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Threading knot diagrams

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1. *Introduction*

Alexander [1] showed that an oriented link K in S^3 can always be represented as a closed braid. Later Markov [5] described (without full details) how any two such representations of K are related. In her book [3], Birman gives an extensive description, with a detailed combinatorial proof of both these results.

In this paper I shall describe a simple method of representing an oriented link K as a closed braid, starting from a knot diagram for K and 'threading' a suitable unknotted curve L through the strings of K so that K is *braided* relative to L, i.e. $K \cup L$ forms a closed braid together with its axis.

I shall then give a straightforward derivation of Markov's result, using the ideas of threading, and a geometric version of the braid moves with which Markov relates two braids representing the same K. The geometric approach is described in terms of links $K \cup L$, in which K forms a closed braid relative to an axis L. Such a link will be called *braided*, and in addition it will be called a *threading* of an explicit diagram for K if it arises from the threading construction. Two braided links which are related by the geometric version of Markov's moves will be called *Markov-equivalent*. Markov's theorem, which says in this geometric translation that braided links $K \cup L$ and $K' \cup L'$ are Markov-equivalent if and only if the oriented links K and K' are isotopic, will then follow from Theorems 2, 3 and 4, on threadings.

These results on threadings, whose proofs are not elaborate, are as follows:

Theorem 2. *Any braided link $K \cup L$ arises as a threading of some diagram for K.*

Theorem 3. *Any two threadings of a given diagram of K are Markov-equivalent.*

Theorem 4. *Two different diagrams of K have Markov-equivalent threadings.*

Remarks. Bennequin [2] gives a geometric proof of Markov's theorem in the course of his work on contact structures, using suitably positioned spanning surfaces for the closed braids; see also Rudolph [7] for a discussion of such surfaces.

Markov's result itself, and also the representation of a knot as a closed braid, have had attention recently following Jones' use of braid groups in constructing his new polynomial knot-invariant [4].

2. *Notation and definitions*

For definitions and notation concerning braids I shall refer to [3]. In particular, given an n-string braid $\beta \in B_n$ I shall refer to the *closure* of β, written $\hat{\beta}$, to mean an oriented link which arises from an explicit geometric representative of β in $D^2 \times I$ by identifying the ends of the cylinder. The closure of β is determined by β up to isotopy in S^3. In fact β determines up to isotopy a link $\hat{\beta} \cup L_\beta$ consisting of its closure $\hat{\beta}$, lying in the complement of L_β, an unknotted curve called the *axis* of $\hat{\beta}$; here the exterior of L_β is an unknotted solid torus $D^2 \times S^1$ in which the curve $\hat{\beta}$ lies regularly with respect to the projection to S^1. I shall refer to the link $\hat{\beta} \cup L_\beta$, with a natural choice of orientation, as the *complete closure* of β.

If an oriented link $K \cup L$ is given in which L is unknotted, and K projects regularly to S^1 under some choice of product projection $p_L: S^3 - L \to S^1$ then I call $K \cup L$ *braided* (relative to L). Then the complete closure of β is braided, and conversely any braided link is the complete closure of some β. Geometrically the complete closure captures $\beta \in B_n$ very well, for β and γ have isotopic complete closures (respecting orientation) if and only if β and γ are conjugate in B_n, see e.g. [**6**].

Consequently a braided link $K \cup L$ determines $\beta \in B_n$, $(n = \mathrm{lk}(K, L))$, up to conjugacy.

A test for a braided link. *If L is unknotted, and a product projection*

$$p_L: S^3 - L \to S^1$$

is found in which K is mapped monotonically, i.e. for a suitable orientation of S^1 the map p_L is locally increasing on K, but not necessarily strictly *increasing, and in addition p_L is not constant on any component of K, then $K \cup L$ is braided.*

Proof. Under these conditions an arbitrarily small isotopy of K can be made in $S^3 - L$ to ensure that p_L becomes strictly increasing.

Markov moves

A Markov move replaces a braid $(\beta, n) \in B_n$ by

(1) (γ, n), with γ conjugate to β in B_n, or

(2) $(\beta\sigma_n^{\pm 1}, n+1) \in B_{n+1}$, or

(3) $(\beta', n-1) \in B_{n-1}$, if $\beta = \beta'\sigma_{n-1}^{\pm 1}$, and β' is a word in $\sigma_1, \ldots, \sigma_{n-2}$.

The complete closures of two braids related by a type (1) move are isotopic; conversely we have noted that a braided link determines a braid up to type (1) moves.

I shall shortly describe a relation, *simple Markov-equivalence*, between two braided links which will ensure that they are complete closures of two braids related by a move of type (2), or its inverse, of type (3).

If *Markov-equivalence* is defined as the relation on braided links generated by isotopy and simple Markov-equivalence we have then the geometric reformulation of Markov's theorem which follows.

Theorem 5. *If β and γ are two braids whose closures are isotopic as oriented links, then their complete closures are Markov-equivalent.*

Definition. Two braided links $K \cup L$ and $K' \cup L'$ are *simply Markov-equivalent* if the second one can be isotoped so that $L' = L$ and K' agrees with K except for arcs α of K and α' of K' with the following properties:

(1) we can find a projection $p_L: S^3 - L \to S^1$ which is constant on α, strictly monotone on the rest of K and monotone of degree 1 on α'.

(2) there is a disc A spanning $\alpha \cup \alpha'$ whose interior meets L transversely in one point and avoids $K \cup \alpha'$.

