Using the formula for the derivative of f^{-1}.

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

- **Note** To use the above formula for $(f^{-1})'(a)$, you do not need the formula for $f^{-1}(x)$, you only need the value of f^{-1} at a and the value of f' at $f^{-1}(a)$.
- Example Consider the function $f(x) = \sqrt{4x+4}$. Find $(f^{-1})'(4)$.
- Using $a = 4$, the formula says $(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))}$
- We calculate $f'(f^{-1}(4))$ from the inside out starting with $f^{-1}(4)$.
- Recall our method:
  $f^{-1}(4) = b$ if and only if $f(b) = 4$ if and only if $\sqrt{4b+4} = 4$ if and only if $4b + 4 = 16$ if and only if $b = 3$; so $f^{-1}(4) = 3$
- Therefore $f'(f^{-1}(4)) = f'(3)$. $f'(x) = \frac{1}{2}(4x+4)^{-1/2}4 = \frac{2}{\sqrt{4x+4}}$ by the chain rule and we get $f'(f^{-1}(4)) = f'(3) = 1/2$.
- Finally, we have $(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = 1/(1/2) = 2$.

Using the formula for the derivative of f^{-1}.

Example Find the equation of the tangent line to the graph of the function $f^{-1}(x)$ at $x = 4$ where $f(x) = \sqrt{4x+4}$.

- The equation of the tangent line to $f^{-1}(x)$ at $x = 4$
$$(y - f^{-1}(4)) = (f^{-1})'(4)(x-4)$$
- We've already figured out that $f^{-1}(4) = 3$ and $(f^{-1})'(4) = 2$.
- Therefore the equation of the tangent line to $f^{-1}(x)$ at $x = 4$
$$(y-3) = 2(x-4) \quad \text{or} \quad \boxed{y = 2x - 5}.$$

Using the formula for the derivative of f^{-1}.

Example Let $f(x) = \sqrt{x+1} + \tan(x)$. Find $(f^{-1})'(1)$.

- Using $a = 1$, the formula says $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$
- We calculate $f'(f^{-1}(1))$ from the inside out starting with $f^{-1}(1)$.
- We have $f^{-1}(1) = x$ is the same as saying that $1 = \sqrt{x+1} + \tan(x)$.
- It is very difficult to solve for x in the above equation, however we can use a little guesswork.
- Since $1 = \sqrt{0+1} + \tan(0)$, we must have $x = 0$ is the unique value of x which solves the equation.
- Thus $f^{-1}(1) = 0$.
- Hence $f'(f^{-1}(1)) = f'(0)$.
- We have $f'(x) = \frac{1}{\sqrt{x+1}} + \sec^2(x)$ and therefore $f'(0) = 1 + 1 = 2$.
- Thus we have $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{2}$

Using the formula for the derivative of f^{-1}.

Example If f is a one-to-one function with the following properties:

$$f(10) = 21, \quad f'(10) = 2, \quad f^{-1}(10) = 4.5, \quad f'(4.5) = 3.$$

Find $(f^{-1})'(10)$.

- Using $a = 10$, the formula says $(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))}$
- We calculate $f'(f^{-1}(10))$ from the inside out starting with $f^{-1}(10)$.
- We know that $f^{-1}(10) = 4.5$, therefore $f'(f^{-1}(10)) = f'(4.5)$ which we know to be 3.
- Therefore $(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{3}$