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ALGEBRAIC FUNCTIONS AND CLOSED BRAIDS

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§1. INTRODUCTION

LET $f(z,w) \equiv f_0(z)w^n + f_1(z)w^{n-1} + \cdots + f_n(z) \in \mathbb{C}[z,w]$. Classically, the equation $f(z,w) = 0$ was said to define w as an (n-valued) *algebraic function* of z, provided that $f_0(z)$ was not identically 0 and that $f(z,w)$ was squarefree and without factors of the form $z-c$. Then, indeed, the *singular set* $B = \{z\colon$ there are not n distinct solutions w to $f(z,w) = 0\}$ is finite; and as z varies in any simply-connected domain avoiding B, the n distinct solutions $w_1, \ldots, w_n$ of $f(z,w) = 0$ will be analytic functions of z. Now let γ be a simple closed curve in $\mathbb{C} - B$. In the open solid torus $\gamma \times \mathbb{C} \subset \mathbb{C}^2$, the set $K_\gamma = V_f \cap \gamma \times \mathbb{C}$ (where $V_f = \{(z,w)\colon\ f(z,w) = 0\}$) is evidently a closed 1-manifold, as smooth as γ, on which the projection to γ is an n-sheeted (possibly disconnected) covering map. A 1-manifold in a solid torus, which projects as a covering onto the circle factor, is called a *closed braid*. When the torus is embedded (in the standard way) in a 3-sphere (as $\gamma \times \mathbb{C}$ will be, shortly), the closed braid becomes a knot or link in that sphere; if the circle factor is oriented, there is a natural way to orient that knot or link. Which such oriented links, we may ask, arise from algebraic functions (when γ is oriented counterclockwise)?

The points $z_0 \in B$ are of two kinds (some may be of both). If, for some w_0 such that $f(z_0,w_0) = 0$, it also happens that $(\partial f/\partial w)(z_0,w_0)$, we call z_0 a *singular point* of the algebraic function. (Either (z_0,w_0) is a singular point, in the usual sense, of the algebraic curve V_f, or it is a regular point at which the tangent line is the vertical line $z = z_0$.) At a singular point z_0, some solution w to $f(z_0,w) = 0$ has multiplicity greater than 1. On the other hand, z_0 may be a root of $f_0(z)$; then there are not n solutions, even counting multiplicities, to $f(z_0,w) = 0$. A root of $f_0(z)$ is a *pole* of the algebraic function.

The set K_γ, being compact, actually lies in some closed solid torus $\gamma \times D_r = \{(z,w)\colon z \in \gamma, |w| \le r\}$. Let B^4 be the bicylinder $D \times D_r$ where D is the bounded region in $\mathbb{C}$ with $\partial D = \gamma$; then B^4 is homeomorphic to a 4-ball, and its boundary 3-sphere is decomposed in the usual way into two solid tori, $\gamma \times D_r$ and $D \times \partial D_r$. *If no pole of $f(z,w)$ lies in D, then K_γ is the entire intersection of V_f with ∂B*; that is, V_f does not meet $D \times \partial D$. (This may be seen by an appeal to the maximum modulus principle.) Below (except in §3, Remark 2) we will assume $f_0(z)$ is a (non-zero) constant, that is, that there are no poles. This is only for convenience; everything would work as well just assuming that no poles lie in D.

In §2 we recall the definition of *positive* closed braids, and define a strictly larger class, the *quasipositive* closed braids. The definition is purely braid-theoretic. Several mathematicians (including Murasugi, Stallings [9], and Birman [1]) have observed that many positive closed braids, in particular all those which are knots (rather than links), are *fibred links*; there are quasipositive closed braids which are knots and not fibred.

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In §3 we give one proof that the closed braid K_γ is quasipositive. The proof is real semi-algebraic geometry, and gives a method (which is, alas, far from practicable in most cases) of explicitly calculating the braid type of K_γ in terms of one's knowledge of γ and $f(z,w)$.

In §4 we briefly discuss those loops in $M-V$, where M is a simply-connected algebraic variety and V is an algebraic subset, which are freely homotopic to loops which bound analytic (possibly singular) disks in all of M. In many cases, the free homotopy classes of "analytic boundaries" turn out to be precisely those classes which are "quasipositive" in an appropriate sense. When M is the space of unordered n-tuples of (not necessarily distinct) complex numbers, and V is the so-called "discriminant locus" of n-tuples with not all members distinct, the theory applies (to check one hypothesis, I use the method of §3), and we have the following theorem.

THEOREM. *The closed braids K_γ that arise from algebraic functions without poles are precisely the quasipositive closed braids.*

Here are some consequences of the theorem. Many more fibred links occur as K_γ than just those associated to singular points of curves (as in [6])—these "links of singularities" may be recovered as a special case (γ is a small circle enclosing a single point of B, for suitable $f(z,w)$). Many non-fibred knots and links occur as K_γ's. And in each concordance class of links that appears at all, infinitely many distinct links occur; for instance (even for $f(z,w)$ as special as $w^3-3w+2z^m$, $m=1,2,3\ldots$), infinitely many distinct slice knots occur—a marked contrast to the links of singularities.

Remarks and examples conclude the paper.

§2. POSITIVE AND QUASIPOSITIVE BRAIDS AND CLOSED BRAIDS

A general reference for the braid theory used here is [1] (where a polyhedral approach is taken).

For $n \geq 2$ the algebraic n-string braid group B_n is generated by $n-1$ standard generators $\sigma_1,\ldots,\sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $(i=1,\ldots,n-2)$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j|>1$. A word $\sigma_{k(1)}^{e(1)}\cdots\sigma_{k(m)}^{e(m)}$ (each $\varepsilon(j)=\pm1$) in the generators and their inverses is *positive* if each $\varepsilon(j)=+1$, *strictly positive* if also every index from 1 to $n-1$ occurs as some $k(j)$; an element p of B_n is (strictly) positive if it can be represented as a (strictly) positive word.

