THE BRAID GROUP AND OTHER GROUPS

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THE braid group B_{n+1} was first defined by Artin in a paper published in 1926 (**1**). The word problem for the group was solved by Artin (**1, 2**), and the centre was given by Chow (**3**). The present paper incorporates the results of my D.Phil. Thesis (Oxford, November 1965), under the supervision of Professor G. Higman, whose help and advice I acknowledge with gratitude. The primary concern will be to give the solution of the conjugacy problem in B_{n+1}. A new solution of the word problem is also given, and a new method of finding the centre. In the last section a connection is traced between the braid groups and the truncated octahedron and higher dimensional polytopes. Examples are given of further groups connected with other even-faced Archimedean solids and polytopes, which can be dealt with in the same manner as that developed for the braid groups.

1. Positive words

1.1. Definitions and notation

The Braid Group B_{n+1}. We define B_{n+1} as the group generated by $a_1, a_2, \ldots, a_n$ subject to the relations

$$\left.\begin{array}{ll} a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} & (1 \leqslant i \leqslant n-1) \\ a_i a_k = a_k a_i & (|i-k| \geqslant 2) \end{array}\right\}. \tag{1.1}$$

Words. If A, B are words in the generators and their inverses, then $A = B$ will mean that A can be transformed into B by the use of the defining relations, $A \equiv B$ will mean the two words are identical letter by letter, $A \sim B$ will mean A is conjugate to B. A word consisting of an ordered sequence of the generators only, in which no inverse of any generator occurs will be called a *positive word*. We shall denote by $L(W)$ the *word-length* of a word W.

Positively equal. Two positive words A, B will be said to be *positively equal*, if (*a*) they are identically equal, or (*b*) they are transformable into each other through a sequence of positive words, such that each word of the sequence is obtained from the preceding one by a single direct application of the defining relations (1.1), so that at no stage of the transformation does the inverse of any one of the generators occur.

If A is positively equal to B we shall write

$$A \doteqdot B,$$

and such a statement will imply that A and B are both positive.

If $A \doteqdot B$, then $L(A) = L(B)$.

If A is transformed positively into B by a sequence of t single applications of the defining relations (1.1), then the whole transformation will be said to be of *chain-length t*.

Reverse. If $P \equiv x_1 x_2 \dots x_t$ be any word, where each x_i is a generator or its inverse, the x_i being not necessarily distinct, then by the *reverse* of P we shall mean the word $x_t \dots x_2 x_1$. We shall write the reverse of P as rev P, and note that rev PQ = rev Q rev P. It is easily seen that if $P \doteqdot Q$, then rev $P \doteqdot$ rev Q.

1.2. THEOREM H. *In B_{n+1}, for i, $k = 1, 2,\dots, n$, given $a_i X \doteqdot a_k Y$, it follows that*

(i) *if $k = i$, then $X \doteqdot Y$,*

(ii) *if $|k-i| \geqslant 2$, then $X \doteqdot a_k Z$, $Y \doteqdot a_i Z$, for some Z,*

(iii) *if $|k-i| = 1$, then $X \doteqdot a_k a_i Z$, $Y \doteqdot a_i a_k Z$, for some Z.*

The theorem for words X, Y of word-length s will be referred to as H_s. For $s = 0, 1$ the theorem takes the simpler forms which follow trivially from (1.1):

H_0. When X, Y are the empty word

(i) if $a_i X \doteqdot a_i Y$, then $X \doteqdot Y$ ($i = 1, 2,\dots, n$),

(ii) if $i \neq k$, then $a_i X$ cannot be positively equal to $a_k Y$.

H_1. When X, Y are of word-length 1, for $i = 1, 2,\dots, n$,

(i) if $a_i X \doteqdot a_i Y$, then $X \doteqdot Y$,

(ii) if $a_i X \doteqdot a_k Y$ ($|k-i| \geqslant 2$), then $X \equiv a_k$, $Y \equiv a_i$,

(iii) if $|k-i| = 1$, then $a_i X$ cannot be positively equal to $a_k Y$.

The proof of the general theorem now follows by induction. For our induction hypothesis we assume

(α) H_s is true for $0 \leqslant s \leqslant r$ for transformations of all chain-lengths, and

(β) H_{r+1} is true for all chain-lengths $\leqslant t$.

Let X, Y be of word-length $r+1$, and let $a_i X \doteqdot a_k Y$ through a transformation of chain-length $t+1$. Let the successive words of the transformation be

$$W_1 \equiv a_i X, \quad W_2 \equiv \dots, \quad W_{t+2} \equiv a_k Y.$$

Choose arbitrarily any intermediate word $W_g \equiv a_m W$, say, from the middle of the chain somewhere. The transformations $a_i X \to a_m W$, $a_m W \to a_k Y$ are each of chain-length $\leqslant t$, and we can therefore apply (β) to them. We have then

$$a_i X \doteqdot a_m W \doteqdot a_k Y. \tag{1.2}$$

For the complete proof (8) we have to consider all possible variations in the values of i, m, k. The general pattern of the proof is, however, exactly the same for each variation, and it will be sufficient here to deal with two cases only, as typical examples of the common method of proof.

(1) $k = i$, $|m-i| \geqslant 2$. From (1.2) we have

$$a_i X \doteqdot a_m W; \qquad a_m W \doteqdot a_i Y \quad (|m-i| \geqslant 2).$$

By (β) $\quad X \doteqdot a_m P, \quad W \doteqdot a_i P$ for some P;

and $\quad W \doteqdot a_i Q, \quad Y \doteqdot a_m Q$ for some Q.

The two expressions for W give $a_i P \doteqdot a_i Q$, and hence by (α), $P \doteqdot Q$. Hence $X \doteqdot a_m P \doteqdot a_m Q \doteqdot Y$ as required.

(2) $|k-i| \geqslant 2$, $|m-i| \geqslant 2$, $|k-m| = 1$. From (1.2) we have

$$a_i X \doteqdot a_m W; \qquad a_m W \doteqdot a_k Y.$$

By (β) $\quad X \doteqdot a_m P, \quad W \doteqdot a_i P \quad$ for some P;

and $\quad W \doteqdot a_k a_m Q, \quad Y \doteqdot a_m a_k Q$ for some Q.

By (α) the two expressions for W give

$$P \doteqdot a_k R, \qquad a_m Q \doteqdot a_i R \quad \text{for some } R.$$

The last equation now gives

$$Q \doteqdot a_i S, \qquad R \doteqdot a_m S \quad \text{for some } S.$$

Therefore $\quad X \doteqdot a_m a_k a_m S, \quad Y \doteqdot a_m a_k a_i S.$

Hence, using the defining relations, we have

$$X \doteqdot a_k a_m a_k S, \qquad Y \doteqdot a_m a_i a_k S \doteqdot a_i a_m a_k S,$$

i.e. $\quad X \doteqdot a_k Z, \quad Y \doteqdot a_i Z$ as required, where $Z \equiv a_m a_k S$.

The proofs for other variations in the values of i, m, k are similar.

Since H_{r+1} is true for chain length 1, an induction proves it for all chain lengths, and a further induction (on r) completes the proof of the theorem.