From the definition it is clear that if $\mathrm{lk}(K, L) = n$ then $\mathrm{lk}(K', L') = n+1$, so the braided links will be complete closures of braids in B_n, B_{n+1} respectively.

Lemma 1. *If $K \cup L$ and $K' \cup L'$ are simply Markov-equivalent braided links then they are the complete closures of some $\beta \in B_n$ and $\beta\sigma_n^{\pm 1} \in B_{n+1}$ respectively where $n = \mathrm{lk}(K, L)$.*

Proof. Suppose that the links have been isotoped to agree except on arcs α of K and α' of K' as specified in the definition, and suppose that α lies at the level $p_L = \theta_0$. Look at the way in which the disc A bounded by $\alpha \cup \alpha'$ meets the level disc D, with boundary L, for the level $p_L = \theta_0$.

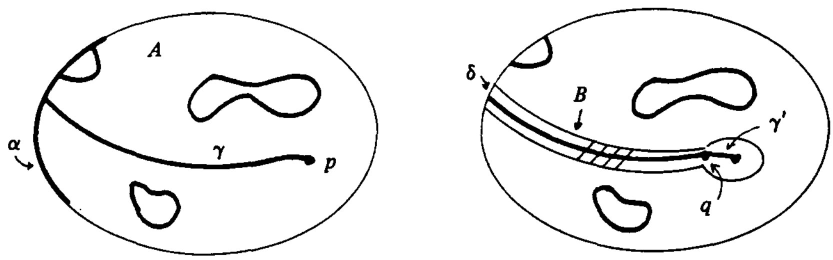

Fig. 1

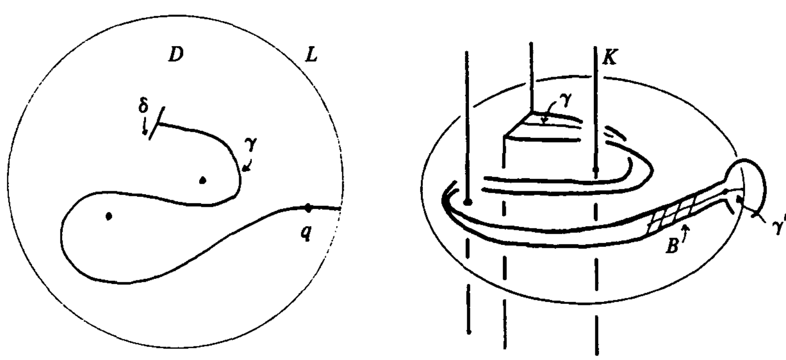

Fig. 2

After a slight isotopy of the interior of A we may assume that it meets D transversely and that $A \cap D$ consists of the arc α in ∂A together with a finite number of disjoint closed curves and arcs as illustrated in Fig. 1. Where an arc meets α we may assume that p_L behaves locally on A like the restriction to one side of a saddle point.

The single point, p, of transverse intersection of A with $L = \partial D$ will be the end-point of exactly one arc γ of $A \cap D$, whose other end must lie on α. Choose a small disc in A, centre p, which we can assume (after isotopy of A) to lie, except for p, as a product of a subarc $\gamma' \subset \gamma$ with S^1 in the solid torus $S^3 - L$. The boundary of this disc will eventually form a single braid string over the end-point q of γ' in D.

Adjoin to this disc a thin ribbon of A about γ small enough to contain no critical points for p_L and to have p_L monotone on its edge. This ribbon, whose end on the arc α will be called δ, may be chosen to lie arbitrarily close to γ, say within the levels $[\theta_0 - \epsilon, \theta_0 + \epsilon]$ of p_L. Together with the disc about p it makes up a disc $B \subset A$. We now use the isotopy determined by A from ∂A to ∂B in the complement of L to isotop K and K' into new positions where they are related by the disc B in place of A. The disc D and its relation with K, K' and B are illustrated in Fig. 2.

To make an explicit correspondence of K, K' with the closure of two braids, choose $n+1$ reference points $q_1, \ldots, q_n, q$ in D, and a standard arc from q_n through q to p on ∂D, which extends the arc γ' from q to p. These points (with or without q) will be the starting and finishing points for the braids on n (or $n+1$) strings in $D \times I$ which become K, or K', when closed.

We now complete the proof by isotoping D, keeping $L = \partial D$ and γ' fixed so that α becomes the standard arc from q_n to p, δ becomes a small arc through q_n and K meets D in this small arc, together with the points $q_1, \ldots, q_{n-1}$. Extend this by a level preserving

