Midpoint Approximation

Sometimes, we need to approximate an integral of the form $\int_a^b f(x)dx$ and we cannot find an antiderivative in order to evaluate the integral. Also we may need to evaluate $\int_a^b f(x)dx$ where we do not have a formula for $f(x)$ but we have data describing a set of values of the function.

Review

We might approximate the given integral using a Riemann sum. Already we have looked at the left end-point approximation and the right end point approximation to $\int_a^b f(x)dx$ in Calculus 1. We also looked at **the midpoint approximation M:**

Midpoint Rule If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx \approx M_n = \sum_{i=1}^{n} f(\bar{x_i})\Delta x = \Delta x(f(\bar{x_1}) + f(\bar{x_2}) + \cdots + f(\bar{x_n})),$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \text{ and } \bar{x_i} = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

Midpoint Approximation

Example Use the midpoint rule with $n = 6$ to approximate $\int_1^4 \frac{1}{x} dx$.
$(= \ln(4) = 1.386294361)$
Fill in the tables below:

- $\Delta x = \frac{4-1}{6} = \frac{1}{2}$
-

x_i	$x_0 = 1$	$x_1 = 3/2$	$x_2 = 2$	$x_3 = 5/2$	$x_4 = 3$	$x_5 = 7/2$	$x_6 = 4$

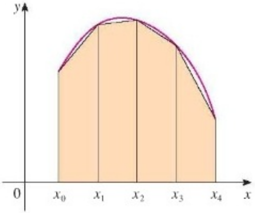
-

$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$	$\bar{x}_1 = 5/4$	$\bar{x}_2 = 7/4$	$\bar{x}_3 = 9/4$	$\bar{x}_4 = 11/4$	$\bar{x}_5 = 13/4$	$\bar{x}_6 = 15/4$
$f(\bar{x}_i) = \frac{1}{\bar{x}_i}$	$4/5$	$4/7$	$4/9$	$4/11$	$4/13$	$4/15$

- $M_6 = \sum_1^6 f(\bar{x}_i)\Delta x = \frac{1}{2}\left[\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \frac{4}{13} + \frac{4}{15}\right] = 1.376934177$

Trapezoidal Rule

We can also approximate a definite integral $\int_a^b f(x)dx$ using an approximation by trapezoids as shown in the picture below for $f(x) \geq 0$



The area of the trapezoid above the interval $[x_i, x_{i+1}]$ is $\Delta x \left[\frac{(f(x_i)+f(x_{i+1})}{2}\right]$.

Trapezoidal Rule If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + +2f(x_{n-1}) + f(x_n))$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \text{ and.}$$

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$f(x_i) = \frac{1}{x_i}$							

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x_i	$x_0 = 1$	$x_1 = 3/2$	$x_2 = 2$	$x_3 = 5/2$	$x_4 = 3$	$x_5 = 7/2$	$x_6 = 4$
$f(x_i) = \frac{1}{x_i}$	1	2/3	1/2	2/5	1/3	2/7	1/4

- $T_6 = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6))$
- $= \frac{1}{4}(1 + 2\left(\frac{2}{3}\right) + 2\left(\frac{1}{2}\right) + 2\left(\frac{2}{5}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{7}\right) + \frac{1}{4})$
- $= 1.405357143.$

Error of Approximation

The **error** when using an approximation is the difference between the true value of the integral and the approximation.

- The error for the midpoint approximation above above is

$$E_M = \int_1^4 \frac{1}{x} dx - M_6 = 1.386294361 - 1.376934177 = 0.00936018$$

  The error for the trapezoidal approximation above is

$$E_T = \int_1^4 \frac{1}{x} dx - T_6 = 1.386294361 - 1.405357143 = -0.0190628$$

- **Error Bounds** If $|f''(x)| \leq K$ for $a \leq x \leq b$. Let E_T and E_M denote the errors for the trapezoidal approximation and midpoint approximation respectively, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

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Example (a) Give an upper bound for the error in the trapezoidal approximation of $\int_1^4 \frac{1}{x} dx$ when $n = 10$.

- $f(x) = \frac{1}{x}$, $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$
- We can use the above formula for the error bound with any value of K for which $|f''(x)| \leq K$ for $1 \leq x \leq 4$.
- Since $|f''(x)| = f''(x) = \frac{2}{x^3}$ is a decreasing function on the interval $[1, 4]$, we have that $|f''(x)| \leq f''(1) = 2$ on the interval $[1, 4]$. So we can use $K = 2$ in the formula for the error bound above.
- Therefore when $n = 10$,

$$|T_{10} - \int_1^4 \frac{1}{x} dx| = |E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(4-1)^3}{12(10)^2} = 0.045$$

- Note that the bound for the error given by the formula is conservative since it turns out to give $|E_T| \leq 0.045$ when $n = 10$, compared to a true error of $|E_T| = 0.00696667$.

