## Lecture 14 : Approximating an integral

Sometimes, we need to approximate an integral of the form $\int_a^b f(x)dx$ and we cannot find an antiderivative in order to evaluate the integral. Also we may need to evaluate $\int_a^b f(x)dx$ where we do not have a formula for $f(x)$ but we have data describing a set of values of the function.

### Review

We might approximate the given integral using a Riemann sum. Already we have looked at the left end-point approximation and the right end point approximation to $\int_a^b f(x)dx$ in Calculus 1. We also looked at **the midpoint approximation M:**

**Midpoint Rule** If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^{n} f(\bar{x_i})\Delta x = \Delta x(f(\bar{x_1}) + f(\bar{x_2}) + \cdots + f(\bar{x_n})),$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \ \text{ and } \ \bar{x_i} = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

**Example** Use the midpoint rule with $n = 6$ to approximate $\int_1^4 \frac{1}{x}dx$. $(= \ln(4) = 1.386294361)$
Fill in the tables below:
$\Delta x =$

| $x_i$ | $x_0 = 1$ | $x_1 = 3/2$ | $x_2 = 2$ | $x_3 = 5/2$ | $x_4 = 3$ | $x_5 = 7/2$ | $x_6 = 4$ |
|---|---|---|---|---|---|---|---|

| $\bar{x_i} = \frac{1}{2}(x_{i-1} + x_i)$ | $\bar{x_1} = 5/4$ | $\bar{x_2} = 7/4$ | $\bar{x_3} = 9/4$ | $\bar{x_4} = 11/4$ | $\bar{x_5} = 13/4$ | $\bar{x_6} = 15/4$ |
|---|---|---|---|---|---|---|
| $f(\bar{x_i}) = \frac{1}{\bar{x_i}}$ | 4/5 | 4/7 | 4/9 | 4/11 | 4/13 | 4/15 |

$M_6 = \sum_1^6 f(\bar{x_i})\Delta x = \frac{1}{2}\left[\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \frac{4}{13} + \frac{4}{15}\right] = 1.376934177$

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We can also approximate a definite integral $\int_a^b f(x)dx$ using an approximation by trapezoids as shown in the picture below for $f(x) \geq 0$

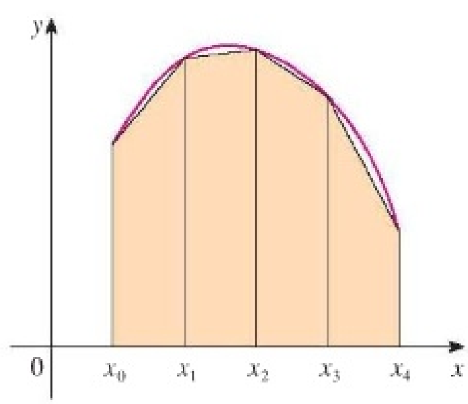


**FIGURE 2**
Trapezoidal approximation

**Trapezoidal Rule** If $f$ is integrable on $[a, b]$, then

$$\boxed{\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + +2f(x_{n-1}) + f(x_n))}$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \ \text{ and.}$$

**Example** Use the trapezoidal rule with $n = 6$ to approximate $\int_1^4 \frac{1}{x}dx$. $(= \ln(4) = 1.386294361)$
Fill in the tables below:

| $x_i$ | $x_0 = 1$ | $x_1 = 3/2$ | $x_2 = 2$ | $x_3 = 5/2$ | $x_4 = 3$ | $x_5 = 7/2$ | $x_6 = 4$ |
|---|---|---|---|---|---|---|---|
| $f(x_i) = \frac{1}{x_i}$ | | | | | | | |

$\Delta x =$

$$T_6 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + +2f(x_5) + f(x_6)) =$$

The **error** when using an approximation is the difference between the true value of the integral and the approximation.