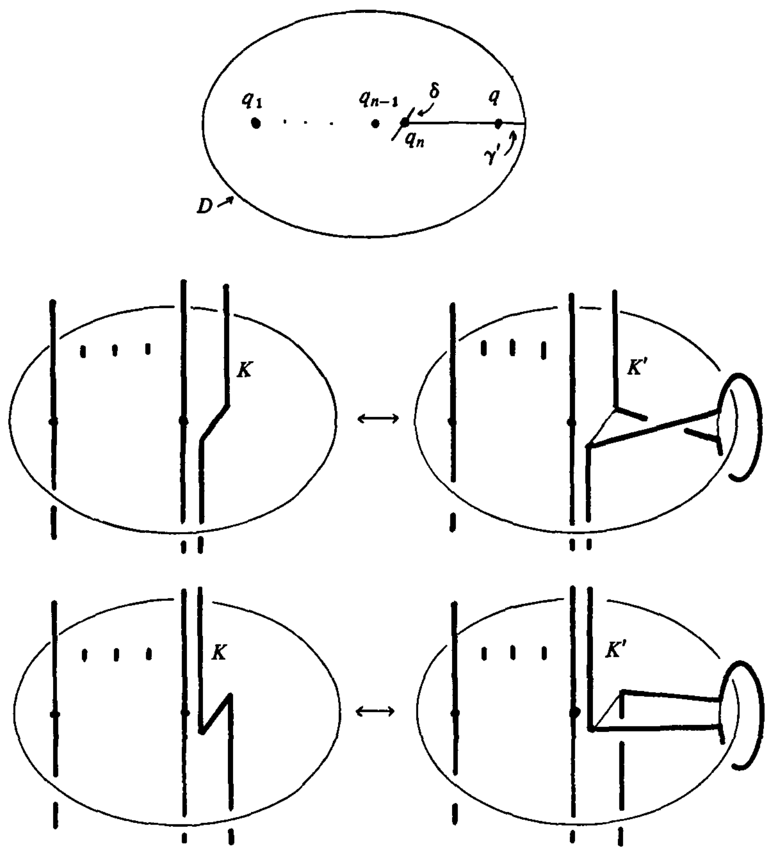

Fig. 3

isotopy which is the identity outside the levels $[\theta_0 \ldots 2\epsilon, \theta_0 + 2\epsilon]$ of p_L and carries B to a ribbon within these levels lying close to the standard arc together with the unchanged disc transverse to L determined by γ'. Assume that the ribbon lies within the levels $[\theta_0 - \epsilon, \theta_0 + \epsilon]$, and that K passes through $q_1, \ldots, q_{n-1}$ vertically between these levels, so that K' differs from K after this isotopy only by the addition of the ribbon edges close to level θ_0 between q_n and q and an extra straight string above q in the levels beyond the ribbon.

A slight adjustment to K and K' is still strictly necessary to realise both as the closure of braids based on the reference points when cut open at level θ_0, or better at $\theta_0 - \epsilon$.

Fig. 3 shows the difference of K and K' between levels $\theta_0 - \epsilon$ and $\theta_0 + \epsilon$ for the two possible orientations of δ in D. It is then clear that the braids given by cutting open at $\theta_0 - \epsilon$ differ simply by the addition of an extra string about q and a generator $\sigma_n^{\pm 1}$ which comes from the string exchange close to the standard arc joining q_n to q. ∎

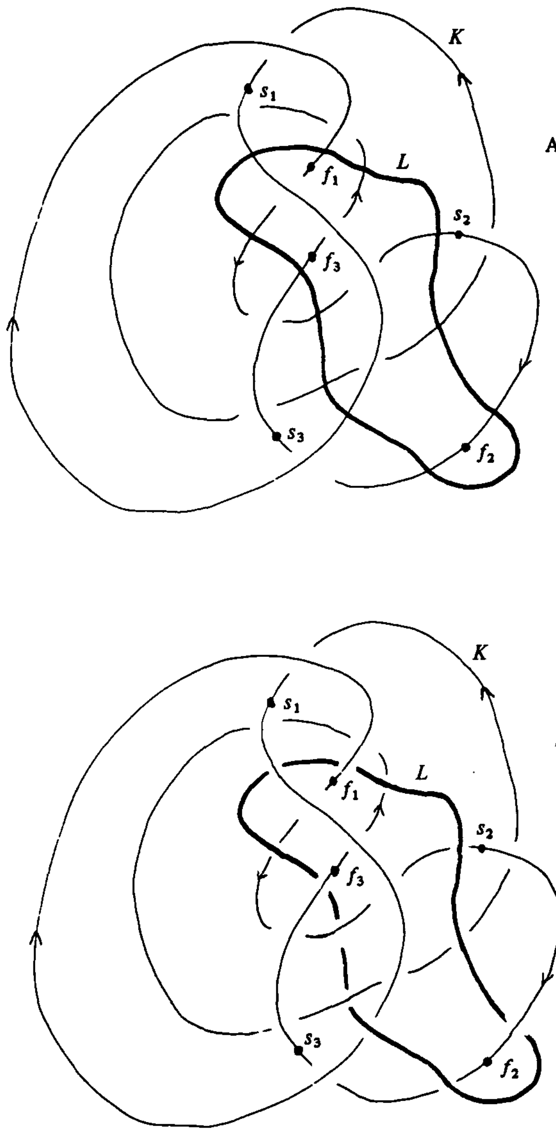

Fig. 4

3. *Threadings*

Starting from a diagram for an oriented link K I shall describe how to find unknotted curves L resulting in a variety of braided links $K \cup L$ which will be called *threadings* of the diagram for K.

By a diagram for K I shall mean a simple projection of K to some plane P in which a finite number of over-crossing and under-crossing points in K are distinguished – these are the inverse images of the double-points of the projection.

Definition. A *choice of overpasses* for a diagram will consist of (S, F), two finite subsets $S = \{s_1, \ldots, s_k\}$, $F = \{f_1, \ldots, f_k\}$ of K forming the 'starting' and 'finishing' points of overpasses, which alternate in K, and which divide K into arcs of the form $[s, f]$ called *overpasses* containing no undercrossings, and $[f, s]$, *underpasses*, containing no overcrossings.

There is no need, however, for an overpass to contain any overcrossing.

We can alter K by isotopy without changing its projection to P so that its overpasses and underpasses lie in planes parallel to P, (horizontal) just above and below it, except for vertical segments near S and F, where K goes upwards or downwards respectively.

