Trigonometric Substitution

To solve integrals containing the following expressions;

$$\sqrt{a^2 - x^2} \qquad \sqrt{x^2 + a^2} \qquad \sqrt{x^2 - a^2},$$

it is sometimes useful to make the following substitutions:

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\theta = \sin^{-1}\frac{x}{a}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\theta = \tan^{-1}\frac{x}{a}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ or $\theta = \sec^{-1}\frac{x}{a}$	$\sec^2\theta - 1 = \tan^2\theta$

Note The calculations here are much easier if you use the substitution in reverse: $x = a\sin\theta$ as opposed to $\theta = \sin^{-1}\frac{x}{a}$.

Integrals involving $\sqrt{a^2 - x^2}$

We make the substitution $x = a\sin\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $dx = a\cos\theta d\theta$,
$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a|\cos\theta| = a\cos\theta$ (since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ by choice.)

Example

$$\int \frac{x^3}{\sqrt{4 - x^2}}\, dx$$

- *Let* $x = 2\sin\theta$, $dx = 2\cos\theta d\theta$, $\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\theta} = 2\cos\theta$.
- $\int \frac{x^3 dx}{\sqrt{4-x^2}} = \int \frac{8\sin^3\theta(2\cos\theta d\theta)}{2\cos\theta} = \int 8\sin^3\theta d\theta = \int 8\sin^2\theta \sin\theta\, d\theta =$ $8\int (1 - \cos^2\theta)\sin\theta\, d\theta$.
- *Let* $w = \cos\theta$, $dw = -\sin\theta\, d\theta$,

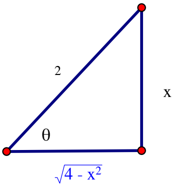
$$8\int (1-\cos^2\theta)\sin\theta\, d\theta = -8\int (1-w^2)\, dw = 8\int (w^2-1)\, dw = \frac{8w^3}{3} - 8w + C$$

$$= \frac{8(\cos\theta)^3}{3} - 8\cos\theta + C = \frac{8(\cos(\sin^{-1}\frac{x}{2}))^3}{3} - 8\cos(\sin^{-1}\frac{x}{2}) + C.$$

Integrals involving $\sqrt{a^2 - x^2}$

$$\int \frac{x^3}{\sqrt{4 - x^2}}\,dx = \frac{8(\cos(\sin^{-1}\frac{x}{2}))^3}{3} - 8\cos(\sin^{-1}\frac{x}{2}) + C.$$

- *To get an expression for* $\cos(\sin^{-1}\frac{x}{2})$*, we use an appropriate triangle*



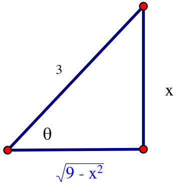
From the triangle, we get
$\cos(\sin^{-1}\frac{x}{2}) = \frac{\sqrt{4-x^2}}{2}$

- *hence* $\int \frac{x^3 dx}{\sqrt{4-x^2}} = \frac{8\left(\frac{\sqrt{4-x^2}}{2}\right)^3}{3} - 8\frac{\sqrt{4-x^2}}{2} + C = \frac{(4-x^2)^{3/2}}{3} - 4\sqrt{4-x^2} + C$

Integrals involving $\sqrt{a^2 - x^2}$

Example $\int \frac{dx}{x^2\sqrt{9-x^2}}$

- *Let* $x = 3\sin\theta$, $dx = 3\cos\theta d\theta$, $\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta$.
- $\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta d\theta}{(9\sin^2\theta)3\cos\theta} = \int \frac{1}{9\sin^2\theta} d\theta = \frac{-\cot\theta}{9} + C = \frac{-\cot(\sin^{-1}\frac{x}{3})}{9} + C$
- *To get an expression for* $\cot(\sin^{-1}\frac{x}{3})$, *we use an appropriate triangle*



From the triangle, we get
$\cot(\sin^{-1}\frac{x}{3}) = \frac{\sqrt{9-x^2}}{x}$ *and hence*

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \frac{-\sqrt{9-x^2}}{9x} + C$$

- **Note** *You can also use this method to derive what you already know*

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C}$$

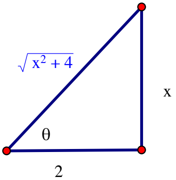
Integrals involving $\sqrt{x^2 + a^2}$

We make the substitution $x = a\tan\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $dx = a\sec^2\theta d\theta$,
$\sqrt{x^2 + a^2} = \sqrt{a^2\tan^2\theta + a^2} = a|\sec\theta| = a\sec\theta$ (since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ by choice.)