Let $K \subset \gamma\times\mathbb{C}$ be a closed braid in an open solid torus, with K, the simple closed curve γ, and $\mathbb{C}$ all oriented, and the projection from K to γ smooth and orientation preserving of degree n. It is well-known that the isotopy classes of such K (say, ambient isotopy preserving the product structure of the solid torus) are in 1-1 correspondence with conjugacy classes in B_n. The correspondence is implemented by the choice of a diffeomorphism (preserving orientations) $h:\gamma\times\mathbb{C}\to S^1\times\mathbb{R}\times\mathbb{R}$ of the form $h(z,w)=(h_0(z),h_1(z,w),h_2(z,w))$ together with a basepoint $\exp i\theta_0$ on S^1. Any such h can be changed by an arbitrarily small isotopy, if necessary, to make it yield a "good" braid diagram $d(K)$ in the half-open rectangle $[\theta_0,\theta_0+2\pi]\times\mathbb{R}$ (project onto S^1 and take logarithms for the first coordinate, project onto the first $\mathbb{R}$ factor for the second coordinate, and at multiple points use the second $\mathbb{R}$ factor to determine under- and over-crossings)—"good" in the sense that: $d(K)$ is the union of n properly embedded arcs, on each of which the projection to $[\theta_0,\theta_0+2\pi]$ is a diffeomorphism; there are no triple points of $d(K)$; there are only finitely many double points, all interior to the rectangle, and at each of which the tangent lines to the two arcs are distinct;

and the θ coordinates of distinct double points are distinct. From such a good braid diagram $d(K)$ a word in the letters σ_j and their inverses may be read off, as follows. Let the θ coordinates of the double points be $\theta_1 < \theta_2 < \cdots < \theta_m$. For each $j = 1, \ldots, m$, there are precisely $n-1$ points in $\{\theta_j\} \times \mathbb{R} \cap d(K)$. Let the double point be the $k(j)$th among them, in increasing order of $\mathbb{R}$ coordinate. Let ψ_θ and $\psi_2(\theta)$ parametrize the two arcs that cross at the double point in question, so labelled that $\psi_1'(\theta_j) > \psi_2'(\theta_j)$. Near θ_j there are smooth functions $\phi_1(\theta), \phi_2(\theta)$ so that $\theta \mapsto (\exp i\theta, \psi_l(\theta), \phi_l(\theta))$ $(l = 1, 2)$ parametrize intervals on $h(K)$. Let $\varepsilon(j) = \operatorname{sgn}(\phi_2(\theta_j) - \phi_1(\theta_j))$. Then the word to be read off from $d(K)$ is $\prod_{j=1}^{m} \sigma_{k(j)}^{\varepsilon(j)}$.

A closed braid is *positive* if its corresponding conjugacy class in B_n contains a positive braid. If K has a braid diagram $d(K)$, as above, in which each exponent $\varepsilon(j)$ is 1, certainly K is positive.

Let $w_1, \ldots, w_m$ be arbitrary words in $\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$. We will say that the word $w_1\sigma_{k(1)}w_1^{-1}w_2\sigma_{k(2)}w_2^{-1} \cdots w_m\sigma_{k(m)}w_m^{-1}$ is *quasipositive*, and that $q \in B_n$ is *quasipositive* if it can be represented as a quasipositive word.

A closed braid is *quasipositive* if the corresponding conjugacy class in B_n contains a quasipositive braid.

Now let $\gamma \times \mathbb{C}$ be embedded in S^3 as a tubular neighborhood of an unknotted circle, and let K be a closed n-string braid in that neighborhood. Corresponding to any good braid diagram $d(K)$, in which there are m double points, there is a natural Seifert surface $S \subset S^3$ for K (i.e. an oriented surface with $\partial S = K$) made up of n disks connected by m bands—the disks are "stacked" (they may be taken to be meridional disks of the complementary solid torus to $\gamma \times \mathbb{C}$) and each band connects two adjacent disks in the stack, with a half-twist in one sense or the other depending on the sign $\varepsilon(j)$ of the corresponding double point. (This construction by "bands", following Murasugi, is expounded in Stallings's paper [9]. A general "band representation" which constructs "Seifert ribbons" instead of Seifert surfaces, is discussed in [7].) As in [9], when K is positive *and so displayed by* $d(K)$, any connected component S_0 of S has the property that the push-off map $\pi_1(S_0) \to \pi_1(S^3 - S_0)$ (defined by taking a nowhere-zero normal vectorfield on S_0 and using it to push any loop on S_0 into the complement of S_0) is a bijection. It then follows from a theorem of Neuwirth and Stallings that the boundary of S_0, a union of components of the link K, *is a fibred link*. In particular, K is fibred if either K is a knot or S is connected, which last happens if and only if the word of $d(K)$ is strictly positive. Details of the proof appear in [2]. ①

§3. THE CLOSED BRAIDS K_γ ARE QUASIPOSITIVE

Until further notice, our algebraic functions will not have any poles.

Let $\pi = \mathrm{pr}_1 | V_f : V_f \to \mathbb{C}$. We begin by observing that there is no loss of generality, for the purposes of studying all the braids K_γ, in assuming that V_f is a non-singular curve and that for each $z_0 \in B$, the fibre $\pi^{-1}(z_0)$ consists of $n-1$ distinct points, at one of which V_f has a vertical tangent. Indeed, if this is not so already, any sufficiently small change in the constant term of $f_{n-1}(z)$ will make it so; while the closed braids lying over a fixed γ on the two curves V_f and $V_{f+\varepsilon w}$ are surely isotopic (by a vertical isotopy) for all sufficiently small ε.

Now suppose that γ_0 and γ_1 are isotopic in the complement of B. The differential $D\pi$ is surjective off $\pi^{-1}(B)$; so the isotopy lifts to an isotopy of embeddings between $K_{\gamma_0} \hookrightarrow \gamma_0 \times \mathbb{C}$ and $K_{\gamma_1} \times \mathbb{C}$. In the special case that γ_0 and $-\gamma_1$ cobound an annulus A in the complement of B, then the union of annuli $\pi^{-1}(A) \subset V_f$ is the trace of an isotopy between the closed braids.

