{k_1+l_1}^{k_1}\langle\underbrace{2^m,\dots,2^m}_{k_m+l_m},\dots,\underbrace{2,\dots,2}_{k_1+l_1}\rangle. \tag{5}$$

Proof. It is convenient, instead of the ring $H^*(B(\infty);\ Z_2)$ with multiplier $\mu: H^*(B(\infty);\ Z_2)\otimes H^*(B(\infty);\ Z_2\to H^*(B(\infty);\ Z_2)$ and "comultiplier" $\Delta: H^*(B(\infty);\ Z_2\to H^*(B(\infty);\ Z_2\otimes H^*(B(\infty);\ Z_2)$ to consider the conjugate ring $F=\mathrm{Hom}(H^*(B(\infty);\ Z_2),Z_2)$ with multiplier $\Delta': F\otimes F\to F$ and comultiplier $\mu': F\to F\otimes F$. We consider a basis in F conjugate to the basis in $H^*(B(\infty);Z_2)$, consisting of elements $\langle 2^{l_1},\dots,2^{l_s}\rangle$; the element of the conjugate basis dual to $\langle\underbrace{2^m,\dots,2^m}_{k_m},\dots,\underbrace{2,\dots,2}_{k_1}\rangle$, we denote by $x_1^{k_1}\dots x_m^{k_m}$. This notation is justified by the fact, as follows from Eq. (4), that the mapping Δ' translates $(x_1^{k_1}\dots x_m^{k_m})\otimes(x_1^{l_1}\dots x_m^{l_m})$ to $x_1^{k_1+l_1}\dots x^{k_m+l_m}$, i.e., F actually is a ring of polynomials in the generators $x_1, x_2,\dots$, where moreover the dimension of the generator x_m is equal to 2^m-1. From Assumption 8.1 it follows that

$$\mu'(x_m)=1\otimes x_m+x_m\otimes 1. \tag{6}$$

Indeed, Formula (6) indicates that the class of cohomologies $\langle 2^m\rangle\in H^{2^m-1}(B(\infty);\ Z_2)$ does not reduce to a product of elements of lesser positive dimension (i.e., that all such products present themselves as sums of generators different from $\langle 2^m\rangle$). But since all generators of the group $H^{2^m-1}(B(\infty);\ Z_2)$, except $\langle 2^m\rangle$, pass over into zero under the homomorphism $H^{2^m-1}(B(\infty);\ Z_2\to H^{2^m-1}(B(2^m);\ Z_2)$, it follows that the latter was an indication that the element $\langle 2^m\rangle\in H^{2^m-1}(B(2^m);\ Z_2)$ decomposes into factors of positive dimension, thus contradicting Assertion 8.1. Using the multiplicativeness of the mapping μ', we get from (6):

$$\mu'(x_1^{k_1}\dots x_m^{k_m})=\sum_{0\leqslant j_s\leqslant k_s;\,s=1,\dots,m} C_{k_1}^{j_1}\dots C_{k_m}^{j_m}(x_1^{j_1}\dots x_m^{j_m})\otimes(x_1^{k_1-j_1}\dots x_m^{k_m-j_m}).$$

Formula (5) is deduced from the last equality automatically: it is sufficient to transpose the matrix of the mapping μ'.

9.2. The ring $H^*(B(\infty);\ Z_2)$ is generated by the generators $a_{m,k}=\langle\underbrace{2^m,\dots,2^m}_{2^k}\rangle$. These generators are connected, besides the usual relations of anticommutativity, only by the relation $a_{m,k}^2=0$. The dimensionality of the mapping $a_{m,k}$ is equal to $2^k(2^m-1)$.

This follows from Eq. (5).

9.3. The ring $H^*(B(n);\ Z_2)$ we get from $H^*(B(\infty);\ Z_2)$ covering the complementary relations $a_{m_1,k_1}\dots a_{m_s,k_s}=0$, where

This follows from the previous theorem and from 4.8.

9.4. The homomorphism $H^*(O(\infty);\ Z_2)\to H^*(B(\infty);\ Z_2)$ is an epimorphism.

Proof. It follows from 9.2 that in every dimension the ring $H^*(B(\infty);\ Z_2)$ has not more than one multiplicative generator. If all multiplicative generators having dimension less than dim $a_{m,k}$ are expressed by means of Stifel classes of fiberings ξ_n, then the same would be true for $a_{m,k}$ since the sum of all additive generators of the group $H^{2^k(2^m-1)}(B(\infty);\ Z_2)$ is $W_{2^k(2^m-1)}$ (see section 5.2).

9.5. The homomorphism $H^*(O(n);\ Z_2) \to H^*(B(n);\ Z_2)$ is an epimorphism for every n.

There follows from Theorem 9.4 the commutative diagram

$$\begin{array}{ccc} H^*(O(\infty);\ Z_2) & \to & H^*(B(\infty);\ Z_2) \\ \downarrow & & \downarrow \\ H^*(O(n);\ Z_2) & \to & H^*(B(\infty);\ Z_2) \end{array}$$

and the fact that for $k < n$ the homomorphism $H^k(O(\infty);\ Z_2) \to H^k(O(n);\ Z_2)$ is an isomorphism, and, for $k \geq n$ the group $H^k(B(n);\ Z_2)$ is trivial.

APPENDIX

The cohomologies of the group O(n) have been thoroughly studied and therefore the epimorphic character of the mapping $H^*(O(n);\ Z_2) \to H^*(B(n);\ Z_2)$ permits us to get a great variety of information concerning $H^*(B(n);\ Z_2)$; for example this permits us even to calculate the action of Steenrod squares there. For example, calculating the action of the operation Sq^1 (and this can be done without resort to the group O(n) but using the fact that Sq^1 is a Bokshtein homomorphism), we can also demonstrate the

Proposition. In the group $H^*(B(n);\ Z)$ there are no elements of order 4.

The author is indebted to V. I. Arnol'd for his fruitful attention to the results here set forth.

LITERATURE CITED

1. V. I. Arnol'd, "On some topological invariants of algebraic functions," Trudy Mosk. Matem. Obshch., 21, 28-43 (1970).