It will be helpful to think of S and F as points lying in P, in the projection of K.

A *threading* of the diagram for K with the given choice of overpasses (S, F) is constructed from any closed curve L in P which separates S and F as follows:

Arrange that L crosses the image of K transversely, and alter K in the neighbourhood of each crossing point so that K crosses *over* L if it is passing from the side of L which contains S, and K crosses *under* L if it passes from the side of L which contains F, to give a link $K \cup L$ called a *threading*. See Fig. 4 for an example of a choice of overpasses and a separating curve L, with the resulting threading.

Theorem 1. *Any threading of a diagram for K is a braided link.*

Proof. Select overpasses (S, F) for the diagram, and a curve L in the plane P of the diagram which separates S from F. Now straighten out L in the plane P by an isotopy of P, carrying the projected image of K along, so that L becomes (almost) a straight line, with points of S lying to one side and F to the other. We can suppose that K is isotoped at the same time so that the overpasses and underpasses lie in planes parallel to P, just above and below their projected image.

We now change our point of view, and imagine that P forms the xz-plane and L forms the z-axis (having sent one point of L to infinity on P). Using polar coordinates based on L as axis, the plane P splits into two half-planes, one, given by $\theta = 0$ say, containing the points of F, and the other, $\theta = \pi$, containing the points of S.

Project the overpasses of K to the half-planes $\theta = -\epsilon$ and $\theta = \pi + \epsilon$, and similarly project the underpasses to the half-planes $\theta = \epsilon$ and $\theta = \pi - \epsilon$, for some suitably small ϵ. These curves (which cross the axis L in various points corresponding to the crossings of L with the projected image of K) are then joined up by vertical arcs (i.e. in the direction of projection) through the points of S and F to give a closed curve isotopic to K with the same projected image.

Apart from the points where this curve crosses L the polar coordinate increases monotonically, since it is constant, except on the vertical arcs, where it increases from $-\epsilon$ to ϵ for those through a point of F, because of moving from an overpass to an underpass, and it increases from $\pi - \epsilon$ to $\pi + \epsilon$ for the arcs through points of S.

As illustrated in Figs. 5 and 6, which show the process for a simple knot diagram, the threading construction now diverts the curve K near its crossings with L to run around L in the direction of increasing θ. We can arrange explicitly that pieces of K which pass through the cylinder $r \leqslant \delta$ are diverted to run around $r = \delta$, with pieces of K which come from the S-side $(\theta = \pi \pm \epsilon)$ being taken above L (i.e. through $\theta = \frac{1}{2}\pi$) and pieces which come from the F-side $(\theta = \pm\epsilon)$ being taken beneath. Thus, where, say, an overpass crosses L from side S to side F the polar coordinate after threading will increase by less than π, from $\pi + \epsilon$ to $-\epsilon$, while on an overpass which crosses from side F to side S the increase will be $\pi + 2\epsilon$.

Consequently the threading has been isotoped so that, with L as axis, the curve K

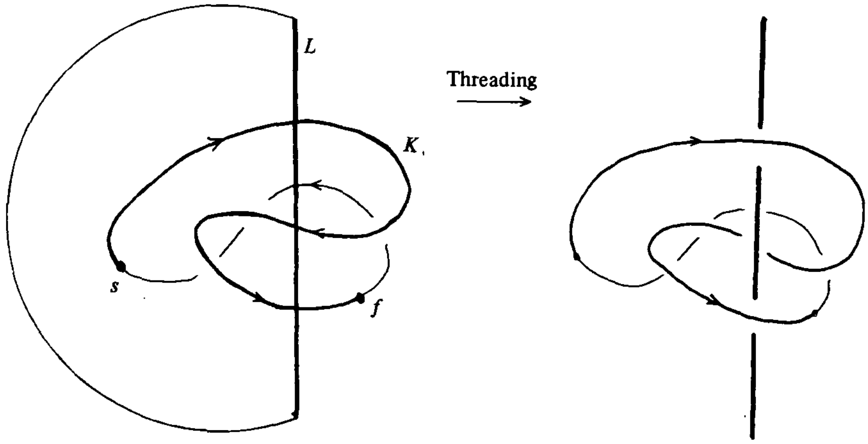
L
Threading
K
s
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Fig. 5