To show that K_γ is quasipositive we will isotope γ to a more-or-less normal form for which the conclusion will be obvious. (All the unbridgeable gap between positive and quasipositive lies in that “more-or-less”!)

We will begin by constructing an oriented graph (smoothly embedded in the plane) with vertices including all the points of B. Let $z_1, \dots, z_l$ be the points of B, and for ②
$j = 1, \dots, l$ let $w_{j,l}, \dots, w_{j,n-1}$ be the $n-1$ distinct roots of $f(z_j, w) = 0$. Then for all but finitely many $\theta \in [0, 2\pi]$ the $n-1$ real numbers $\Re((\exp i\theta) w_{j,k})$, $k = 1, \dots, n-1$, are pairwise distinct, for each $j = 1, \dots, l$. Changing the w-coordinate by a rotation, then, we may assume without loss of generality that $\theta = 0$ works, that is, that at each point z_j the $n-1$ real parts $\Re w_{j,k}$ are pairwise distinct. Let $B^+ = B \cup \{z \in \mathbb{C} - B\colon$ for some two distinct solutions w_1, w_2 of $f(z,w) = 0, \Re w_1 = \Re w_2\}$. Then B^+ is the projection of a real algebraic set, so on general principles it is a real semialgebraic set, evidently of dimension 1, and so a graph; we will see this directly in the course of establishing its local structure. We will find a locally-finite (actually finite) subset B_0 of B^+, containing B, so that $\mathbb{C}$ is stratified by B_0, $B^+ - B_0$, $\mathbb{C} - B^+$. Let us consider the intersection of B^+ with a disk around an arbitrary point of $\mathbb{C}$. If this point z_0 does not belong to B, let $\varepsilon > 0$ be sufficiently small that the disk $D_\varepsilon(z_0)$ is disjoint from B. Then on this disk there are analytic functions $w_j(z)$ so that $\pi^{-1}(D_\varepsilon(z_0))$ is the union of the graphs of the functions w_j. Thus $B^+ \cap D_\varepsilon$ is the union of sets $A_{j,k} = \{z \in D_\varepsilon(z_0)\colon \Re(w_j(z) - w_k(z)) = 0\}$. Each difference $w_j - w_k$ is analytic, not identically 0, and so, near any point of $D_\varepsilon(z_0)$, $w_j - w_k$ is a branched cover of its image; so the real analytic set $A_{j,k}$ is a 1-complex, smoothly embedded near its manifold points, and near its finitely many non-manifold points (which we assign to B_0) smoothly equivalent to a union of diameters in a disk. Likewise, distinct sets $A_{j,k}, A_{g,h}$ cross only finitely often; put their intersections in B_0 too.

If we look near a point z_j of B the situation is slightly different. Here, for small $\varepsilon > 0$, $\pi^{-1}(D_\varepsilon(z_j))$ consists of not n but $n-1$ smooth disks. There are $n-2$ functions $w_k(z)$ analytic on $D_\varepsilon(z_j)$ whose graphs are $n-2$ of these disks; the last disk is parametrized by $t \mapsto (z_j + t^2, w(t))$, where $|t|^2 < \varepsilon$, $w(t)$ is analytic, and $w'(0) \neq 0$ (we are at a simple vertical tangent). Since we have assumed $\Re w_i(z_j), \dots, \Re w_{n-2}(z_j), \Re w(0)$ are distinct, after possibly shrinking ε we can guarantee that $B^+ \cap D_\varepsilon(z_j)$ has no contributions from the interaction of any of the $w_k(z)$ with each other or with $w(t)$: we will have simply $B^+ \cap D_\varepsilon(z) = \{z_j + t^2\colon |t|^2 < \varepsilon, \Re(w(t) - w(-t) = 0\}$. But, like $w(t)$, $w(t) - w(-t)$ has non-zero derivative at $t = 0$, so (shrinking again if necessary) we see that $\{t\colon |t|^2 < \varepsilon, \Re(w(t) - w(-t) = 0\}$ is smoothly (and equivariantly) equivalent to a diameter of the t-disk, and its image in B^+ is smoothly equivalent to a radius of $D_\varepsilon(z_j)$.

We now orient B^+, at the same time labelling each edge with one of the symbols $\sigma_1, \dots, \sigma_{n-1}$. Let A be an arc in $B^+ - B_0$. Then anywhere in the interior of A, one may find a short transverse arc which intersects A only in one point, and B^+ nowhere else. Over such an arc the n branches of $w(z)$ are distinct, and even their real parts are distinct except where the transverse arc crosses A: at that point, for some k, $1 \leq k \leq n-1$, the branches with real parts kth-greatest and $(k+1)$st-greatest among all the branches have equal real part; label A with σ_k. (Clearly this label is independent of the transverse arc.) Orient A so that, when the orientation of the transverse arc, following the orientation of A, gives the complex orientation of $\mathbb{C}$, the braid diagram over the transverse arc is one for σ_k (rather than for σ_k^{-1}).

Let γ be a smooth simple closed curve in $\mathbb{C} - B$, oriented counterclockwise, and bounding the bounded region D. Let $z_1, \dots, z_s$ be the points of $B \cap D$, let $D_j = D_\varepsilon(z_j)$, and let $C_j = \partial D_j$ oriented counterclockwise, for $j = 1, \dots, s$. For sufficiently small ε the disks D_j lie in D and are pairwise disjoint. By a traditional construction of the theory of algebraic functions, there is a disk $D_0 = D_{\varepsilon_0}(z_0) \subset D - \bigcup_{j=1}^{s} D_j$ with boundary C_0 (oriented counterclockwise), and pairwise disjoint smooth embeddings $a_j : [0,1] \to D$ $(j = 1, \ldots, s)$ with $a_j(0) \in C_0, a_j(1) \in C_j, a(]0,1[) \subset D - \bigcup_{k=0}^{s} D_k$, and a_j perpendicular to C_0 and C_j at its ends, all so that γ is isotopic in $D - B$ to a simple closed curve γ' which "follows the arcs and circles." Formally, $\gamma' = \partial(D_0 \cup \bigcup_{j=1}^{s} N_j \cup \bigcup_{j=1}^{s} D_j)$, where the sets N_j are "strips"—pairwise disjoint product neighborhoods of the arcs $a_j([0,1])$, say $N_j = \nu_j([-1,1] \times [0,1])$, where ν_j is an embedding such that $\nu_j(0,t) = a_j(t)$ $(t \in [0,1])$, etc.