THEOREM K. *In B_{n+1}, for $i, k = 1, 2,..., n$, given $Xa_i \doteqdot Ya_k$, it follows that*

(i) *if $k = i$, then $X \doteqdot Y$,*

(ii) *if $|k-i| \geqslant 2$, then $X \doteqdot Za_k$, $Y \doteqdot Za_i$, for some Z,*

(iii) *if $|k-i| = 1$, then $X \doteqdot Za_i a_k$, $Y \doteqdot Za_k a_i$, for some Z.*

The theorem follows from Theorem H and the fact that $X \doteq Y$ implies $\operatorname{rev} X \doteq \operatorname{rev} Y$.

As an immediate consequence of Theorems H (i), K (i) follows

THEOREM 1. *In* B_{n+1}, *if* $A \doteq P$, $B \doteq Q$, $AXB \doteq PYQ$ ($L(A) \geqslant 0$, $L(B) \geqslant 0$), *then* $X \doteq Y$.

2. The fundamental word Δ

2.1. Definitions and notation

The word $a_r a_{r+1} \dots a_s$ ($a_r a_{r-1} \dots a_s$), where all the generators from a_r to a_s inclusive occur in ascending (descending) sequence will be denoted by the abbreviation $(a_r \dots a_s)$. By the notation Π_s we shall mean the word $(a_1 \dots a_s)$.

In B_{n+1}, if $\mathfrak{R}$ is the mapping of $(a_1, a_2, \dots, a_n)$ onto itself given by $\mathfrak{R}a_i = a_{n+1-i}$, then by inspection of the relations $\mathfrak{R}$ extends to an automorphism of B_{n+1}. This automorphism we continue to denote by $\mathfrak{R}$, and call it *reflection in* B_{n+1}. We note that if $P \doteq Q$, then $\mathfrak{R}P \doteq \mathfrak{R}Q$.

Associated with the ordered sequence of generators $a_1, a_2, \dots, a_r$ is the word

$$\Delta_r \equiv \Pi_r \Pi_{r-1} \dots \Pi_1,$$

which is of fundamental importance in what follows. We shall refer to Δ_r as the *fundamental word of order* $r+1$. When we are considering B_{n+1} we shall normally abbreviate Δ_n to the simpler form Δ.

LEMMA 1. *In* B_{n+1}, *for* $1 < s \leqslant t \leqslant n$, $a_s \Pi_t \doteq \Pi_t a_{s-1}$.

For by use of the defining relations

$$\begin{aligned} a_s \Pi_t &\equiv a_s (a_1 \dots a_{s-2}) a_{s-1} a_s (a_{s+1} \dots a_t) \\ &\doteq (a_1 \dots a_{s-2}) a_s a_{s-1} a_s (a_{s+1} \dots a_t) \\ &\doteq (a_1 \dots a_{s-2}) a_{s-1} a_s a_{s-1} (a_{s+1} \dots a_t) \\ &\doteq (a_1 \dots a_{s-2}) a_{s-1} a_s (a_{s+1} \dots a_t) a_{s-1} \\ &\doteq \Pi_t a_{s-1} \quad \text{as required.} \end{aligned}$$

LEMMA 2. *In* B_{n+1} (i) $a_1 \Delta_t \doteq \Delta_t a_t$ ($t = 1, 2, \dots, n$); (ii) $a_s \Delta \doteq \Delta \mathfrak{R} a_s$, (iii) $a_s^{-1} \Delta = \Delta (\mathfrak{R} a_s)^{-1}$, (iv) $a_s \Delta^{-1} = \Delta^{-1} \mathfrak{R} a_s$, (v) $a_s^{-1} \Delta^{-1} = \Delta^{-1} (\mathfrak{R} a_s)^{-1}$ ($s = 1, 2, \dots, n$).

(i) For $t = 1$,

$$a_1 \Delta_1 \equiv a_1 a_1 \doteq \Delta_1 a_1 \quad \text{as required.}$$

For $t = 2, 3, \ldots, n$,

$$\begin{aligned} a_1 \Delta_t &\equiv a_1\{\Pi_t\}(a_1 \ldots a_{t-1})\Delta_{t-2} \\ &\doteq a_1(a_2 \ldots a_t)\{\Pi_t\}\Delta_{t-2}, \quad \text{by Lemma 1,} \\ &\doteq \Pi_t\{\Pi_{t-1} a_t\}\Delta_{t-2} \\ &\doteq \Pi_t \Pi_{t-1} \Delta_{t-2} a_t \\ &\doteq \Delta_t a_t \quad \text{as required.} \end{aligned}$$

(ii) For $s = 1$, by (i) above,

$$a_1 \Delta \doteq \Delta a_n \equiv \Delta \Re a_1 \quad \text{as required.}$$

For $s = 2, 3, \ldots, n$,

$$\begin{aligned} a_s \Delta &\equiv a_s \Pi_n \Pi_{n-1} \ldots \Pi_{n-s+2} \Delta_{n-s+1} \\ &\doteq \Pi_n \Pi_{n-1} \ldots \Pi_{n-s+2} a_1 \Delta_{n-s+1}, \quad \text{by Lemma 1,} \\ &\doteq \Pi_n \Pi_{n-1} \ldots \Pi_{n-s+2} \Delta_{n-s+1} a_{n-s+1}, \quad \text{by (i),} \\ &\equiv \Delta a_{n-s+1}, \end{aligned}$$

i.e. $\quad a_s \Delta \doteq \Delta \Re a_s \quad$ as required.

(iii), (iv), and (v) follow easily from (ii).

THEOREM 2. *In* B_{n+1},

(i) $P\Delta^{2m} \doteq \Delta^{2m}P$, $P\Delta^{2m+1} \doteq \Delta^{2m+1}\Re P$ *for all positive words* P $(m \geqslant 0)$,

(ii) $Q\Delta^{2m} = \Delta^{2m}Q$, $Q\Delta^{2m+1} = \Delta^{2m+1}\Re Q$ *for all words* Q, m *positive or negative.*

This follows immediately from repeated applications of Lemma 2, remembering that $\Re^2 P \equiv P$, $\Re^2 Q \equiv Q$.

2.2. LEMMA 3. *In* B_{n+1}, (i) $\Re\Delta \doteq \Delta$, (ii) $\operatorname{rev}\Delta \doteq \Delta$.

(i) By Theorem 2,

$$(\Re\Delta)\Delta \doteq \Delta\Re(\Re\Delta) \doteq \Delta\Delta.$$

Hence, by Theorem 1,

$$\Re\Delta \doteq \Delta \quad \text{as required.}$$

(ii) The proof is by induction. Assume that for any particular r that $\operatorname{rev}\Delta_r \doteq \Delta_r$. Then

$$\begin{aligned} \operatorname{rev}\Delta_{r+1} &\equiv \operatorname{rev}\{(a_1 \ldots a_{r+1})\Delta_r\} \\ &\equiv \operatorname{rev}\Delta_r \operatorname{rev}(a_1 \ldots a_{r+1}) \\ &\doteq \Delta_r(a_{r+1} \ldots a_1), \quad \text{using the induction hypothesis,} \end{aligned}$$

i.e. $\quad \operatorname{rev}\Delta_{r+1} \doteq \Pi_r \Pi_{r-1} \ldots \Pi_1(a_{r+1} \ldots a_1)$.