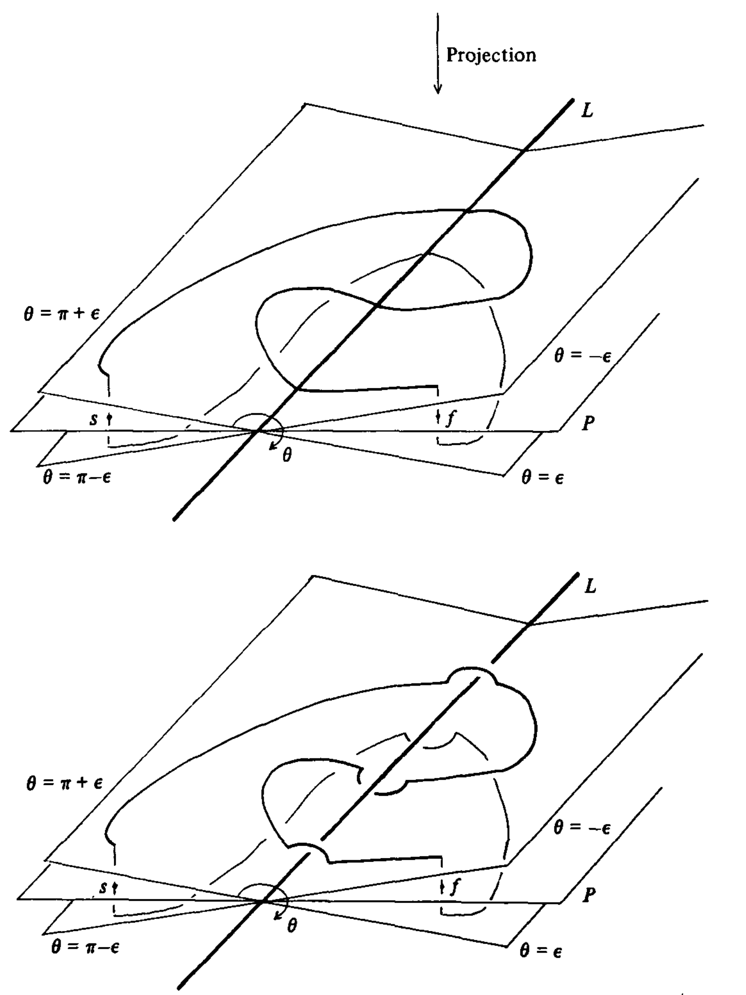
Projection
L
$\theta = \pi + \epsilon$
$\theta = -\epsilon$
s
f
P
θ
$\theta = \pi - \epsilon$
$\theta = \epsilon$
L
$\theta = \pi + \epsilon$
$\theta = -\epsilon$
s
f
P
θ
$\theta = \pi - \epsilon$
$\theta = \epsilon$

Fig. 6

runs monotonically (althought not strictly so) with respect to the polar coordinate. This is enough to guarantee, by the monotone test, that $K \cup L$ is braided. ∎

Remark. A threading of K will exhibit K as the closure of a braid on n strings, where n is the number of times that K crosses the curve L from the S-side to the F-side; consequently n is at least as large as the number of overpasses.

In an attempt to reduce n for a given K it would be natural to present K so that a small number of overpasses can be chosen (the minimum possible is the bridge number). It is clearly always possible to find an L which meets each overpass exactly once; the resulting threading may however have many more strings, for there is no guarantee that the underpasses all cross L once, and L will have to be threaded under them every time they cross from side S to side F.

Nevertheless the threading process is a very quick means of exhibiting any K as a closed braid, starting from any diagram of K.

Alexander's theorem, that any oriented link K can be represented as a closed braid, is an immediate corollary of Theorem 1, since any diagram of K can be threaded in many ways.

I shall now show that all braided links arise from threadings in the next Theorem.

Theorem 2. *The complete closure of a braid $\beta \in B_n$ is a threading of some diagram of its closure $\hat{\beta}$.*

Proof. A braid β corresponds to a homeomorphism h of the disc D^2 leaving ∂D^2 fixed and n points $p_1, \ldots, p_n$ invariant. In this correspondence we choose an isotopy of h to the identity, rel ∂D^2, to give a level-preserving homeomorphism $H: D^2 \times I \to D^2 \times I$, with $h = H|D^2 \times \{1\}$. The image of $\{p_1, \ldots, p_n\} \times I$ form the strings of a representative braid for β.

Choose disjoint arcs $a_1, \ldots, a_n$ in D^2 joining points $r_i \in \partial D^2$ to p_i. The closure of β is then isotopic to $H(\{p_1, \ldots, p_n\} \times I)$ together with

$$(a_1 \cup \ldots \cup a_n) \times \{0\} \cup (a_1 \cup \ldots \cup a_n) \times \{1\} \cup \{r_1, \ldots, r_n\} \times I,$$

and the axis of $\hat{\beta}$ can be represented by a circle just inside $\partial D^2 \times \{\frac{1}{2}\}$; as shown in Fig. 7.

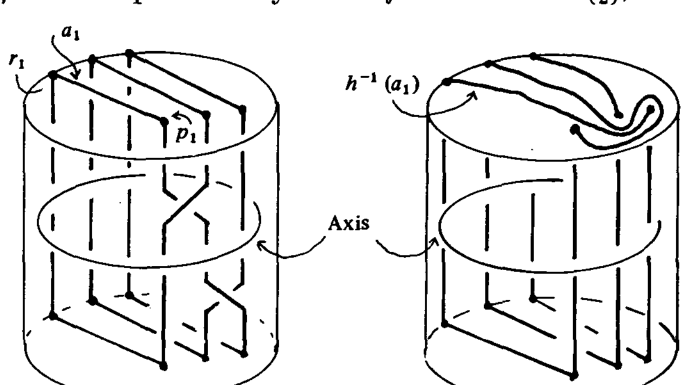

Fig. 7

Apply the homeomorphism H^{-1} to $D^2 \times 1$ to show that $\hat{\beta}$ is isotopic to the vertical lines $\{p_1, \ldots, p_n\} \times I \cup \{r_1, \ldots, r_n\} \times I$ together with the arcs $a_1 \cup \ldots \cup a_n$ in level 0 overlaid with the arcs $h^{-1}(a_1) \cup \ldots \cup h^{-1}(a_n)$ in level 1. The axis remains as before, and $\hat{\beta}$ is represented as the threading of a diagram in D^2 with $S = (p_1, \ldots, p_p\}$, $F = \{r_1, \ldots, r_n\}$, underpasses $a_1, \ldots, a_n$ and overpasses $h^{-1}(a_1), \ldots, h^{-1}(a_n)$, where the threading curve is a circle just inside ∂D^2. ∎

4. *Markov's Theorem*

As shown by Lemma 1, Markov's theorem, which says that any two braids with isotopic closures are related by a sequence of Markov moves, can be given in geometric form as follows.