Now we involve B^+. Without loss of generality, we assume that D_j $(j = 1, \ldots, s)$ intersects B^+ only in an arc that joins z_j to C_j, and that D_0 is disjoint from B^+. It is clear that, in performing the traditional construction, we may so arrange things that the embeddings a_j are transverse to the stratification—they miss B_0 and cross the manifold points of B^+ transversely in the ordinary sense—and then make the product neighborhoods N_j so narrow that $N_j \cap B^+$ is itself a product $[-1,1] \times (a_j([0,1]) \cap B^+)$.

Let $h_0 : \gamma' \to S^1$ be a diffeomorphism so that $h_0^{-1}(1)$ is a point on C_0; define $h : \gamma' \times \mathbb{C} \to S^1 \times \mathbb{R} \times \mathbb{R}$ by $h(z,w) = (h_0(z), \Re w, \Im w)$. I claim that applying the construction of §2 to this h (with base-point 1 on S^1) yields a good braid diagram $d(K'_\gamma)$ for which the braid word is already in the form $\prod_{j=1}^{m} \alpha_j \sigma_{k(j)} \alpha_j^{-1}$; so that K'_γ and K_γ are quasipositive. Indeed, the diagram $d(K'_\gamma)$ is the "product" in an obvious sense of diagrams for the (non-closed) braids which correspond to the successive arcs $\nu_{j(1)}(\{1\} \times [0,1])$, $C_{j(1)} - \nu_{j(1)}(]-1,1[\times \{1\})$, $\nu_{j(1)}(\{-1\} \times [0,1]), \ldots$ of γ' (where the order in which the points of $B \cap D$ are gone around is $z_{j(1)}, \ldots, z_{j(s)}$, and where the arcs $\nu_{j(k)}(\{-1\} \times [0,1])$ are of course traversed from the 1 end to the 0 end). Each arc contributes, in turn, the word in the symbols $\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$ which is given by its successive crossings of the labelled arcs of $B^+ - B_0$ (a crossing which, following the orientation of the arc, gives the wrong orientation to $\mathbb{C}$, is what merits the exponent -1). Obviously, by our construction, the two edges of a strip N_j give (up to orientation) the same word as the central arc $a_j([0,1])$, call it α_j. So the claim of quasipositivity is proved once one sees that the diagrams corresponding to the arcs on the circles C_j $(j = 1, \ldots, s)$ contribute exactly a generator $\sigma_{k(j)}$, and not the inverse of a generator. (Certainly by construction each such arc meets B^+ in just one point.) The exponent is seen to be $+1$ in all cases; it suffices to study just one example, for instance $f(z,w) = w^2 + z$, where $B = \{0\}$, B^+ is the non-negative real numbers, and the conclusion is obvious.

We have proved that *if $f(z,w)$ has no poles inside γ, the closed braid K_γ is quasipositive.* A converse will be proved in the next section.

Remarks. (1) The *exponent sum $e(w)$* of a braid word $\prod_{j=1}^{m} \sigma_{k(j)}^{\varepsilon(j)}$ is $\sum_{j=1}^{m} \varepsilon(j)$. From the form of the relations in B_n, this is actually defined on braids; clearly it is conjugation invariant, so it is an isotopy invariant of closed braids. The exponent sum of a quasipositive braid is non-negative. The proof above actually shows that the exponent sum of $K\gamma$ is the number of points of B enclosed by γ (counting multiplicities appropriately if $f(z,w)$ is not restricted to simple vertical tangents and no singularities). It is easy to see that the exponent sum of a closed braid K equals $sw(K)$, the *self-winding* defined by Laufer [4]. The proof above readily generalizes to analytic (rather than simply algebraic curves), and some theorems of [4] can be recovered quickly.

(2) We have excluded from consideration simple closed curves γ enclosing poles of our algebraic function. This is because, on the one hand, if γ does enclose any poles of $f(z,w)$ then the closed braid K_γ is not the whole intersection of V_f with $\partial(D\times D_r^2)$ for any r—there are always components in $D\times\partial D_r^2$ corresponding to the poles; while, on the other hand, if we allow poles then *every isotopy class of closed braid* can be realized as the braided part K_γ of that intersection, for appropriate $f(z,w)$ and γ. The proof is by the theory of rational approximation. Let $\gamma=\{z\colon |z|=1\}$. Let $K_0\subset\gamma\times\mathbb{C}$ be a closed braid, not necessarily smoothly embedded, with components $C_1,\ldots,C_d$ of degrees $n_1,\ldots,n_d$. For a suitable large constant M, the polynomial $p(z)=M(z-1)^{n_1}\cdots(z-d)^{n_d}$ is such that the compact set $P=\{z: |p(z)|\le 1\}$ is the union of d components, each diffeomorphic to a disk, on the boundaries of which $p(z)$ has degrees $n_1,\ldots,n_d$ respectively. Evidently, there is a unique continuous function $q_0(z)$ defined on ∂P such that the pair $(p,q_0):\partial P\to\gamma\times\mathbb{C}$ parametrizes K_0. According to the Hartogs-Rosenthal Theorem [3], on any compact subset of $\mathbb{C}$ with measure 0 (e.g. ∂P) the rational functions with poles off the compact set are uniformly dense in the continuous functions. Let $q(z)$ be a rational approximation to $q_0(z)$ so close that $K=(p,q)(\partial P)$ lies inside a tubular neighborhood of K_0 in $\gamma\times\mathbb{C}$ (which exists, even though K_0 may not be smooth, because K_0 is a closed braid); then K and K_0 are isotopic (by a vertical isotopy). But $(p,q)(\mathbb{C})=V$ is an algebraic curve in $\mathbb{C}^2$ (generally with many singularities), that is, $V=V_f$ for some $f(z,w)$.