Example

$$\int \frac{dx}{\sqrt{x^2+4}}$$

- *Let* $x = 2\tan\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $dx = 2\sec^2\theta d\theta$,
  $\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sqrt{\sec^2\theta} = 2\sec\theta$.
- $\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta d\theta}{2\sec\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C =$
  $\ln|\sec(\tan^{-1}\frac{x}{2}) + \tan(\tan^{-1}\frac{x}{2})| + C = \ln|\sec(\tan^{-1}\frac{x}{2}) + \frac{x}{2}| + C$
- *To get an expression for* $\sec(\tan^{-1}\frac{x}{2})$, *we use an appropriate triangle*



From the triangle, we get
$\sec(\tan^{-1}\frac{x}{2}) = \frac{\sqrt{x^2+4}}{2}$ *and hence*

$$\int \frac{dx}{\sqrt{x^2+4}} = \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C.$$

Integrals involving $\sqrt{x^2 + a^2}$

We make the substitution $x = a\tan\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $dx = a\sec^2\theta d\theta$,
$\sqrt{x^2 + a^2} = \sqrt{a^2\tan^2\theta + a^2} = a|\sec\theta| = a\sec\theta$ (since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ by choice.)

Note You can also use this substitution to get the familiar

$$\boxed{\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.}$$

Integrals involving $\sqrt{x^2 + a^2}$, Completing the square.

Sometimes we can convert an integral to a form where trigonometric substitution can be applied by completing the square.

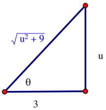
Example Evaluate

$$\int \frac{dx}{\sqrt{x^2 - 4x + 13}}.$$

- $x^2 - 4x + 13 = x^2 - 2(2)x + 2^2 - 2^2 + 13 = (x-2)^2 + 9.$
- $\int \frac{dx}{\sqrt{x^2-4x+13}} = \int \frac{dx}{\sqrt{(x-2)^2+9}} = \int \frac{du}{\sqrt{(u)^2+9}}$, *where* $u = x - 2$.
- *Now we apply the substitution* $u = 3\tan\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $du = 3\sec^2\theta d\theta$, $\sqrt{u^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\sec^2\theta} = 3\sec\theta.$
- $\int \frac{du}{\sqrt{(u)^2+9}} = \int \frac{3\sec^2\theta d\theta}{3\sec\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C =$ $\ln|\sec(\tan^{-1}\frac{u}{3}) + \tan(\tan^{-1}\frac{u}{3})| + C.$

Integrals involving $\sqrt{x^2+a^2}$, Completing the square.

- $\int \frac{dx}{\sqrt{x^2-4x+13}} = \ln|\sec(\tan^{-1}\frac{u}{3}) + \tan(\tan^{-1}\frac{u}{3})| + C$, *where* $u = x - 2$.
- *To get an expression for* $\sec(\tan^{-1}\frac{u}{3})$, *we use an appropriate triangle*



From the triangle, we get
$\sec(\tan^{-1}\frac{u}{3}) = \frac{\sqrt{u^2+9}}{3}$

- $\int \frac{dx}{\sqrt{x^2-4x+13}} = \ln|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}| + C = \ln|\frac{\sqrt{(x-2)^2+9}}{3} + \frac{(x-2)}{3}| + C.$

Integrals involving $\sqrt{x^2 - a^2}$

We make the substitution $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, (This amounts to saying $\theta = \sec^{-1}\frac{x}{a}$), $dx = a\sec\theta\tan\theta d\theta$,
$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = a|\tan\theta| = a\tan\theta$ (since $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ by choice)