Now a_{r+1} commutes with $\Pi_1, \Pi_2, \ldots, \Pi_{r-1}$; a_r commutes with $\Pi_1, \Pi_2, \ldots, \Pi_{r-2}$; ..., etc. Hence

$$\operatorname{rev} \Delta_{r+1} \doteq \Pi_r a_{r+1} \Pi_{r-1} a_r \ldots \Pi_2 a_3 \Pi_1 a_2 a_1 \equiv \Delta_{r+1}.$$

The induction is now established, since the hypothesis is clearly true for $r = 1$, and the result follows.

LEMMA 4. *In B_{n+1} there exist positive words X_r, Y_r such that*

$$a_r X_r \doteq \Delta \doteq Y_r a_r \quad (r = 1, 2, \ldots, n).$$

By definition $\Delta \equiv \Pi_n \Pi_{n-1} \ldots \Pi_2 \Pi_1$,

i.e. $$\Delta \doteq Y_1 a_1, \quad \text{where} \quad Y_1 \equiv \Pi_n \Pi_{n-1} \ldots \Pi_2. \tag{2.1}$$

We now observe that if $f(a_2, a_3, \ldots, a_t)$ is any positive word involving the generators $a_2, a_3, \ldots, a_t$ only, then by Lemma 1

$$\Pi_t f(a_1, a_2, \ldots, a_{t-1}) \doteq f(a_2, a_3, \ldots, a_t) \Pi_t.$$

Let a_t be any particular one of the generators $a_2, a_3, \ldots, a_n$. Then, denoting $\Pi_{t-1} \Pi_{t-2} \ldots \Pi_1$ by $f(a_1, a_2, \ldots, a_{t-1})$, we have

$$\begin{aligned} \Delta &\equiv \Pi_n \Pi_{n-1} \ldots \Pi_{t+1} \Pi_t f(a_1, a_2, \ldots, a_{t-1}) \\ &\doteq \Pi_n \Pi_{n-1} \ldots \Pi_{t+1} f(a_2, a_3, \ldots, a_t) \Pi_t \\ &\doteq \Pi_n \Pi_{n-1} \ldots \Pi_{t+1} f(a_2, a_3, \ldots, a_t)(a_1 \ldots a_{t-1}) a_t \\ &\equiv Y_t a_t, \quad \text{say.} \end{aligned} \tag{2.2}$$

(2.1) and (2.2) show that words Y_r exist for $r = 1, \ldots, n$. Now putting $X_r = \operatorname{rev} Y_r$, we have, for $r = 1, 2, \ldots, n$,

$$a_r X_r \equiv a_r \operatorname{rev} Y_r \equiv \operatorname{rev}(Y_r a_r) \doteq \operatorname{rev} \Delta \doteq \Delta, \quad \text{by Lemma 3.}$$

Hence words X_r also exist, and the proof is complete.

COROLLARY. *In B_{n+1}, if A is any positive word, then for $r = 1, 2, \ldots, n$, there exist words A_r such that $\Delta A \doteq A_r a_r$.*

For $$\Delta A \doteq (\Re A)\Delta \doteq (\Re A) Y_r a_r \equiv A_r a_r, \quad \text{say.}$$

LEMMA 5. *Let a_i be any one of the n generators in B_{n+1}, and let $x_1, x_2, \ldots, x_t$ be generators, not necessarily distinct, such that each x_r permutes with a_i. Then, if $a_i P \doteq x_1 x_2 \ldots x_t Q$, there exists some R such that $Q \doteq a_i R$.*

We have $a_i P \doteq x_1 x_2 \ldots x_t Q$. Hence, by making successive applications of Theorem H (ii), we have $x_2 x_3 \ldots x_t Q \doteq a_i R_2$ for some R_2; $x_3 x_4 \ldots x_t Q \doteq a_i R_3$ for some R_3; ...; $x_t Q \doteq a_i R_t$ for some R_t; and finally $Q \doteq a_i R$ for some R, as required.

LEMMA 6. *In B_{n+1}, if $a_{i+1} P \doteq \Pi_i Q$, then $Q \doteq a_{i+1} a_i R$, for some R ($i = 1, 2, \ldots, n-1$).*

By hypothesis $a_{i+1} P \doteqdot a_1 a_2 \dots a_i Q$ and hence, by Lemma 5,

$$a_i Q \doteqdot a_{i+1} T$$

for some T. Hence, by Theorem H (iii), it follows that $Q \doteqdot a_{i+1} a_i R$ for some R, as required.

THEOREM 3. *If W is any positive word in B_{n+1} such that either*

$$\text{(i)} \quad W \doteqdot a_1 X_1 \doteqdot a_2 X_2 \doteqdot \dots \doteqdot a_n X_n,$$

or

$$\text{(ii)} \quad W \doteqdot Y_1 a_1 \doteqdot Y_2 a_2 \doteqdot \dots \doteqdot Y_n a_n,$$

then $W \doteqdot \Delta Z$ for some Z.

(i) The proof is by induction. Let r be any natural number $\leqslant n-1$. Then as our induction hypothesis we assume that, in B_{n+1}, if

$$W \doteqdot a_1 X_1 \doteqdot a_2 X_2 \doteqdot \dots \doteqdot a_r X_r,$$

then $W \doteqdot \Delta_r P_r$ for some P_r. Now suppose that

$$W \doteqdot a_1 X_1 \doteqdot a_2 X_2 \doteqdot \dots \doteqdot a_r X_r \doteqdot a_{r+1} X_{r+1}. \tag{2.3}$$

Then from (2.3) and the induction hypothesis it follows that

$$a_{r+1} X_{r+1} \doteqdot W \doteqdot \Delta_r P_r \equiv (a_1 \dots a_r) \Delta_{r-1} P_r.$$

Hence, by Lemma 6,

$$\Delta_{r-1} P_r \doteqdot a_{r+1} a_r Q_r \quad \text{for some } Q_r,$$

so that

$$W \doteqdot (a_1 \dots a_r) a_{r+1} a_r Q_r,$$

or, putting

$$T \equiv a_r Q_r \tag{2.4}$$

$$W \doteqdot (a_1 \dots a_{r+1}) T \equiv \Pi_{r+1} T. \tag{2.5}$$

From (2.3) and (2.5) we now have, for $i = 1, 2, \dots, r-1$,

$$a_{i+1} X_{i+1} \doteqdot (a_1 \dots a_i)(a_{i+1} \dots a_{r+1}) T,$$

so that, by Lemma 6,

$$(a_{i+1} \dots a_{r+1}) T \doteqdot a_{i+1} a_i S_i, \quad \text{for some } S_i.$$

Therefore, by Theorem 1,

$$(a_{i+2} \dots a_{r+1}) T \doteqdot a_i S_i.$$

Applying Lemma 5 it follows that for some Q_i

$$T \doteqdot a_i Q_i \quad (i = 1, 2, \dots, r-1). \tag{2.6}$$

From (2.4), (2.6), and the induction hypothesis, it now follows that

$$T \doteqdot \Delta_r P_{r+1}, \quad \text{for some } P_{r+1},$$

and hence, by (2.5)

$$W \doteqdot \Pi_{r+1} \Delta_r P_{r+1} \equiv \Delta_{r+1} P_{r+1}.$$

Remarking that the induction hypothesis is clearly true for $r = 1$, the induction is now established, and the result follows.