Of course, when q has poles interior to P as well as in $\mathbb{C}-P$, there will be poles of $f(z,w)$ enclosed by γ.

(3) For later use, and intrinsic interest, we give some calculations of sets B^+ in particular examples.

Example 3.1. $f(z,w)=w^2-z$. Here $w_1=\sqrt{(z)}$, $w_2=-\sqrt{(z)}$, and $\Re w_1=\Re w_2$ iff w_1 and w_2 are pure imaginary iff z is negative real; thus B^+ is the ray $]-\infty,0]$ ending in 0, the only point of B; the ray is oriented away from 0, and labelled σ_1. More generally, if $f(z,w)=w^2-z^n-1$, then $B^+=\{z\colon z^n+1 \text{ is negative real}\}$ is the union of n rays, oriented outward, emanating from the nth roots of 1, all labelled σ_1. Of course, in the 2-string braid group, which is infinite cyclic, quasipositive is the same as positive.

Example 3.2. $f(z,w)=w^3-3w+2z^n$. If w_1, w_2, and w_3 are the three roots of $f(z,w)=0$, then $w_1+w_2+w_3=0$, $w_1w_2+w_1w_3+w_2w_3=-3$, and $w_1w_2w_3=-2z^n$. Eliminating w_3 between the first two equations, we get the quadratic relation $w_2^2+w_1w_2+(w_1^2-3)=0$, whence $\{w_2,w_3\}=\{\frac{1}{2}(-w_1+\sqrt{(-3w_1^2+12)},\frac{1}{2}(-w_1-\sqrt{(-3w_12+12}\}$. The indices are irrelevant; there is perfect symmetry, and we see that $B^+=\{z\colon \Re w_2=\Re w_3\}=\{z\colon \sqrt{(-3w_1^2+12)} \text{ is pure imaginary}\}=\{z\colon -3w_1^2\in\,]-\infty,-12]\}$. For $n=1$, B^+ is thus the two rays $]-\infty,-1]$ and $[1,\infty[$; in general, B^+ is the union of $2n$ rays, oriented outward, emanating from the $2n$th roots of 1, and labelled alternately σ_2 and σ_1. For $n=4$, we get an example of a quasipositive, not positive, knot K_γ for the curve pictured in Figure 1; the braid word here is $\sigma_1\sigma_2^3\sigma_1\sigma_2^{-3}$. This knot is 8_{20} of the Alexander-Briggs table; it is slice—indeed, ribbon—and non-trivial; it cannot be positive because, for instance, according to [8] a non-trivial positive closed braid has signature greater than 0.

Example 3.3. (This example will be used in the next section to establish that all quasipositive closed braids occur as K_γ's.) Consider the reducible polynomial $f(z,w)=P(w)(w-z)$, where P is a polynomial in w without double roots. Here $B=z\colon P(z)=0$

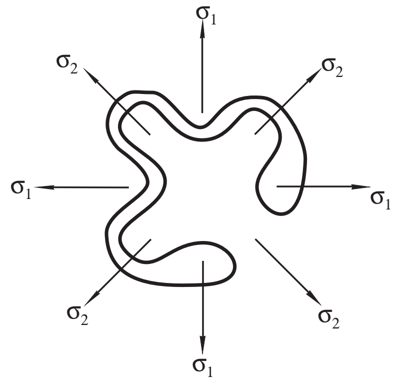

Fig. 1.

is just the set of roots of P, and B^+ will either be all of $\mathbb{C}$ (in the unfortunate case, ruled out in the discussion above by a rotation of w when necessary, that some two distinct roots of P have equal real part) or, generically, the union of n straight (real) lines $\Re z = r_j$ ($j = 1, \ldots, n$), where r_j is the real part of a (unique) root of P: B_0 here is just B. Now suppose P has real coefficients, and consider, for $\varepsilon \neq 0$ small and real, the set B^+_ε corresponding to $f(z,w) + \varepsilon$, and its distinguished subset B_ε. Evidently these sets are invariant under complex conjugation of the variable z. One sees that, in fact, the points of the original B were "to be counted twice" and that as ε moves away from 0 these points of multiplicity two alternately (with increasing r_j) bifurcate to two real points and to two conjugate, non-real points. Further, it is not much harder to see that the interval of the real line between the points of a real pair itself lies entirely in B^+_ε. Only in the simplest case, when P is linear, have I been able to get an explicit description of the full set B^+_ε; but this suffices to give an adequate qualitative description in the general case. Namely, if $P(w) = w$, say, then $B^+_\varepsilon = \{z\colon\ w^2 - wz + \varepsilon\}$ has two real roots with equal real parts$\} = \{z\colon\ \sqrt{(z^2 - 4\varepsilon)}$ is pure imaginary$\} = \{z\colon\ z^2 \in\ ]-\infty, 4\varepsilon]\}$. When $\varepsilon < 0$, this is the union of two rays lying on the imaginary axis, oriented outward; when $\varepsilon > 0$, however, it is a cross, containing the whole imaginary axis and a short interval of the real axis—the short arms oriented towards the crossing point, the long arms out to infinity. Now for a polynomial P of higher degree, there is a neighborhood N of B which is a union of disjoint disks around the roots of P, so that for ε sufficiently small (and real) the set B^+_ε looks like B^+ outside N (that is, it consists of two proper arcs leaving each disk of N and going to infinity without crossing) while inside alternate disks of N (from left to right) B^+_ε looks like the case $P(w) = w$, with an ε of the same or opposite sign. So the whole set B^+_ε is, qualitatively, a sequence of alternate crosses and double-rays; Figure 2 gives a sketch in case $P(w) = w(w-1)(w+1)$. The orientations are as in the linear model, and from left to right the arcs of B^+_ε are labelled $\sigma_1, \ldots, \sigma_n$ (where n is the degree of P) in batches. For later use note that, from an arbitrary basepoint $*$ (off B^+) for each j a loop can be drawn whose word in the labels σ_j and σ_j^{-1} is freely equal (in the free group on the labels) to σ_j. For instance, for $*$ to the far left in Fig. 2, a loop for σ_1 is obvious; a loop for σ_2 can slip between the two rays labelled σ_1, do the obvious, and slip back; a loop for σ_3 will have to intersect the cross labelled σ_2, but if it goes through the gap between the two ends of the short arm it will pick up successively σ_2 and σ_2^{-1}; and so on.

