Functions

Recall **A function f is** a rule which assigns to each element x of a set D, exactly one element, $f(x)$, of a set E.

- A function can be viewed as a machine like object which acts on a variable to transform it.
- For example, the function $f(x) = 2x + 1$, transforms the number x by multiplying it by 2 and adding 1.
- We can gain a lot of information about the behavior of a function by using algebra and by calculating derivatives if they exist.
- We can also gain a lot of information about a function by sketching its graph either using the basic graphing techniques from precalculus or the more sophisticated ones from Calculus 1.

- The graph of every function passes the vertical line test i.e. when we graph the equation $y = f(x)$ every vertical line cuts the graph at most once.
- In fact if the graph of an equation passes this test, the graph is the graph of some function and we can (theoretically) solve for y in terms of x.

One-To One Functions

One-to-one Functions A function f is 1-to-1 if it never takes the same value twice or for every pair of numbers x_1 and x_2 in the domain of f;

$$\boxed{f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.}$$

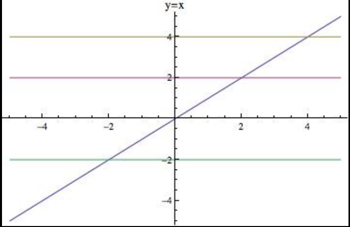
- **Example** The function $f(x) = x$ is one to one,
- because if $x_1 \neq x_2$, then $(x_1 =) f(x_1) \neq f(x_2) (= x_2)$.
- On the other hand the function $g(x) = x^2$ is not a one-to-one function, because $g(-1) = g(1)$.

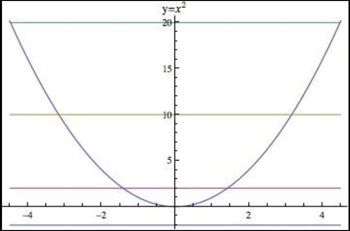
- Note that to prove that a function is not one-to-one, it is enough to find just one pair of numbers x_1 and x_2 with $x_1 \neq x_2$ for which $f(x_1) = f(x_2)$ whereas to prove that a function is one to one, we must show that $f(x_1) \neq f(x_2)$ for every such pair.

Graph of a one-to-one function

If f is a one to one function then no two points (x_1, y_1), (x_2, y_2) have the same y-value. This is equivalent to the geometric condition that no horizontal line cuts the graph of the equation $y = f(x)$ more than once.

- **Example** We can draw the same conclusions about the functions we looked at in the previous slides from the graphs:



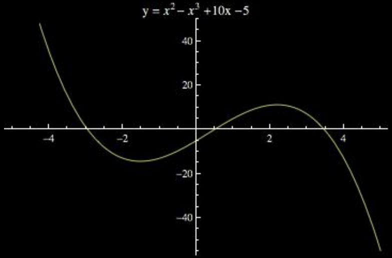


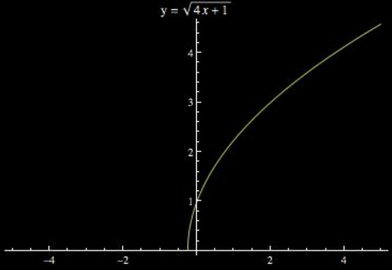
- Note that the lines $y = 2$, $y = 10$ and $y = 20$ all cut the graph of $y = x^2$ twice, showing that it is not a 1-to-1 function.

Determining if a function is one-to-one geometrically

Horizontal Line test (HLT) : A graph passes the Horizontal line test if each horizontal line cuts the graph at most once.

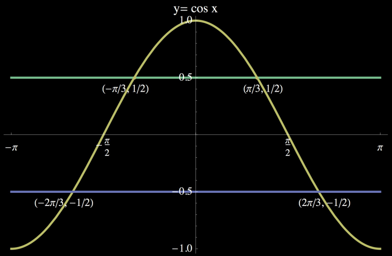
- A function f is one-to-one if and only if the graph $y = f(x)$ passes the Horizontal Line Test (HLT).
- **Example** Which of the following functions are one-to-one?





Example: Cosine

Is the function $f(x) = \cos x$ a one-to-one function?

- 

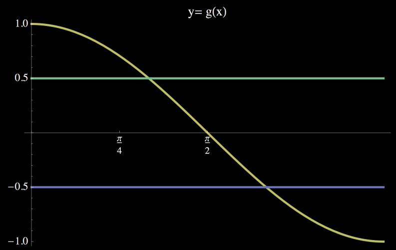
- We see that there are several horizontal lines that cut the graph more than once, So the cosine function is not one-to-one

Example: Restricted Cosine Function

The following piecewise defined function, is called the restricted cosine function because its domain is restricted to the interval $[0, \pi]$.

$$g(x) = \begin{cases} \cos x & 0 \leq x \leq \pi \\ \text{undefined} & \text{otherwise} \end{cases}$$

We have Domain(g) $= [0, \pi]$ and Range(g) $= [-1, 1]$.



- Is $g(x)$ a one-to-one function?
- The answer is yes, because each horizontal line cuts the graph at most once.