(ii) Now suppose

$$W \doteq Y_1 a_1 \doteq Y_2 a_2 \doteq \ldots \doteq Y_n a_n.$$

Then

$$\operatorname{rev} W \doteq a_1 \operatorname{rev} Y_1 \doteq a_2 \operatorname{rev} Y_2 \doteq \ldots \doteq a_n \operatorname{rev} Y_n \doteq \Delta P, \quad \text{by (i).}$$

Hence

$$W \doteq \operatorname{rev} P \operatorname{rev} \Delta \doteq (\operatorname{rev} P)\Delta, \quad \text{by Lemma 3,}$$

$$\doteq \Delta\mathfrak{R}(\operatorname{rev} P), \quad \text{by Theorem 2,}$$

and the result follows.

2.3. Lemma 7. *If X, Y are any two positive words in B_{n+1}, then there exist words U, V such that $UX \doteq VY$.*

For let $X \equiv r_1 r_2 \ldots r_l$, $Y \equiv s_1 s_2 \ldots s_m$ be any two positive words, where the r_i and s_i are generators, not necessarily distinct. Then, by repeated application of the Corollary to Lemma 4,

$$\Delta^m X \doteq \Delta^{m-1} A_1 s_m \doteq \Delta^{m-2} A_2 s_{m-1} s_m \doteq \ldots \doteq A_m Y.$$

The result follows on putting $U \equiv \Delta^m$, $V \equiv A_m$.

Theorem 4. *In B_{n+1} if two positive words are equal they are positively equal.*

Let S be the semi-group generated by $a_1, a_2, \ldots, a_n$ subject to the relations

$$\left.\begin{array}{ll} a_i a_{i+1} a_i \doteq a_{i+1} a_i a_{i+1} & (1 \leqslant i \leqslant n-1) \\ a_i a_k \doteq a_k a_i & (|i-k| \geqslant 2) \end{array}\right\}. \tag{2.7}$$

By Theorem 1 and Lemma 7, S is cancellative and right-reversible, and hence, by Öre's Theorem (**4**, **5**), can be embedded in a group, G_{n+1}, say. Let $\bar{G}_{n+1}$ be the subgroup of G_{n+1} generated by $a_1, a_2, \ldots, a_n$. Then $\bar{G}_{n+1}$ embeds S, and in virtue of (2.7) its relations include

$$\left.\begin{array}{ll} a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} & (1 \leqslant i \leqslant n-1) \\ a_i a_k = a_k a_i & (|i-k| \geqslant 2) \end{array}\right\}, \qquad [(1.1)]$$

which are precisely the relations of B_{n+1}.

Now suppose X, Y are any two equal positive words in B_{n+1}. The equality $X = Y$ in B_{n+1} must be a consequence of the relations (1.1). These are also relations in $\bar{G}_{n+1}$, and hence $X = Y$ in $\bar{G}_{n+1}$. Since $\bar{G}_{n+1}$ embeds S, $X = Y$ in S also, i.e. $X \doteq Y$, and the theorem is proved.

3. Cayley diagrams

3.1. Any group G with given generators and defining relations can be represented in a drawn diagram, called its *Cayley diagram* (**6, 7**). In the sequel, although all the proofs given will be purely algebraic, considerable use will be made of the general concept of the Cayley diagram, and in one or two instances actual diagrams will be drawn. In order to preserve algebraic rigour we proceed to make certain formal definitions.

3.2. Definitions and notation

Links. The successive generators of a positive word will be called ***links***. Thus the initial link of the word $a_2 a_1 a_4 a_3$ is a_2; the third link is a_4; etc. In the drawn diagram the link a_r will be represented as

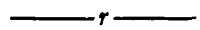

FIG. 1

No arrow will be put in, as it will be understood always that the positive direction is left to right. The drawn figure will show the initial link on the left, the successive other links extending in order to the right.

Diagram. Let W be any positive word, and let $W, W_1, W_2, \ldots, W_m$ be the complete set of distinct words which are positively equal to W (see Lemma 8). Then we shall refer to this set as the *diagram of* W, and write it $D(W)$. Clearly $D(W) \equiv D(W_1) \equiv \ldots \equiv D(W_m)$. The words $W, W_1, \ldots, W_m$ will be called the *routes* of $D(W)$. The process of enumerating the routes of $D(W)$ will be called *drawing the diagram* $D(W)$. In the drawn figure the diagram of W is the Cayley diagram of all words positively equal to W. The name Cayley will be omitted from now on.

Nodes of $D(W)$. Let W be any positive word, and $D(W)$ its diagram. If A, X are any two positive words such that $W \doteq AX$ $(0 \leqslant L(A), L(X))$, then we shall call $D(A)$ a *node* of $D(W)$. When we are considering nodes we shall frequently write the node $D(A)$ as the node $\dot{A}$, or simply $\dot{A}$. If $L(A) = t$ we shall say the node $\dot{A}$ is of *order* t.

Sub-routes of $D(W)$. If $W \doteq AXB$ $(L(A) \geqslant 0,\ L(B) \geqslant 0)$, we shall say that X is a *sub-route of* $D(W)$. If $L(A) = 0$, we shall say X is an *initial sub-route of* $D(W)$. If $W \doteq PXQ$ $(L(P) \geqslant 0, L(Q) \geqslant 0)$, we shall say the sub-route X *starts* at $\dot{P}$. If $W \doteq RQ \doteq PXQ$ we shall say the sub-route X *ends* at $\dot{R}$.

Incidence. If the link a_r either (i) starts at $\dot{P}$, or (ii) ends at $\dot{P}$, we shall say the link a_r is *incident* at $\dot{P}$. If the links a_r, a_s are both incident at $\dot{P}$, we shall say they *meet* at $\dot{P}$. We shall also say that $\dot{P}$ is the *meet* of the links a_r, a_s. If a link a_r ends at $\dot{P}$ and a link a_r starts at $\dot{P}$, we shall say the link a_r is *repeated* at $\dot{P}$.

W contains Δ. *W is prime to* Δ. If any sub-route of $D(W)$ is Δ, i.e. if $W \doteqdot A\Delta B$ ($L(A) \geqslant 0$, $L(B) \geqslant 0$), we shall say Δ is a *factor* of W, or simply W *contains* Δ. It follows from Theorem 2 that if W contains Δ, then $W \doteqdot \Delta X$ for some X. If W is any positive word which does not contain Δ, we shall say W is *prime to* Δ.

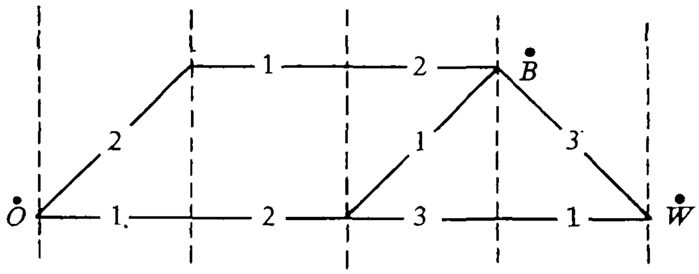

FIG. 2

Base of $D(W)$. In B_{n+1} suppose W is of word-length L, and suppose $D(W)$ consists of the t words $W_1 \equiv a_i a_j a_k \ldots$, $W_2 \equiv a_p q_q a_r \ldots$, ..., $W_t \equiv a_x a_y a_z \ldots$. Then there is a one to one correspondence between the words W_1, W_2,..., W_t and the set of numbers $P_1 \equiv ijk\ldots$, $P_2 \equiv pqr\ldots$, ..., $P_t \equiv xyz\ldots$, where each number P is expressed in the scale of $n+1$, and consists of L digits. The numbers P are all distinct. Suppose the smallest is P_r. Then the corresponding word W_r, which is uniquely defined, will be called the *base* of $D(W)$. If A is a positive word prime to Δ, we shall sometimes denote the base of A by $\bar{A}$. The use of this notation will imply that A is positive and prime to Δ.