The error for the midpoint approximation above is

$$E_M = \int_1^4 \frac{1}{x}dx - M_6 =$$

The error for the trapezoidal approximation above is

$$E_T = \int_1^4 \frac{1}{x}dx - T_6 =$$

**Error Bounds** If $|f''(x)| \leq K$ for $a \leq x \leq b$. Let $E_T$ and $E_M$ denote the errors for the trapezoidal approximation and midpoint approximation respectively, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

**Example** (a) Give an upper bound for the error in the trapezoidal approximation of $\int_1^4 \frac{1}{x}dx$ when $n = 10$.

(b) Give an upper bound for the error in the midpoint approximation of $\int_1^4 \frac{1}{x}dx$ when $n = 10$.

(c) Using the error bounds given above determine how large should $n$ be to ensure that the trapezoidal approximation is accurate to within $0.000001 = 10^{-6}$ ?

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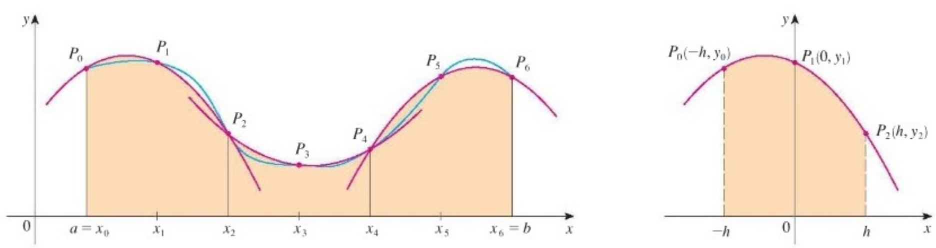
We can also approximate a definite integral using parabolas to approximate the curve as in the picture below.




Three points determine a unique parabola. We draw a parabolic segment using the three points on the curve above $x_0, x_1, x_2$. We draw a second parabolic segment using the three points on the curve above $x_2, x_3, x_4$ etc... We estimate the integral by summing the areas of the regions below these parabolic segments to get **Simpson's Rule** for even $n$:

$$\boxed{\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))}$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \quad \text{and,} \quad \text{n even}$$

In fact we have

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n.$$

**Example** Use Simpson's rule with $n = 6$ to approximate $\int_1^4 \frac{1}{x} dx$. $(= \ln(4) = 1.386294361)$
Fill in the tables below:
$\Delta x =$

| $x_i$ | $x_0 = 1$ | $x_1 = 3/2$ | $x_2 = 2$ | $x_3 = 5/2$ | $x_4 = 3$ | $x_5 = 7/2$ | $x_6 = 4$ |
|---|---|---|---|---|---|---|---|
| $f(x_i) = \frac{1}{x_i}$ | | | | | | | |

$$S_6 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)) =$$

The error in this estimate is

$$E_S = \int_1^4 \frac{1}{x} dx - S_6 =$$

**Extra Example (data)** The following table gives the speed of a runner during the first 5 seconds of a race, use Simpson's rule to estimate the distance covered by the runner in those 5 seconds

| $t(s)$ | $v(m/s)$ | $t(s)$ | $v(m/s)$ |
|---|---|---|---|
| 0 | 0 | 3.0 | 10.51 |
| 0.5 | 4.67 | 3.5 | 10.67 |
| 1.0 | 7.34 | 4.0 | 10.76 |
| 1.5 | 8.86 | 4.5 | 10.81 |
| 2.0 | 9.73 | 5.0 | 10.81 |
| 2.5 | 10.22 | | |

$$\text{Distance Travelled} = \int_0^5 v(t)dt \approx$$

$$\frac{\Delta x}{3}(f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)) =$$

$$= \frac{.5}{2}[0+4(4.67)+2(7.34)+4(8.86)+2(9.73)+4(10.22)+2(10.51)+4(10.67)+2(10.76)+4(10.81)+10.81]$$

$$= 44.735m$$

**Error Bound for Simpson's Rule** Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If $E_S$ is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

**Example** How large should $n$ be in order to guarantee that the Simpson rule estimate for $\int_1^4 \frac{1}{x} dx$ is accurate to within $0.000001 = 10^{-6}$?