Error of Approximation

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Example(b) Give an upper bound for the error in the midpoint approximation of $\int_1^4 \frac{1}{x}dx$ when $n = 10$.

- As above, we can use $K = 2$ to get

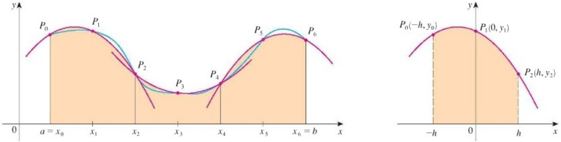
$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{2(3)^3}{24(10)^2} = 0.0225.$$

(c) Using the error bounds given above determine how large should n be to ensure that the trapezoidal approximation is accurate to within 0.000001 $= 10^{-6}$?

- We want $|E_T| \leq 10^{-6}$.
- We have $|E_T| \leq \frac{K(b-a)^3}{12n^2}$, where $K = 2$ since $|f''(x)| \leq 2$ for $1 \leq x \leq 4$.
- Hence we will certainly have $|E_T| \leq 10^{-6}$ if we choose a value of n for which $\frac{2(4-1)^3}{12n^2} \leq 10^{-6}$.
- That is $\frac{(10^6)2(27)}{12} \leq n^2$
- or $n \geq \sqrt{\frac{(10^6)2(27)}{12}} = 2121.32$, $n = 2122$ will work.

Simpson's Rule

We can also approximate a definite integral using parabolas to approximate the curve as in the picture below. **[note n is even]**.



Three points determine a unique parabola. We draw a parabolic segment using the three points on the curve above x_0, x_1, x_2. We draw a second parabolic segment using the three points on the curve above x_2, x_3, x_4 etc... The area of the parabolic region beneath the parabola above the interval $[x_{i-1}, x_{i+1}]$ is $\frac{\Delta x}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})]$. We estimate the integral by summing the areas of the regions below these parabolic segments to get **Simpson's Rule** for even n:

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \ \text{ and.}$$

In fact we have $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$.

Simpson's Rule

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

Example Use Simpson's rule with $n = 6$ to approximate $\int_1^4 \frac{1}{x} dx$. ($= \ln(4) = 1.386294361$)

Fill in the tables below:

- $\Delta x = \frac{4-1}{6} = \frac{1}{2}$
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x_i	$x_0 = 1$	$x_1 = 3/2$	$x_2 = 2$	$x_3 = 5/2$	$x_4 = 3$	$x_5 = 7/2$	$x_6 = 4$
$f(x_i) = \frac{1}{x_i}$	1	2/3	1/2	2/5	1/3	2/7	1/4

- $S_6 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)) =$
- $\frac{1}{6}\left[1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{2}{3} + \frac{8}{7} + \frac{1}{4}\right] = 1.387698413$
- The error in this estimate is

$$E_S = \int_1^4 \frac{1}{x} dx - S_6 =$$

$$1.386294361 - 1.387698413 = -0.00140405$$

Error Bound Simpson's Rule

Error Bound for Simpson's Rule Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Example How large should n be in order to guarantee that the Simpson rule estimate for $\int_1^4 \frac{1}{x}dx$ is accurate to within $0.000001 = 10^{-6}$?

- $f(x) = \frac{1}{x}$, $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f^{(3)}(x) = \frac{(-3)2}{x^4}$, $f^{(4)}(x) = \frac{4\cdot3\cdot2}{x^5} \leq 24$ (for $1 \leq k \leq 4$) $= K$
- We have $|E_S| \leq \frac{24(3)^5}{180n^4}$
- We want $|E_S| \leq 10^{-6}$, hence if we find a value of n for which $\frac{24(3)^5}{180n^4} \leq 10^{-6}$ it is guaranteed that $|E_S| \leq 10^{-6}$.
- From $\frac{24(3)^5}{180n^4} \leq 10^{-6}$ we get that $10^6\frac{24(3)^5}{180} \leq n^4$ or $n \geq \sqrt[4]{10^6\frac{24(3)^5}{180}} = 75$. $n = 76$ will work.
- This is a conservative upper bound of the error, the actual error for $n = 76$ is -8×10^{-8}