Example. We proceed to give an example to illustrate the correspondence between these definitions and the drawn figure. In B_4 consider the word $W \equiv a_1 a_2 a_3 a_1$. The drawn diagram of $D(W)$ is shown in Fig. 2.

Algebraically, $D(W)$ is the set $a_1 a_2 a_1 a_3$, $a_1 a_2 a_3 a_1$, $a_2 a_1 a_2 a_3$. The node $\dot{O}$, of order 0, is the empty set. The node $\dot{B}$, of order 3, is the set $a_1 a_2 a_1$, $a_2 a_1 a_2$. The links a_2, a_1 end at $\dot{B}$. The links a_1, a_2, a_3 are incident at $\dot{B}$. $\overline{W}$, the base of $D(W)$, is $a_1 a_2 a_1 a_3$ ..., etc.

LEMMA 8. *The diagram of any positive word W in B_{n+1} can be systematically drawn, and is finite.*

Let the set of all distinct words positively equal to W through a transformation of chain-length 1 be W_1,..., W_t. It is clear that this set can be enumerated, and is finite. Now consider the set of words positively equal to W_1 through a transformation of chain-length 1. Denote those which are distinct from $W, W_1, \ldots, W_t$ and from each other, by $W_{t+1}, W_{t+2}, \ldots$. Continue to repeat the process successively for $W_2, W_3, \ldots, W_{t+2}, \ldots$, etc. Clearly the number of positive words of word-length equal to $L(W)$ is finite, and hence the set of words positively equal to W is finite. Hence the sequence $W, W_1, \ldots$ ultimately terminates. It is clear that any word which is positively equal to W must ultimately be reached through the process outlined above, and the lemma is proved.

3.3. Solution of the word problem

THEOREM 5. *In B_{n+1} every word W can be expressed uniquely in the form $\Delta^m \bar{A}$.*

(i) First suppose P is any positive word. From the set $D(P)$ select any route starting with as many consecutive sub-routes Δ as possible, equal to t, say ($t \geqslant 0$). Suppose $P \doteqdot \Delta^t A$. Then A is prime to Δ, as otherwise there would be a route of $D(P)$ starting with more than t consecutive sub-routes Δ. Denoting the base of A by $\bar{A}$, we have $P \doteqdot \Delta^t \bar{A}$.

(ii) Now let W be any word in B_{n+1}. Then clearly we may put

$$W \equiv W_1(x_1)^{-1}W_2(x_2)^{-1}\ldots(x_s)^{-1}W_{s+1},$$

where each W_r is a positive word of word-length $\geqslant 0$, and the x_r are generators. Now for each x_r there exists, by Lemma 4, a positive word X_r such that $x_r X_r \doteqdot \Delta$, so that $(x_r)^{-1} = X_r \Delta^{-1}$, and hence

$$W = W_1 X_1 \Delta^{-1} W_2 X_2 \Delta^{-1} \ldots W_s X_s \Delta^{-1} W_{s+1}.$$

Hence, moving the factors Δ^{-1} to the left by Theorem 2, we have $W = \Delta^{-s} P$, where P is positive. Now using (i) above to express P in the form $\Delta^t \bar{A}$, we have $W = \Delta^{-s}\Delta^t \bar{A}$, or, putting $t-s = m$,

$$W = \Delta^m \bar{A}. \tag{3.1}$$

(iii) It now merely remains to show that the form (3.1) is unique. Suppose

$$\Delta^m \bar{A} = \Delta^p \bar{B}. \tag{3.2}$$

First suppose $p < m$, and let $m-p = t$, where $t > 0$. Then (3.2) gives $\Delta^t \bar{A} = \bar{B}$ and hence, by Theorem 4, $\Delta^t \bar{A} \doteqdot \bar{B}$. Hence $\bar{B}$ contains Δ, which is impossible. Therefore $p \nless m$, and similarly $m \nless p$. Hence $p = m$, and from (3.2) we now have $\bar{A} = \bar{B}$, and on using Theorem 4, $\bar{A} \doteqdot \bar{B}$. But any positive word has one and only one base. Hence $\bar{A} \equiv \bar{B}$, and the uniqueness of the form (3.1) is established.

Definitions. Any word W of B_{n+1} expressed in the unique form $\Delta^m \bar{A}$ of Theorem 5 will be said to be in *standard form*. The index m will be called the *power* of W.

THEOREM 6. *The necessary and sufficient condition that two words in B_{n+1} are equal is that their standard forms are identical.*

The condition is clearly sufficient. The necessity has been shown in section (iii) of the proof of Theorem 5.

3.4. The centre of B_{n+1}

THEOREM 7. (i) *When $n = 1$ the centre of B_{n+1} is generated by Δ.* (ii) *When $n > 1$ the centre of B_{n+1} is generated by Δ^2.* (3)

(i) This case is trivial.

(ii) Let W be any word in the centre. Then, by the definition of centre, if X is any word in B_{n+1}, $X^{-1}WX = W$, so that

$$WX = XW. \tag{3.3}$$

There are three possible forms for W: (*a*) $W = \Delta^p \bar{A}$, where $L(\bar{A}) > 0$; (*b*) $W = \Delta^{2m+1}$; (*c*) $W = \Delta^{2m}$. We proceed to consider each in turn.

(*a*) $W = \Delta^p \bar{A}$ $(L(\bar{A}) > 0)$.

Let $\bar{A} \doteq a_i A_i$ $(L(A_i) \geqslant 0)$. Let $|s-i| = 1$. Considering first the case p even, put $X \equiv a_s a_i$. Then (3.3) gives

$$\Delta^p a_i A_i a_s a_i = a_s a_i \Delta^p a_i A_i = \Delta^p a_s a_i a_i A_i.$$

Hence $a_i A_i a_s a_i = a_s a_i a_i A_i$. Applying Theorem 4,

$$a_i A_i a_s a_i \doteq a_s a_i a_i A_i,$$

and hence by Theorem H, $a_i a_i A_i \doteq a_i a_s A_s$ for some A_s, so that by Theorem 1

$$a_i A_i \doteq a_s A_s. \tag{3.4}$$

The case p odd gives exactly the same result on putting $X = \Re(a_s a_i)$. Repeated application of (3.4) now gives

$$a_1 A_1 \doteq a_2 A_2 \doteq \dots \doteq a_n A_n \doteq \bar{A}.$$

Hence by Theorem 3, $\bar{A}$ contains Δ, which is impossible. Therefore there are no words in the centre of the form (*a*).

(*b*) $W = \Delta^{2m+1}$.

Putting $X \equiv a_1$, (3.3) gives $\Delta^{2m+1} a_1 = a_1 \Delta^{2m+1} = \Delta^{2m+1} \Re a_1$ by Theorem 2. Hence $a_1 = \Re a_1$, which is impossible since $n > 1$. Therefore there are no words in the centre of the form (*b*).

(*c*) $W = \Delta^{2m}$.