THEOREM 5. (Markov). *If $K \cup L$ and $K' \cup L'$ are braided links, and K is isotopic to K', as oriented links, then $K \cup L$ and $K' \cup L'$ are Markov-equivalent.*

The theorem will follow by using Theorem 2 to show that $K \cup L$ and $K' \cup L'$ are both threadings of some diagrams of K. Then Theorems 3 and 4 will complete the proof.

THEOREM 3. *Any two threadings of a given diagram for K are Markov-equivalent.*

THEOREM 4. *Any two diagrams for K have Markov-equivalent threadings.*

Proof of Theorem 5. By Theorem 2, $K \cup L$ is a threading of some diagram of K. Again by Theorem 2, $K' \cup L'$ is a threading of some diagram of K'; since K' is isotopic to K this is a threading of a second diagram of K.

By Theorem 4 we can choose threadings of the first and second diagrams of K which are Markov-equivalent. By Theorem 3 the chosen threading of the first diagram is Markov-equivalent to $K \cup L$, since $K \cup L$ is another threading of the same diagram of K; the chosen threading of the second diagram is similarly Markov-equivalent to $K' \cup L'$. ∎

To prove Theorem 3 we show it first with a given choice of overpasses (S, F), in Lemma 2. Independence of the choice of overpasses follows, using Lemma 3 to construct a choice of overpasses (S'', F'') with $S, S' \subset S''$; $F, F' \subset F''$, for any two given choices (S, F) and (S', F'). Then any threading of (S'', F'') will give threadings of (S, F) and (S', F') which are isotopic.

For Theorem 4 is is enough to show that two diagrams of K which differ by a Reidemeister move have isotopic, hence Markov-equivalent, threadings for some choice of overpasses. This is done by choosing (S, F) and the threadings to be identical outside the region of the move, and only to involve the region very simply, if at all.

I now complete the proof of Theorem 5 by proving Lemmas 2 and 3, and Theorem 4. In the accompanying diagrams the curve L to be threaded is drawn more thickly than K.

LEMMA 2. *Given a diagram for an oriented link K in a plane P, with choice of overpasses (S, F), then the threadings defined by any two simple closed curves L, L' which separate S and F are Markov-equivalent.*

Proof. We may suppose, without loss of generality, that F lies in the bounded component of $P - L$ and of $P - L'$.

(*a*) Suppose that L and L' are isotopic in $P - (S \cup F)$. Then L, L' and K are related by a sequence of moves of type 1 or type 2 shown in Fig. 8, in which no points of S or F appear.

The two threadings in type 2 are clearly isotopic, except when the orientations require them to look as in Fig. 9. A picture like that in Fig. 9 can indeed occur, as seen

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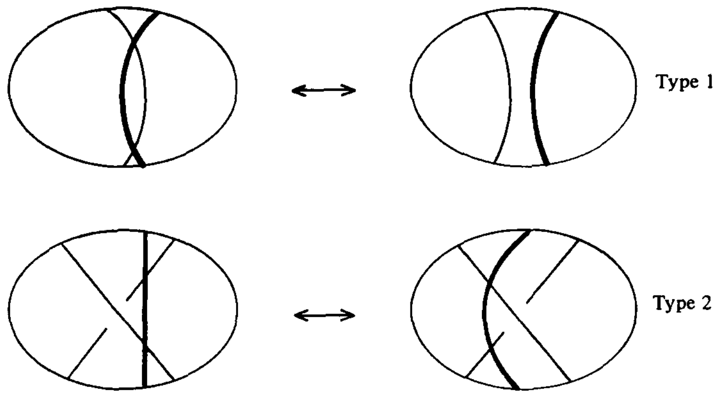

Fig. 8.

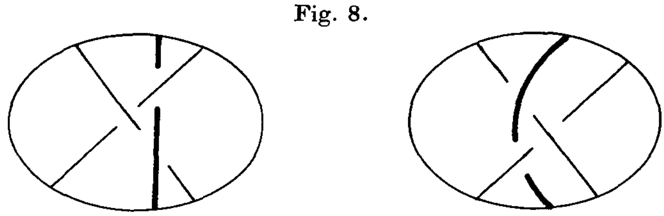

Fig. 9.

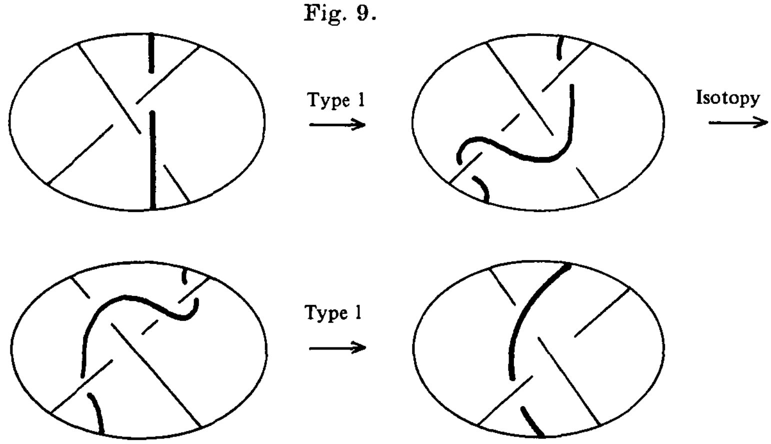

Fig. 10.