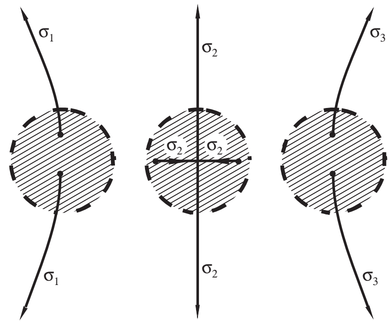

Fig. 2.

§4. ANALYTIC LOOPS IN THE CONFIGURATION SPACE

Throughout this section let D be the closed unit disk in $\mathbb{C}$, $S^1 = \partial D$ its boundary oriented counterclockwise.

If X is a complex analytic space, an *analytic disk* in X is a map $i : D \to X$ which is the restriction to D of a complex analytic map on some slightly larger open disk; an *analytic loop* is the oriented boundary of an analytic disk. Suppose X is simply connected, and $V \subset X$ is a closed analytic subset such that $X - V$ is connected but no longer simply connected. We may ask, which non-trivial homotopy classes of loops in $X - V$ contain representatives which are analytic loops in X?

Even when the question is asked in such generality, partial answers can be given. For our present purposes, however, it is enough to have the answer with X and V considerably restricted. So, let $X = \mathbb{C}^n$ be affine space, and let $V \subset \mathbb{C}^n$ be an algebraic hypersurface $V = V_f = \{\mathbf{z} \in \mathbb{C}^n\colon\ f(\mathbf{z}) = 0\}$, possibly singular and/or reducible (but without multiple components). The complex manifold $R(V)$ of regular points of V is of (real) codimension 2 in $\mathbb{C}^n$, and is everywhere dense in V; let its connected components be $R_1, \dots, R_s$. For some arbitrary point on each R_i, let D_i be an oriented normal 2-disk intersecting V only at that point, and there positively (with respect to the complex orientations of $R(V)$ and $\mathbb{C}^n$); for some fixed basepoint $*$ not on V, let a_i be an arc in $\mathbb{C}^n - V$ from $*$ to a point on ∂D_i; let l_i be a loop which runs from $*$ along a_j to ∂D_i, once around ∂D_i countercloskwise, and back along a_j to $*$; and let $[l_i]$ be the class of l_i in $\pi_1(\mathbb{C}^n - V)$; all for $i = 1, \dots, s$. For later use, in the particular case that $n = 1$ and V is a finite set of points, each one a component R_i, let us demand further that the disks D_i be pairwise disjoint from each other and from $*$, and that the arcs a_i be simple, pairwise disjoint except for their common endpoint $*$, and outside the union of the D_i (except for their other endpoints).

An element of $\pi_1(\mathbb{C}^n - V; *)$ which can be written as a product $\prod_{j=1}^{m} w_i[l_j(i)]w_i^{-1}$ of conjugates of the classes $[l_i]$ will be called a *quasipositive* element of the fundamental group. Quasipositivity is invariant under conjugation, and thus is really a property of free homotopy classes of loops.

LEMMA 1. *An analytic loop in $\mathbb{C}^n - V$ represents a quasipositive conjugacy class in $\pi_1(\mathbb{C}^n - V; *)$.*

Proof. Let $i : D \to \mathbb{C}^n$ be an analytic disk in $\mathbb{C}^n$ with $i(S^1) \cap V = \emptyset$. Replacing i by a sufficiently close approximation (for instance, a high-order Taylor polynomial at 0) we may assume i is the restriction to D of a (vector-valued) complex polynomial $p(t)$ of a single complex variable t, without changing the (free) homotopy class of $i(S^1)$ in the complement of V. In $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n$ let Z be the set $\{(t, \varepsilon, \mathbf{z})\colon\ \mathbf{z} = p(t) + \varepsilon$ belongs to $S(V)\}$, where $S(V) =$

$V-R(V)$ is the singular set of V, an algebraic set of complex dimension no greater than $n-2$. Then Z is an algebraic subset of $\mathbb{C}^{2n+1}$. Its complex dimension is no greater than $n-1$, for $\mathbf{z}$ varies in a set of dimension at most $n-2$, $p(t)$ is on a curve, and ε is determined by $\mathbf{z}$ $p(t)$. Then the projection of Z onto the second factor, $pr_2(Z)\subset\mathbb{C}^n$, is again an algebraic set of dimension at most $n-1$. Then almost any ε, in particular, almost any ε sufficiently close to $\mathbf{0}$, is not in $pr_2(Z)$. Translating $i(D)$ by an appropriate small ε will not change the free homotopy class of the analytic loop $i(S^1)$ while ensuring that $p(\mathbb{C})$ and its subset the new analytic disk meet $S(V)$ nowhere. Now the whole intersection of the analytic disk and V is in the manifold $R(V)$ and it is a simple matter to make the intersection transverse, when it will appear that each point of intersection counts $+1$ because $p(\mathbb{C})$ and $R(V)$ are complex manifolds. Since the boundaries of two normal disks (positively oriented) at any two points of a component R_i are freely homotopic, the analytic loop is a product of conjugates of the loops l_i. ■

LEMMA 2. *Conversely, when $n=1$, every quasipositive conjugacy class in $\pi_1(\mathbb{C}-\{z_1,\dots,z_s\})$ is represented by an analytic loop in $\mathbb{C}$.*

I do not know if Lemma 2 is true when $n\neq 1$. However, the following immediate consequence of Lemma 2 suffices to replace the putative stronger version for our purposes.