Clearly $X^{-1}WX = W$ for all words X in virtue of Theorem 2. Hence any word of the form Δ^{2m} is in the centre, and no other words, i.e. the centre of B_{n+1} is generated by Δ^2.

3.5. The structure of $D(\Delta)$

THEOREM 8. *In B_{n+1}, if $W \doteq \Delta V$ is any positive word containing Δ, then each of the n links a_r $(r = 1, 2,\ldots, n)$ is incident at each node of $D(\Delta)$.*

By Lemma 4, $W \doteq a_1 W_1 \doteq a_2 W_2 \doteq \ldots \doteq a_n W_n$, so the theorem is certainly true for the initial node $\dot{O}$. The proof of the theorem will be by induction. As our induction hypothesis we assume the theorem is true for all nodes of $D(W)$ of order $\leqslant m$. Let $\dot{C}$ be any node of order m, and let a_s be any link of the diagram starting at $\dot{C}$ and ending at $\dot{D}$.

(*a*) We first consider the links a_i, where $|i-s| \geqslant 2$. By the induction hypothesis $D(W)$ includes either (i), a link a_i ending at $\dot{C}$, or (ii), a link a_i starting at $\dot{C}$, or (iii), both (i) and (ii) are true.

(i) a_i ends at $\dot{C}$ $(|i-s| \geqslant 2)$. The diagram $D(W)$ includes Fig. 3. By the defining relations this implies Fig. 4, i.e. $D(W)$ includes a link a_i ending at $\dot{D}$.

(ii) a_i starts at $\dot{C}$ $(|i-s| \geqslant 2)$. The diagram $D(W)$ includes Fig. 5. By Theorem H this implies Fig. 6, i.e. $D(W)$ includes a link a_i starting at $\dot{D}$.

(iii) If (i) and (ii) are both true $D(W)$ must include both a link a_i ending at $\dot{D}$, and a link a_i starting at $\dot{D}$.

Hence in all cases, for $|i-s| \geqslant 2$, at least one link a_i is incident at $\dot{D}$.

(*b*) It remains to consider the links a_t, where $|t-s| = 1$. The proof will be omitted. It follows the same pattern as (*a*) above. In all cases, if $|t-s| = 1$, at least one link a_t is incident at $\dot{D}$.

Now by hypothesis there is a link a_s ending at $\dot{D}$. Hence, by (*a*) and (*b*) together, we see that the n links a_r $(r = 1, 2,\ldots, n)$ are incident at $\dot{D}$. The induction is now established, and the result follows.

THEOREM 9. *In B_{n+1} every node of $D(\Delta)$ is the meet of the n links $a_1, a_2,\ldots, a_n$. Furthermore only n links are incident at each node.*

By Theorem 8 it follows at once that each node of $D(\Delta)$ is the meet of the n links $a_1, a_2,\ldots, a_n$. It therefore remains only to prove that we cannot have a repeated link at any node. For suppose the contrary is true, so that for some A, r, B we have $\Delta \doteq A a_r a_r B$. Then

$$A a_r a_r B \mathfrak{R} A \doteq \Delta \mathfrak{R} A = A\Delta,$$

by Theorem 2. Hence

$$a_r a_r X \doteq \Delta, \tag{3.5}$$

where $X \equiv B\mathfrak{R}A$. Using Lemma 4 then, (3.5) gives

$$a_r a_r X \doteqdot a_1 A_1 \doteqdot \dots \doteqdot a_{r-1} A_{r-1} \doteqdot a_{r+1} A_{r+1} \doteqdot \dots \doteqdot a_n A_n,$$

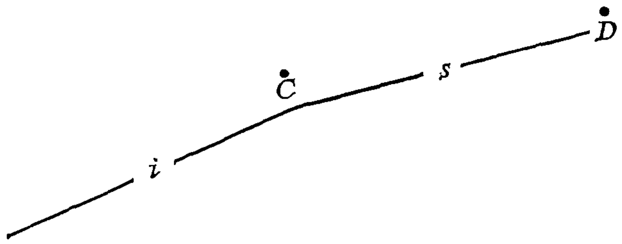

Fig. 3

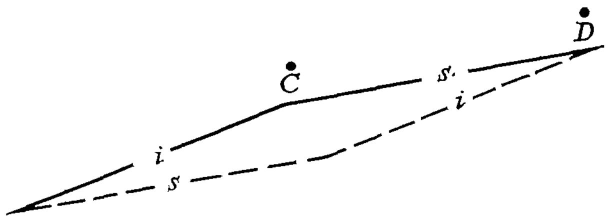

Fig. 4

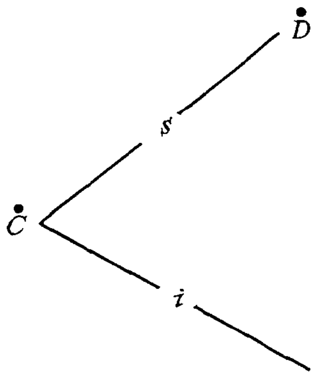

Fig. 5

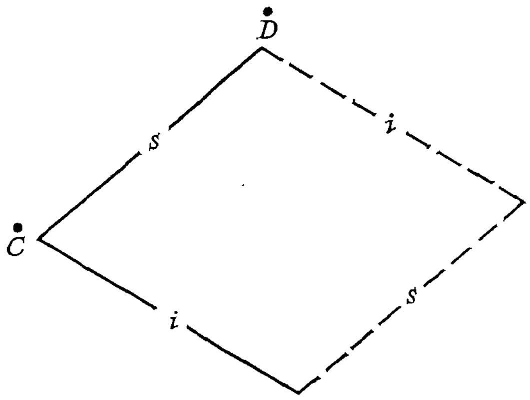

Fig. 6

and Theorem H now gives

$$a_r X \doteqdot a_1 B_1 \doteqdot \dots \doteqdot a_{r-1} B_{r-1} \doteqdot a_{r+1} B_{r+1} \doteqdot \dots \doteqdot a_n B_n.$$

Hence, by Theorem 3, $a_r X$ contains Δ, which is impossible since $L(a_r X) < L(\Delta)$, from (3.5). The theorem therefore follows.

Drawn diagram of Δ_3.

The drawn diagram of Δ_3 is given in Fig. 7.

4. Solution of the conjugacy problem in B_{n+1}

4.1. Index length

The algebraic sum of the indices of any given word will be called its *index length*. For example $(a_1)^{-3}(a_3)^4 a_2$ is of index length 2.

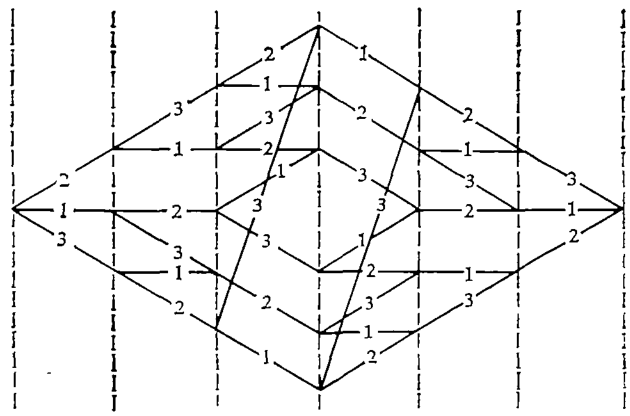

FIG. 7

LEMMA 9. *In B_{n+1} the number of words in standard form of index length t and power $\geqslant p$ is finite.*

Let $\Delta^m \bar{A}$ be any word satisfying the conditions. Then if $L(\Delta) = d$, we have

$$m \geqslant p, \tag{4.1}$$

and

$$t = md + L(\bar{A}). \tag{4.2}$$

Since $L(\bar{A}) \geqslant 0$ and d is positive, the last equation gives

$$m \leqslant t/d. \tag{4.3}$$

(4.1) and (4.3) together show that the number of values of m is finite. (4.2) shows that for any fixed m, $L(\bar{A})$ is constant, and so the number of possible values of $\bar{A}$ is finite. The result now follows.