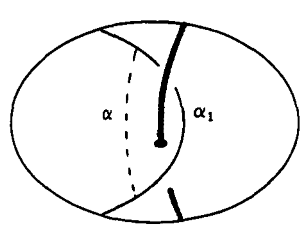

Fig. 11.

for example near the bottom of Fig. 4. In this case one threading converts to the other by a sequence of type 1 moves and isotopies, as in Fig. 10, noting that undercrossings and overcrossings of K with L must alternate on passing around K. It is then enough to show that the two threadings in a type 1 move are simply Markov-equivalent. Now the whole of K in the diagram belongs either to one overpass or to one underpass. In the

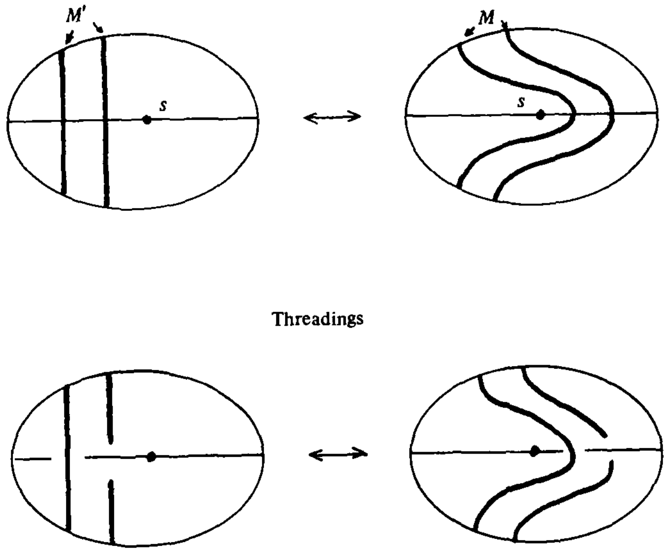

Fig. 12.

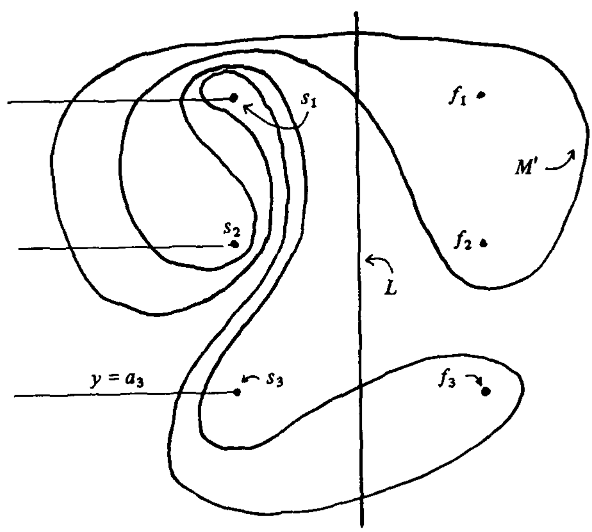

Fig. 13.

threading construction we may then assume that the two parts of K to one side of L lie at the same level of p_L (it will be $\theta = \pi \pm \epsilon$ or $\pm\epsilon$ according to side and whether we have an under or overpass) until reaching the immediate neighbourhood of L. Join them by an arc α at this level, as indicated in Fig. 11. Then α together with the arc α' of K crossing L bounds a disc as required for a simple Markov-equivalence between the two threadings.

(*b*) To deal with the general case, observe that if a curve M' in P separating S and F is isotoped to M by pushing a *pair* of crossings with K past a point of S (or of F) as in Fig. 12, then M, which also separates S and F, and M' define isotopic threadings.

By isotopy of P we may make an explicit choice of S, F and L. Let us take S and F to consist of the points $\{(-1, a_i)\}$ and $\{(1, a_i)\}$ respectively, for some $a_1, \ldots, a_k$, and take

L to be part of the y-axis, completed with a large semicircle to enclose F. We may suppose also that K runs parallel to the x-axis near each point $s_1, \ldots, s_k$ of S.

Let L' be any simple closed curve separating S and F and enclosing F. Assume, after a small isotopy that L' meets transversely the lines $y = a_i$, $x < -1$ running from s_i to infinity away from L. By an isotopy in the complement of S and F, pushing along these lines, we can alter L' to M' whose intersections with the lines all lie close to S. A picture of a typical M' is given in Fig. 13. Each line is met an even number of times by M', since S lies outside M'; we may now isotop M' by moving the intersections across the points of S a pair at a time to reach a curve M which does not meet the lines, and so is isotopic to L in the complement of S and F.

The threadings defined by M and L, and by M' and L', are Markov-equivalent, by (a); we have just observed that the threadings defined by M and M' are isotopic, so Lemma 2 is established. ∎

Lemma 3. *Given a diagram for K and a choice (S, F) of overpasses, and any point s of K not in F, we can make a new choice of overpasses $(\bar{S}, \bar{F})$ with $s \in \bar{S}$, $S \subset \bar{S}$ and $F \subset \bar{F}$.*

Proof. If s lies on an overpass of (S, F) then choose f immediately before s on K, so that $[f, s] \subset K$ contains no overcrossing point, and take $\bar{F} = F \cup \{f\}$, $\bar{S} = S \cup \{s\}$. The original overpass containing s becomes separated into two by the new underpass $[f, s]$

If s lies on an underpass then choose f immediately after s with no undercrossings in $[s, f]$, and take $\bar{F}$, $\bar{S}$ as before. Then $[s, f]$ becomes a new overpass, separating one underpass into two. ∎

A similar argument allows extension of F by any $f \notin S$. Two choices (S, F) and (S', F') of overpasses can then be extended readily (provided that

$$S \cap F' = S' \cap F = \varnothing)$$

to a choice of overpasses (S'', F'') with $S \cup S' \subset S''$ and $F \cup F' \subset F''$.