Example Evaluate

$$\int \frac{1}{x^2\sqrt{x^2 - 25}} dx$$

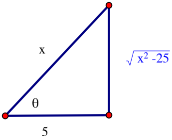
- *Let* $x = 5\sec\theta$ $0 \le \theta < \frac{\pi}{2}$ *or* $\pi \le \theta < \frac{3\pi}{2}$, *then* $dx = 5\sec\theta\tan\theta d\theta$,
  $\sqrt{x^2 - 25} = \sqrt{25\sec^2\theta - 25} = 5\sqrt{\tan^2\theta} = 5|\tan\theta| = 5\tan\theta.$
- $\int \frac{1}{x^2\sqrt{x^2-25}} dx = \int \frac{5\sec\theta\tan\theta}{25\sec^2\theta(5\tan\theta)} d\theta = \int \frac{1}{25\sec\theta} d\theta$

$$= \frac{1}{25}\int \cos\theta\, d\theta = \frac{1}{25}\sin\theta + C = \frac{1}{25}\sin(\sec^{-1}\frac{x}{5}) + C$$

Integrals involving $\sqrt{x^2 - a^2}$

$$\int \frac{1}{x^2\sqrt{x^2-25}}\,dx = \frac{1}{25}\sin(\sec^{-1}\frac{x}{5}) + C$$

- *To get an expression for* $\sin(\sec^{-1}\frac{x}{5})$, *we use an appropriate triangle*



From the triangle, we get
$\sin(\sec^{-1}\frac{x}{5}) = \frac{\sqrt{x^2-25}}{x}$

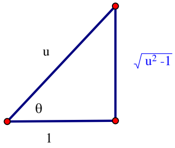
- *Hence* $\int \frac{1}{x^2\sqrt{x^2-25}}\,dx == \frac{1}{25}\cdot\frac{\sqrt{x^2-25}}{x} + C = \frac{\sqrt{x^2-25}}{25x} + C.$
- **Note** *You can also use this substitution to get*

$$\boxed{\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.}$$

Integrals involving $\sqrt{x^2 - a^2}$, Completing the square

Example Evaluate $\int_4^6 \frac{dx}{\sqrt{x^2-6x+8}} dx$.

- *Completing the square* $x^2 - 6x + 8 = x^2 - 2(3)x + 9 - 9 + 8 = (x-3)^2 - 1$
- $\int \frac{dx}{\sqrt{x^2-6x+8}} dx = \int \frac{dx}{\sqrt{(x-3)^2-1}} dx = \int \frac{du}{\sqrt{u^2-1}}$, *where* $u = x - 3$.
- *Let* $u = \sec\theta \quad 0 \le \theta < \frac{\pi}{2}$ *or* $\pi \le \theta < \frac{3\pi}{2}$, $du = \sec\theta \tan\theta d\theta$.
- $= \int \frac{\sec\theta \tan\theta d\theta}{\sqrt{\sec^2\theta - 1}} = \int \frac{\sec\theta \tan\theta d\theta}{\sqrt{\sec^2\theta - 1}} = \int \sec\theta d\theta$
- $= \ln|\sec\theta + \tan\theta| + C = \ln|\sec(\sec^{-1} u) + \tan(\sec^{-1} u)|$
- *To get an expression for* $\tan(\sec^{-1} u)$, *we use an appropriate triangle*



From the triangle, we get
$\tan(\sec^{-1} u) = \sqrt{u^2 - 1}$ *Hence*
$\int \frac{dx}{\sqrt{x^2-6x+8}} dx$
$= \ln|u + \sqrt{u^2-1}| + C$
$= \ln|(x-3) + \sqrt{(x-3)^2 - 1}| + C$.
$\int_4^6 \frac{dx}{\sqrt{x^2-6x+8}} dx$
$= \ln|3 + \sqrt{8}| - \ln|1| = \ln|3 + \sqrt{8}|$