Definitions. In the diagram $D(\Delta)$ in B_{n+1}, let α be any initial sub-route, so that $\Delta \doteq \alpha X$ $(0 \leqslant L(X) \leqslant L(\Delta))$. We shall call such a sub-route an *α-route*. If W is any word in B_{n+1}, the word $\alpha^{-1}W\alpha$, reduced to standard form, will be called an *α-transformation of W*. If $\bar{\alpha}$ is the base of any α-route α, then we shall call $\bar{\alpha}$ an *$\bar{\alpha}$-route* and the transformation $\bar{\alpha}^{-1}W\bar{\alpha}$ an *$\bar{\alpha}$-transformation* of W. It is clear that any α-transformation is equal to the corresponding $\bar{\alpha}$-transformation.

Summit form. Summit set. Summit. Summit power.

Let W be any word in B_{n+1} with standard form $\Delta^m \bar{A} \doteq W_1$, say. Consider now the following chains of α-transformations of W. Take all the α-transformations of W_1 and let those which are of power $\geqslant m$ and which are distinct from W_1 and from each other, be $W_2, W_3, \ldots, W_t$. Now repeat the process for each of the words $W_2, W_3, \ldots, W_t$ in turn, denoting successively by $W_{t+1}, W_{t+2}, \ldots$ any new words occurring, the condition being always that each new word must be of power $\geqslant m$. Continue to repeat the process for every new distinct word arising, as the sequence $W_1, W_2, \ldots, W_{t+2}, \ldots$ expands. Now each word of the sequence is of the same index length as W. Hence, by Lemma 9, the sequence is finite, and ultimately a stage must be reached when further applications of the process will yield no new words.

Suppose that in the set $W_1, W_2, \ldots$ the highest power reached is s, and that the words of power s form the subset $V_1, V_2, \ldots$. Then any V_r will be said to be a *summit form* of W. The set $V_1, V_2, \ldots$ will be called the *summit set* of W, or simply the *summit* of W. The power s of any summit form will be called the *summit power* of W. It is clear from the definitions given above that no single α-transformation of a summit form can be of power greater than the summit power.

LEMMA 10. *In B_{n+1}, if $W = \Delta^m V$, where V is positive, and P is a positive word such that $P^{-1}WP$ is of power $m+r$ $(r > 0)$, then VP contains Δ.*

By hypothesis $P^{-1}\Delta^m VP = \Delta^{m+r}\bar{Q}$, so that

$$VP = \Delta^{-m}P\Delta^{m+r}\bar{Q}. \tag{4.4}$$

Put $\tilde{P} \equiv P$ $(m+r$ even$)$, $\tilde{P} \equiv \Re P$ $(m+r$ odd$)$. Then, by Theorem 2, (4.4) gives $VP = \Delta^r\tilde{P}\bar{Q}$, so that by Theorem 4, $VP \doteq \Delta^r\tilde{P}\bar{Q}$. Hence VP contains Δ.

LEMMA 11. *In B_{n+1}, if $W \sim V$, then there exists a positive word X such that $X^{-1}WX = V$.*

By hypothesis there exists a word A such that $A^{-1}WA = V$. Let $A = \Delta^m\bar{P}$. Then

$$\bar{P}^{-1}\Delta^{-m}W\Delta^m\bar{P} = V. \tag{4.5}$$

If m is even, Theorem 2 now gives $\bar{P}^{-1}W\bar{P} = V$ ($\bar{P}$ positive). If m is odd, (4.5) may be written $\bar{P}^{-1}\Delta^{-1}(\Delta^{-m+1}W\Delta^{m-1})\Delta\bar{P} = V$, i.e. using Theorem 2 again, $(\Delta\bar{P})^{-1}W(\Delta\bar{P}) = V$ ($\Delta\bar{P}$ positive), and the lemma is proved.

LEMMA 12. *In B_{n+1}, suppose* (i) *that $W \equiv \Delta^p\bar{P}$ is a summit form of any given word A,* (ii) *that X is any positive word such that $X^{-1}WX = \Delta^q\bar{Q}$, where $q \geqslant p$, and* (iii) *that $X = uY$ where u is an α-route of maximum length. Then $u^{-1}Wu$, reduced to standard form, is a summit form of A.*

When $u = \Delta$ the proof is trivial. For $u \neq \Delta$, since u is an α-route, there exists a word U ($L(U) > 0$) such that

$$uU \doteqdot \Delta. \tag{4.6}$$

Now, by Theorem 9, every node of $D(\Delta)$ is the meet of the n links $a_1, a_2,..., a_n$, and of these n links only. In the diagram $D(\Delta)$ denote

the links ending at the node $D(u)$ by $x_1, x_2,..., x_s$; (4.7)

and the links starting at $D(u)$ by $y_1, y_2,..., y_{n-s}$. (4.8)

By hypothesis,

$$\Delta^q\bar{Q} = X^{-1}WX = Y^{-1}u^{-1}\Delta^p\bar{P}uY \quad (q \geqslant p). \tag{4.9}$$

Now from (4.6), $u^{-1}Wu \equiv u^{-1}\Delta^p\bar{P}u = u^{-1}uU\Delta^{p-1}\bar{P}u = U\Delta^{p-1}\bar{P}u$, so that, putting $\tilde{U} \equiv \Re U$ (p even), and $\tilde{U} \equiv U$ (p odd), and using Theorem 2,

$$u^{-1}Wu = u^{-1}\Delta^p\bar{P}u = \Delta^{p-1}\tilde{U}\bar{P}u. \tag{4.10}$$

Substituting in (4.9) we now get $Y^{-1}\Delta^{p-1}\tilde{U}\bar{P}uY = \Delta^q\bar{Q}$ ($q \geqslant p$), and hence, by Lemma 10, $\tilde{U}\bar{P}uY$ contains Δ. By Theorem 8, therefore, each of the n links $x_1, x_2,..., x_s, y_1, y_2,..., y_{n-s}$ is incident at each node of $D(\tilde{U}\bar{P}uY)$. Now in the diagram $D(\tilde{U}\bar{P}uY)$ no link y_i can start at the node $D(\tilde{U}\bar{P}u)$. For in this case we would have $Y \doteqdot y_i Z$ for some Z, and hence $X \doteqdot uy_i Z$ where uy_i would be an α-route of length greater than $L(u)$, vitiating condition (iii) of the hypothesis. Hence the $n-s$ links $y_1, y_2,..., y_{n-s}$ all end at the node $D(\tilde{U}\bar{P}u)$. Furthermore, by (4.7), the s links $x_1, x_2,..., x_s$ end at $D(\tilde{U}\bar{P}u)$. Hence by Theorem 3, $\tilde{U}\bar{P}u$ contains Δ, so that, from (4.10), $u^{-1}Wu$ reduced to standard form is of power at least p. Now it cannot be of power $> p$, since it is an α-transformation of the summit form W. Hence $u^{-1}Wu$, reduced to standard form, is an α-transformation of the summit form W of A, of the same power as W, and is therefore itself a summit form of A.