COROLLARY. *If there is a proper analytic map L of $\mathbb{C}$ into $\mathbb{C}^n$ so that the induced homomorphism $\pi_1(\mathbb{C}-L^{-1}(V))\to\pi_1(\mathbb{C}^n-V)$ is surjective, then every quasipositive conjugacy class in $\pi_1(\mathbb{C}^n-V)$ is represented by an analytic loop (which in fact bounds an analytic disk lying on $L(\mathbb{C})$).* ■

Proof of Lemma 2. Let $\alpha=\prod_{j=1}^{m} w_i[l_j(i)]w_i^{-1}\in\pi_1(\mathbb{C}=\{z_1,\dots,z_s\},*)$ be quasipositive. Let the disks D_j ($j=1,\dots,s$) be as above, let D_0 be a disk centered at $*$ and disjoint from all the other D_j, and suppose for neatness that for each $j=1,\dots,s$ the arc a_j intersects D_0 in a radius of D_0, and comes into D_j normally. Let $c(j)\geq 0$ be the number of times the index j appears as $j(i)$ in the given presentation of α, as i runs from 1 to m. Let $D'_{j,c}$ ($j=1,\dots,s,c=1,\dots,c(j)$) and D'_0 be 2-disks which we think of as (2-dimensional) 0-handles, and let N_i ($i=1,\dots,m$) be strips, each homeomorphic to $[-1,1]\times[0,1]$, which we think of as 1-handles. Fix orientations on all the handles. Take m disjoint closed intervals, successive in the cyclic order, on ∂D_0, and one closed interval on each of the $\partial D'_{j,c}$ (of which there are m all together). We form an identification space from the disjoint union of all the 0- and 1-handles as follows: orientedly, attach one end $[-1,1]\times\{0\}$ of N_i to the ith chosen interval on ∂D_0, and the other end $[-1,1]\times\{1\}$ to the chosen interval on $\partial D'_{j(i),c}$ (where c is the number of k with $k\leq i$, $j(k)=j(i)$). Then this identification space D'' is homeomorphic to a disk. We will map D'' into $\mathbb{C}$ handle by handle. First each $D_{j,c}$ is mapped homeomorphically, preserving orientation, onto D_j so that the image of the chosen interval on $\partial D_{j,c}$ is centered at the end of a_j on ∂D_j; and D_0 is mapped homeomorphically, preserving orientation, onto D_i. For each conjugator w_i, find an immersed arc in $\mathbb{C}$ which begins (outward normal) in the image on ∂D_0 of the ith chosen interval on ∂D_0 and represents w_i in $\pi_1(\mathbb{C}-\{z_1,\dots,z_s\},D_0)$; then map the center line $\{0\}\times[0,1]$ of N_i to an arc which follows the arc representing w_i from ∂D_0 back to D_0, then in D_0 to $*$, and then along $a_{j(i)}$ to D_j. Because the exponent of $[l_{j(i)}]$ in α is $+1$ and not -1, the map on this center line can be extended over all of N_i to give an immersed tubular neighborhood of the image of the centerline, which respects the identifications at both ends. The map so constructed is an immersion on the interior $\mathring{D}''$, and on the boundary represents α. By "transport of structure" the interior of D'' becomes a Riemann surface, and by the Riemann Mapping theorem there is an analytic homeomorphism $\mathring{D}_{1+\varepsilon} \to \mathring{D}''$, where $\mathring{D}_{1+\varepsilon} = \{z\colon |z| < 1+\varepsilon\}$, for any $\varepsilon > 0$. For appropriately small ε, if i is the composite $D \subset \mathring{D}_{1+\varepsilon} \to \mathring{D}'' \to \mathbb{C}$, then i is an analytic disk whose boundary $i(S^1)$ represents (the conjugacy class of) α. (A tiny bit more juggling could assure that $i(S^1)$ passed through $*$.) ■ ③

Presumably the hypothesis of the corollary is always true, even with L a linearly parametrized straight line in sufficiently general position (see [5, p. 33]). In any case, consider the following example.

Example 4.1. The group B_n may be defined topologically as the fundamental group of the *configuration space* of unordered n-tuples of distinct points in $\mathbb{R}^2$. Reading $\mathbb{C}$ for $\mathbb{R}^2$, one may recognize that, first, the space $\mathbb{C}^n/\mathbb{S}_n$ (where $\mathbb{S}_n$, the symmetric group on n letters, acts by permuting the coordinates) of unordered n-tuples of complex numbers (distinct or not) is in a natural way equal to $\mathbb{C}^n$ again, by the theorem on symmetric polynomials; and, second, that the so-called "multi-diagonal" or discriminant locus, consisting of unordered n-tuples of which two (at least) are equal, is an algebraic hypersurface V_Δ in the affine space $\mathbb{C}^n/\mathbb{S}_n$. I claim that Example 3.3 provides one with a line L in $\mathbb{C}^n/\mathbb{S}_n$ satisfying the hypothesis of the Corollary to Lemma 2. For, what "is" an element of $\mathbb{C}^n/\mathbb{S}_n$ but the monic polynomial of degree n, in one complex variable w, whose roots are the unordered n-tuple in question? Under this identification, the affine coordinates in $\mathbb{C}^n/\mathbb{S}_n$ are precisely the significant coefficients of that polynomial (to wit, up to sign, the elementary symmetric functions of the roots). Now, if the polynomial $P(w)$ in Example 3.3 is chosen monic of degree $n-1$, then the assignment $L: z \mapsto P(w)(w-z) + \varepsilon \in C[w]$ of a monic polynomial of degree n is clearly a linear parametrization of a straight line in $\mathbb{C}^n/\mathbb{S}_n$. The work done in the example shows that $\pi_1(L(\mathbb{C}) - V_\Delta) \to \pi_1(\mathbb{C}^n/\mathbb{S}_n - V_\Delta) = B_n$ is surjective. Further, the two uses of the word "quasipositive" coincide here.

