Continuity of f^{-1}.

We can derive properties of the graph of $y = f^{-1}(x)$ from properties of the graph of $y = f(x)$, since they are refections of each other in the line $y = x$. For example:

- If f is a one-to-one function, it passes both the HLT and the VLT. Since horizontal lines become vertical lines when reflected in the line $y = x$ and vice-versa, the graph of f^{-1} also passes both tests and is a one-to-one function.
- Thus f^{-1} has an inverse function and since the graph of f is the mirror image of (its mirror image) f^{-1}, f must be the inverse function of f^{-1}.
- If f is continuous, then f^{-1} is also a continuous function. Although it does not constitute a proof, it is intuitively obvious that if you can draw the graph of f without lifting the pen from the paper, you can draw the graph of its mirror image f^{-1} without lifting the pen from the paper also.

Derivative of f^{-1}.

Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$\left.\frac{d(f^{-1})}{dx}\right|_{x=a} = (f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{\left.\frac{d(f)}{dx}\right|_{x=f^{-1}(a)}}.$$

- We can see this in two ways, both of which are important to understand
- **proof using algebra:** Recall that $y = f^{-1}(x)$ if and only if $x = f(y)$.
- Using implicit differentiation we differentiate $x = f(y)$ with respect to x to get
  $1 = f'(y)\frac{dy}{dx}$ or $\frac{1}{f'(y)} = \frac{dy}{dx}$
- or $\frac{1}{f'(y)} = (f^{-1})'(x)$ or $\frac{1}{f'(f^{-1}(x))} = (f^{-1})'(x)$

Derivative of f^{-1}.

Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

- We can also see this geometrically from the slopes of the tangents to the the graphs of f and f^{-1}.
- For any given line with slope m, its reflection in the line $y = x$ will have slope $\frac{1}{m}$.
- Recall if $(a, f^{-1}(a))$ is a point on the curve $y = f^{-1}(x)$, then its reflection in the line $y = x$ is the point $(f^{-1}(a), a)$ and is on the curve $y = f(x)$.
- The slope of the tangent to the curve $y = f^{-1}(x)$ at $(a, f^{-1}(a))$ is $(f^{-1})'(a)$. The slope of the tangent line to the curve $y = f(x)$ at the point $(f^{-1}(a), a)$ is $f'(f^{-1}(a))$.
- Since the above tangents are reflections of each other in the line $y = x$, we have reciprocal slopes:

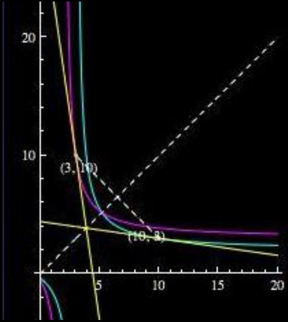
$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Derivative of f^{-1}. Example

Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

- To demonstrate this principle with some familiar graphs, graphs of the function $f(x) = \frac{2x+1}{x-3}$ (blue) and $f^{-1}(x) = \frac{3x+1}{x-2}$ (purple) are shown below.



- You can verify that $-7 = (f^{-1})'(3) = \frac{1}{f'(10)}$.

Using the formula for the derivative of f^{-1}.

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

- We will use this formula in two important ways:
- 1. To find a formula for the derivative of a number of new functions which we define as inverse functions as we did the arccos function. (these will include exponential functions and more inverse trigonometric functions.)
- 2. We will also use this formula for find derivatives for f^{-1} using the formula for f without solving for a formula for f^{-1}. This is particularly useful when solving for a formula for f^{-1} is very difficult or impossible.

Using the formula for the derivative of f^{-1}.

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

- We will look at what is involved in 1 briefly
- Since arccos is the inverse of the restricted cosine function and the derivative of the restricted cosine function is $-\sin$, the formula says that for $x \in [-1, 1]$,

$$\frac{d(\arccos(x))}{dx}(x) = \frac{1}{-\sin(\arccos(x))}$$

.

- We will return to this problem in more detail later lectures, in particular we will use trigonometric identities to find a formula in terms of x for $\sin(\arccos(x))$.
- In the next video, we will look at lots of examples of applications of type 2.