THEOREM 10 (CONJUGACY). *In B_{n+1}, $A \sim B$ if and only if their summit sets are identical.*

(i) If the condition is satisfied, let C be any member of the common summit set. Then $A \sim C$, $B \sim C$. Hence $A \sim B$, so that the condition is certainly sufficient.

(ii) We now proceed to show that the condition is necessary. Suppose

$$A \sim B. \tag{4.11}$$

Let $\Delta^p\bar{P} \sim A$ be any summit form of A, (4.12)

and $\Delta^q\bar{Q} \sim B$ be any summit form of B. (4.13)

First suppose $q \geqslant p$. Clearly $\Delta^p\bar{P} \sim \Delta^q\bar{Q}$, and hence, by Lemma 11, there exists a positive word X such that

$$X^{-1}\Delta^p\bar{P}X = \Delta^q\bar{Q} \quad (q \geqslant p). \tag{4.14}$$

Let $X \doteqdot u_1 X_1$, $X_1 \doteqdot u_2 X_2,\ldots$, etc., and finally $X_s \doteqdot u_{s+1}$, where u_1, $u_2,\ldots$ are defined successively as α-routes of maximum length, and X_1, $X_2,\ldots$ are words of steadily reducing length, so that the final word X_{s+1} is the empty word. Then

$$X \doteqdot u_1 u_2 \ldots u_{s+1}. \tag{4.15}$$

Using (4.15) the transformation (4.14) may be regarded as the product of the $s+1$ successive transformations $(u_1)^{-1}\Delta^p\bar{P}u_1 = W_1$ say, in standard form; $(u_2)^{-1}W_1 u_2 = W_2$ say, in standard form; ...; $(u_{s+1})^{-1}W_s u_{s+1} = \Delta^q\bar{Q}$. Now, by Lemma 12, W_1, $W_2,\ldots$ and finally $\Delta^q\bar{Q}$ are each summit forms of A. Hence we cannot have $q > p$, and similarly we cannot have $p > q$. Hence $q = p$, and by the argument given above $\Delta^q\bar{Q} \equiv \Delta^p\bar{Q}$ is a summit form of A. We have thus proved that any summit form of B is a summit form of A. Similarly any summit form of A is a summit form of B, i.e. the summit sets of A and B are identical.

4.2. Remark on the definition of summit set

In B_{n+1} suppose any word $W = \Delta^p\bar{A}$ has summit power $p+r$, where $r > 0$. Then in the process of finding the summit set of W outlined in § 4.1, we have constantly to include in the words considered all words of powers p, $p+1$, $p+2,\ldots$ until finally the complete set of words of power $p+r$ is established. In the process we must at some stage reach a first word of power $p+1$, W_1 say. Now since $W_1 \sim W$ it follows from Theorem 10 that their summits are the same. Hence it now suffices to find the summit of W_1, and in doing this we can ignore all words of power p. Similarly, when once a word of power $p+2$ is reached we can thereafter ignore words of powers p and $p+1$ . . . etc. . . . . Moreover, since any α-transformation is equal to the corresponding $\tilde{\alpha}$-transformation, it is in fact sufficient to consider $\tilde{\alpha}$-transformations only.

5. Other groups

5.1. Considered as a diagram in 3-space, the drawn Cayley diagram of Δ_3, given in Fig. 7, will be seen to be the 2-skeleton of the truncated octahedron (4.6^2). Similarly, in B_{n+1}, the drawn diagram of Δ_n is the 2-skeleton of the n-dimensional polytope $(4^{\frac{1}{2}(n-1)(n-2)}, 6^{n-1})$.

Groups similar to the braid groups exist whose Cayley Δ-diagrams are the 2-skeletons of the other even-faced Archimedean solids (including the prisms), and their higher-dimensional counterparts. The methods given in the present paper can be applied to solving the word problems and the conjugacy problems of these groups. In the next three sections examples will be given of groups for which it may be verified that the above remarks apply.

For all the examples given, Theorems H, K, 1–6, 8–10, and Lemmas 3, 4, 7–12, are true. The centres are given by methods of the same general pattern as for the braid groups (Theorem 7), but there are considerable differences in detail. In each example given, Δ is the shortest element of the group which can start with each one of the generators.

5.2. The truncated cuboctahedron (4. 6. 8)

The group, T_3 say, is generated by a_1, a_2, a_3 subject to the relations

$$a_1 a_2 a_1 a_2 = a_2 a_1 a_2 a_1, \quad a_2 a_3 a_2 = a_3 a_2 a_3, \quad a_1 a_3 = a_3 a_1.$$

For T_3, $\Delta = (a_1 a_2 a_3)^3$, and $\Delta a_i = a_i \Delta$ $(i = 1, 2, 3)$. The centre is generated by Δ.

5.3. The truncated icosidodecahedron (4. 6. 10)

The group, I_3 say, is generated by a_1, a_2, a_3 subject to the relations

$$a_1 a_2 a_1 a_2 a_1 = a_2 a_1 a_2 a_1 a_2, \quad a_2 a_3 a_2 = a_3 a_2 a_3, \quad a_1 a_3 = a_3 a_1.$$

For I_3, $\Delta = (a_1 a_2 a_3)^5$, and $\Delta a_i = a_i \Delta$ $(i = 1, 2, 3)$. The centre is generated by Δ.

5.4. The hypercube (4^m)

Naming the groups C_m, say, there are two cases according as m is odd or even.

(1) The group C_{2n-1} is generated by $a_1, a_2,\ldots, a_{2n-1}$ subject to the relations

$$\left.\begin{array}{ll} a_r a_{2n-r} = a_{2n-r} a_r & (r = 1, 2,\ldots, 2n-1), \\ a_r a_s = a_s a_{2n-r} & (r, s\colon s \text{ lies between } r \text{ and } 2n-r) \end{array}\right\}.$$

For C_{2n-1}, $\qquad \Delta = (a_1 a_{2n-1})(a_2 a_{2n-2})\ldots(a_{n-1} a_{n+1}) a_n,$

and $\qquad \Delta a_r = a_{2n-r} \Delta \quad (r = 1, 2,\ldots, 2n-1).$

The n products $(a_r a_{2n-r})$ $(r = 1, 2,\ldots, n)$ generate the centre. Δ^2 is in the centre, but Δ is not.

(2) The group C_{2n} is generated by $a_1, a_2,\ldots, a_{2n}$ subject to the relations

$$\left.\begin{array}{ll} a_r a_{2n-r+1} = a_{2n-r+1} a_r & (r = 1, 2,\ldots, 2n) \\ a_r a_s = a_s a_{2n-r+1} & (r, s\colon s \text{ lies between } r \text{ and } 2n-r+1) \end{array}\right\}.$$

For C_{2n}, $$\Delta = (a_1 a_{2n})(a_2 a_{2n-2})\ldots(a_n a_{n+1}),$$
and $$\Delta a_r = a_r \Delta \quad (r = 1, 2,\ldots, 2n).$$
The n products $(a_r a_{2n-r+1})$ $(r = 1, 2,\ldots, n)$ generate the centre, and in this case Δ is in the centre.

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