Lecture 15 : Improper Integrals

In this section, we will extend the concept of the definite integral $\int_a^b f(x)dx$ to functions with an infinite discontinuity and to infinite intervals.

That is, integrals of the type

$$\int_1^\infty \frac{1}{x^3}dx \qquad \int_0^1 \frac{1}{x^3}dx \qquad \int_{-\infty}^\infty \frac{1}{4+x^2}$$

Infinite Intervals

An Improper Integral of Type 1

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

provided that limit exists and is finite.

(c) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

provided that limit exists and is finite.

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and is finite and **divergent** if the limit does not exist.

(c) If (for any value of a) both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

If $f(x) \geq 0$, we can give the definite integral above an area interpretation.

Example Determine whether the integrals $\int_1^\infty \frac{1}{x}dx$, and $\int_{-\infty}^0 e^x dx$ converge or diverge.

Example Determine whether the following integral converges or diverges and if it converges find its value.

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2} dx$$

Theorem

$$\boxed{\int_1^{\infty} \frac{1}{x^p} dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leq 1}$$

Functions with infinite discontinuities

Improper integrals of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if that limit exists and is finite.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if that limit exists and is finite.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example Determine whether the following integral converges or diverges

$$\int_0^2 \frac{1}{x-2}dx$$

Theorem It is not difficult to show that

$$\boxed{\int_0^1 \frac{1}{x^p}dx \quad \text{is divergent if } p \geq 1 \text{ and convergent if } p < 1}$$

Example determine if the following integral converges or diverges and if it converges find its value.

$$\int_0^4 \frac{1}{(x-2)^2} dx$$

Comparison Test for Integrals

Theorem If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

How do I pick g? The hard part of using the comparison test is deciding on what new function to compare it to. Here is some advice and some examples.

1. Decide whether you think $\int_a^\infty f(x)\, dx$ should converge or diverge.

   How do you do this? Generally, you should see if the integral "looks like" something you already know.

   (a) Example:

   $$\int_1^\infty \frac{1}{x^2+x+1}\, dx.$$

   The denominator "looks like" x^2. More precisely, the *sizes* of x^2+x+1 and x^2 are comparable for large x, so you can expect them either to both converge or both diverge. Since $\int_1^\infty \frac{1}{x^2}\, dx$ converges, this suggests that $\int_1^\infty \frac{1}{x^2+x+1}\, dx$ should converge too.

   (b) Example:

   $$\int_1^\infty \frac{1}{x-\frac{1}{2}}\, dx$$

(c) Example:

$$\int_0^\pi \frac{\cos^2 x}{\sqrt{x}}\, dx.$$

(d) Example:

$$\int_0^\infty \frac{e^{-x}}{1+\sin^2 x}\, dx.$$

2. If you decided on convergence, look for a $g(x)$ that is *larger* and *simpler* than f; conversely if you decided on divergence, look for $g(x)$ that is *smaller* and simpler. Often you already have a good candidate based on your thought process in Step 1.

   (a) We decided above that we expect convergence, so we want $g(x)$ so that $\frac{1}{x^2+x+1} \leq g(x)$. Making the denominator *smaller* makes the whole fraction *bigger*. Since $x^2 \leq x^2 + x + 1$ for $x \geq 0$, we find that

   $$\frac{1}{x^2+x+1} \leq \frac{1}{x^2}$$

   on the interval $[1, \infty)$, and so we can use the comparison test with $g(x) = \frac{1}{x^2}$ to conclude that the example in (a) is convergent.

   (b) $\int_1^\infty \frac{1}{x-\frac{1}{2}}\, dx$

(c) $\int_0^\pi \frac{\cos^2 x}{\sqrt{x}}\, dx$

(d) $\int_0^\infty \frac{e^{-x}}{1+\sin^2 x}\, dx$

Extra Example The standard normal probability distribution has the following formula:

$$\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$$

The graph is a bell shaped curve. The area beneath this curve is 1 and it fits well to many histograms from data collected. It is used extensively in probability and statistics. to calculate the integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$$

you need multivariable calculus. However we can see it the integral converges using the comparison test.

$$e^{\frac{-x^2}{2}} \leq e^{-x/2}$$

when $x \geq 1$. Use this to show that $\int_1^\infty e^{\frac{-x^2}{2}}$ converges.