According to this example and the corollary, every quasipositive element of B_n, when considered as a homotopy class in the configuration space, contains an analytic loop in $\mathbb{C}^n/\mathbb{S}_n$. But an analytic disk $i: D \to \mathbb{C}^n/\mathbb{S}_n$ is nothing more nor less than an n-valued analytic function on D, that is, an analytic subset of $D \times \mathbb{C}$ which projects properly and n-to-1 (counting multiplicities) to D. Without changing the free homotopy class of $i(S^1)$ in $\mathbb{C}^n/\mathbb{S}_n - V_\Delta$, one may (as in the proof of Lemma 1) replace the analytic function by (the restriction to D of) a vector-valued polynomial; and a polynomial map from $\mathbb{C}$ to $\mathbb{C}^n/\mathbb{S}_n$ is precisely an n-valued *algebraic* function without poles. We have proved the following.

THEOREM. *The closed braids that arise from algebraic functions without poles are precisely the quasipositive closed braids.*

Remarks. (1) Which classes in $\pi_1(X - V; *)$ are represented by analytic loops depends not only on $X - V$ but very strongly on X as well. For instance, the natural way to complete the affine space $\mathbb{C}^n/\mathbb{S}_n$ is to $(\mathbb{CP}^1)^n/\mathbb{S}_n$, which is canonically $\mathbb{CP}^n$. Let $\overline{V_\Delta}$ be the completion of V_Δ in $\mathbb{CP}^n$ and let ${\mathbb{CP}_\infty}^{n-1}$ be $\mathbb{CP}^n - \mathbb{C}^n$, that is, the unordered n-tuples of extended complex numbers one at least of which is ∞. Then certainly $(\mathbb{C}^n/\mathbb{S}_n) - V_\Delta = ((\mathbb{CP}^1)^n/\mathbb{S}_n) - (\overline{V_\Delta} \cup {\mathbb{CP}_\infty}^{n-1})$. But the loops in this space, which are boundaries of analytic disks in the whole projective space, fall into every homotopy class: everything is quasipositive. Indeed, an analytic disk in the projective space is an n-valued analytic function *with poles allowed*; the poles correspond to intersections of the disk with ${\mathbb{CP}_\infty}^{n-1}$. Then by Remark 2 of §3 we actually have that any loop at all can be perturbed by an arbitrarily small amount, to become the boundary of an analytic disk (probably crossing infinity). In general, it appears that there will be more analytic disks in a projective variety

than in a comparable affine one. (2) If X is a simply connected complex manifold, and V is a non-singular analytic subset with finitely many components, with the components of complex codimension 1 being $R_1, \ldots, R_s$, then it is general knot theory that $\pi_1(X - V; *)$ is normally generated by the classes of loops l_i, $i = l, \ldots, s$, defined as in the case studied earlier of $X = \mathbb{C}^n$. In fact, even when V is singular (without multiple components) and the R_i are the complex-codimension-1 components of its regular set, the same conclusion holds—one need only observe that the union of the singular set $S(V)$ and the regular components of complex codimension 2 or more, as an analytic variety in its own right, has a resolution which is a smooth map of a smooth manifold into X; then any loop in $X - V$ may be made to bound a smooth 2-disk in X transverse to the resolution, and therefore disjoint from its image. Note however that this argument depends on the ambient space X being a smooth manifold with its given structure as analytic space. In this connection it is worth contemplating the example of $X = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4\colon z_1^2 + z_2^3 + z_3^5 = 0\}$. This is the product of $\mathbb{C}$ (the z_4 factor) with the cone on the dodecahedral space [6], and by the celebrated Double Suspension Theorem, X is homeomorpic to $\mathbb{C}^3$. The singular set $S(X)$ is a straight ④
complex line, with real codimension 4. Of course $\pi_1(X - S(X))$ has 120 elements. (It can be shown that each of them is, in fact, represented by analytic loops.)

(3) It was asserted in the introduction that not all quasipositive knots were fibred. Indeed, the first non-fibred knot in the Alexander-Briggs table, 5_2, can be represented as the closure of the quasipositive braid $\sigma_1^2\sigma_2(\sigma_2\sigma_1\sigma_2^{-1})$.

(4) For each n, there is an *analytic* curve V_f in $\mathbb{C}^2$, smooth, and n-sheeted over the z-axis, such that all quasipositve n-string closed braids occur as K_γ for this $f(z,w)$ and an appropriate γ. For $n = 3$, one may take $f(z,w) = w^3 - 3w + 2\exp z$. Here, the points of B are the integral multiples of πi, and B^+ is a union of horizontal rays.

(5) Every oriented link has infinitely many representations as a closed braid (see [1]). It would be interesting to have purely knot-theoretical necessary and/or sufficient conditions that one of the representations be quasipositive. Presumably not every knot or link has such a representation. I hope to return to this and related questions in a future paper. ⑤

Acknowledgements—I wish to thank the referee of an earlier draft for pointing out that, with the conventions I had then established, my braids were actually quasi*negative*. And I am very grateful to the 1980 Georgia Topology Conference, where in giving a talk on some of these results I first realized the significance of analytic disks in the configuration space.

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ADDENDA

Typographical errors in the original publication have been corrected without notice; it is to be hoped that no new ones have been introduced. The following notes provide updates on various points.

(1) [19] gives another proof that the closure of a positive braid is a fibered link, by constructing the fibration explicitly.

(2) Some other applications of the oriented graph B^+ have been given by Orevkov [17], [18] and Dung [13].

(3) I am indebted to Stepan Orevkov for his observation that Sandy Blank's unpublished 1967 thesis (see [11]) contains a proof that (what is here called) the quasipositivity of α is equivalent to the existence of an immersion $\mathring{D}'' \to \mathbb{C}$ like that constructed in the proof of Lemma 2.

(4) The "celebrated Double Suspension Theorem" is expounded in [15].

(5) The existence of a link which has no representation as the closure of a quasipositive braid was first proved using knot polynomials, as a corollary to an inequality of Morton [16] and Franks and Williams [14]. Boileau and Orevkov [12] have characterized such "quasipositive links" as precisely the links isotopic to boundaries of pieces of complex plane curve in D^4, but "purely knot-theoretical necessary and/or sufficient conditions" remain elusive.

(6) [2] was published as [10].

(7) [7] was published as [20].

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