Inverse Functions

Inverse Functions <u>If f is a one-to-one function</u> with domain A and range B, <u>we can define an inverse</u> <u>function f^{-1}</u> (with domain B and range A) by the rule

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y.$$

- This satisfies the requirements for the definition of a function, precisely because each value of y in the domain of f^{-1} has exactly one x in A associated to it by the rule $y = f(x)$.
- We will use this very important equivalence of equations in three ways:
  - To find $f^{-1}(y)$ for specific values of y without finding a formula for f^{-1} itself.
  - To find a formula for f^{-1}.
  - To define new functions as inverses of well known functions.

Finding $f^{-1}(y)$ for specific values of y.

Example: If $f(x) = x^3 + 1$, use the equivalence of equations given above find $f^{-1}(9)$.

- We first write out our equivalence of equations:
  $f^{-1}(y) = x$ if and only if $f(x) = y$.
- Replacing y by 9 tells us that
  $f^{-1}(9) = x$ is the same as saying that $f(x) = 9$.
- Substituting the formula for f tells us that
  $f^{-1}(9) = x$ is the same as saying that $x^3 + 1 = 9$.
- Thus solving for x in
  $f^{-1}(9) = x$ is the same as solving for x in $x^3 = 8$. which gives $x = 2$.
- Thus $f^{-1}(9) = 2$.

- Try to repeat this process to find $f^{-1}(28)$ before you see the solution. (Do not solve this by finding a formula for the inverse function, the purpose of this exercise to is learn the above method.)

Finding $f^{-1}(y)$ for specific values of y.

Example: If $f(x) = x^3 + 1$, find $f^{-1}(28)$.

- $f^{-1}(y) = x$ if and only if $f(x) = y$.
- This tells us that $f^{-1}(28) = x$ if and only if $f(x) = 28$.
- Therefore $f^{-1}(28) = x$ if and only if $x^3 + 1 = 28$.
- Thus $f^{-1}(28) = x$ if and only if $x^3 = 27$. which is the same as saying that $x = 3$.
- Thus $f^{-1}(28) = 3$.
- Note: the statement of equivalence "if and only if" is often abbreviated to iff or $\iff$ in mathematics.

A bit of guesswork

Example: If $g(x) = \cos(x) + 2x$, find $g^{-1}(1)$.

- In this case, we know that g is a one-to-one function (why?)
- Because $g'(x) = 2 - sin(x) > 0$.
- We know $g^{-1}(1) = x \iff 1 = \cos(x) + 2x$ (using $g^{-1}(1) = x$ is the same as $g(x) = 1$).
- It is difficult to solve for x in the equation $1 = \cos(x) + 2x$ but in this case we can guess:
- We know that $\cos(0) = 1$, therefore $\cos(0) + 2(0) = 1$ and $x = 0$ must the unique value of x which fits the equation $1 = \cos(x) + 2x$.
- Thus $g^{-1}(1) = 0$.

Domains and Ranges

Note that the domain of f^{-1} equals the range of f and the range of f^{-1} equals the domain of f.

- **Example** Let $g(x) = \sqrt{4x+4}$.
- What is Domain g? The domain of g is all values of x for which $4x + 4 \geq 0$ i.e. $\{x | x \geq -1\}$.
- What is Range g? The range of g is $\{y | y \geq 0\}$.
- Does g^{-1} exist?
- Yes because g is a 1-1 function.
- What is the domain and range of g^{-1}?
- The domain of g^{-1} is the range of g which is $\{x | x \geq 0\}$.
- The range of g^{-1} is the domain of g which is $\{x | x \geq -1\}$.

- What is $g^{-1}(4)$?
- $g^{-1}(4) = x \iff g(x) = 4 \iff \sqrt{4x+4} = 4 \iff 4x + 4 = 16 \iff 4x = 12 \iff x = 3.$ i.e. $g^{-1}(4) = 3.$