Clearly any curve L in P which separates S'' and F'' will also separate S and F, and so L determines a threading for (S'', F'') and for (S, F). These threadings are actually

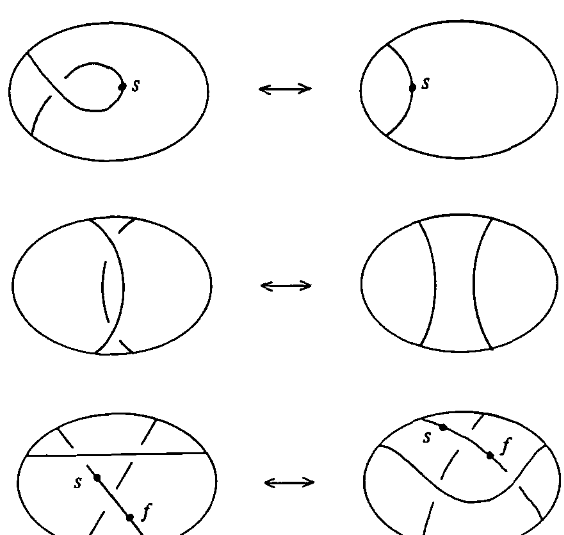

Fig. 14.

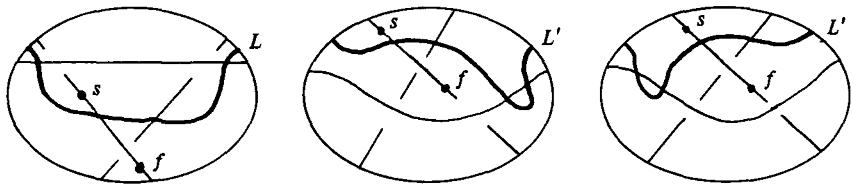

Fig. 15.

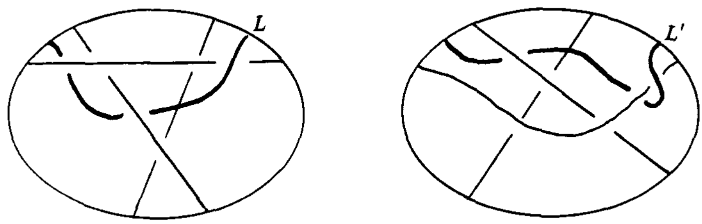

Fig. 16.

the same, since the construction of the threading from the diagram depends only on the sense in which K crosses L. The dissection of K into overpasses is required only to ensure that L is a suitable curve to make $K \cup L$ braided. Consequently for any two choices of overpasses there is some common threading. The Markov-equivalence of any two threadings of a given diagram then follows using Lemma 2.

THEOREM 4. *Any two diagrams for an oriented link K have Markov-equivalent threadings.*

Proof. Any two diagrams for K are related by a sequence of Reidemeister moves, illustrated in Fig. 14, so it is enough to show how, for each Reidemeister move, isotopic threadings can be chosen for the two related diagrams.

By Theorem 3 we may make any convenient choice of overpasses. In choosing (S, F) on any diagram we need only ensure that there is always a point of $S \cup F$ separating an overcrossing from a neighbouring undercrossing.

Place points $s \in S$ and $f \in F$ as indicated in Fig. 14, and choose the rest of $S \cup F$ to lie outside the region altered by the move, with s and f interchanged if the orientation of K is in the opposite sense. In the case of the first two moves we can then choose our separating curve L to lie outside the region of change, so that the resulting threadings are unaltered by the move.

For the third move we must choose L to separate s and f. Fig. 15 shows part of a suitable L which gives a threading isotopic to that from L' or L'', depending on the orientation of the uppermost piece of K. Fig. 16 shows the relevant parts of the threadings in one of the two cases, making it clear that they are isotopic. The crossing of L with the lowest piece of K will not affect the isotopy. The proof of Theorem 4, and so also of Markov's theorem, is complete. ∎

REFERENCES

[1] J. W. ALEXANDER. A lemma on systems of knotted curves. *Proc. Nat. Acad. Sci. U.S.A.* **9** (1923), 93–95.

[2] D. BENNEQUIN. Entrelacements et équations de Pfaff. *Astérisque* **107–108** (1983), 87–161.

[3] J. S. BIRMAN. *Braids, Links and Mapping Class Groups. Ann. of Math. Stud.* **82** (1974).

[4] V. F. R. JONES. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc.* **12** (1985), 103–111.

[5] A. A. MARKOV. Über die freie Äquivalenz geschlossener Zöpfe. *Recueil Mathématique Moscou* **1** (1935), 73–78.

[6] H. R. MORTON. Infinitely many fibred knots with the same Alexander polynomial. *Topology* **17** (1978), 101–104.

[7] L. RUDOLPH. Special positions for surfaces bounded by closed braids. Preprint 1984, Box 251, Adamsville, Rhode Island.