Lecture 16: Direction Fields and Euler's Method

A **Differential Equation** is an equation relating an unknown function and one or more of its derivatives.

Examples Population growth : $\frac{dP}{dt} = kP$, or $\frac{dP}{dt} = kP(1 - \frac{P}{K})$.
Motion of a spring with a mass m attached: $m\frac{d^2x}{dt^2} = -kx$.
Body of mass m falling under the action of gravity g encounters air resistance. The velocity of the falling body at time t satisfies the equation : $m\frac{dv(t)}{dt} = mg - k[v(t)]^2$.
General Examples

$$y' = x - y, \qquad y' = yx, \qquad y' + xy = x^2.$$

The **Order** of a differential equation is the order of the highest derivative that occurs in the equation.

Example The differential equation

$$2\frac{d^2x}{dt^2} = -10x \quad \text{has order} \quad \_\_\_\_$$

The differential equation

$$\frac{dv(t)}{dt} = 32 - 10[v(t)]^2 \quad \text{has order} \quad \_\_\_\_$$

A function $y = f(x)$ is **a solution of a differential equation** if the equation is satisfied when $y = f(x)$ and its appropriate derivatives are substituted into the equation.

Example Match the following differential equations with their solutions:

Equation	**Solution**
$\frac{dP}{dt} = 2P$	$y = x - 1$
$y' = x - y$	$y = \ln\|1 + e^x\|$
$y' = \frac{e^x}{1+e^x}$	$P(t) = 10e^{2t}$
	$y = x - 1 + \frac{1}{e^x}$

When asked to **Solve** a differential equation we aim to find all possible solutions. Our solution will be a family of functions. A **General Solution** is a solution involving constants which can be specialized to give any particular solution. **Example** The general solutions to the differential equations given above are

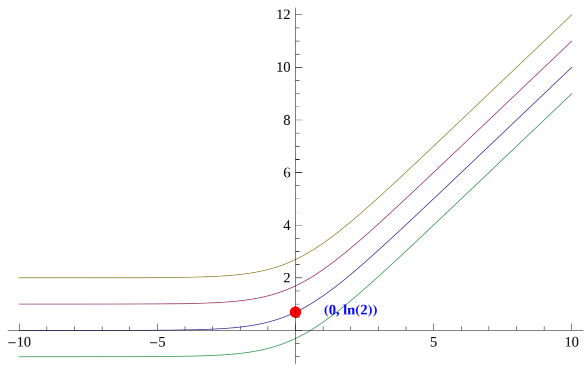
Equation	**General Solution**
$\frac{dP}{dt} = 2P$	$P(t) = Ke^{2t}$
$y' = x - y$	$y = x - 1 + \frac{C}{e^x}$
$y' = \frac{e^x}{1+e^x}$	$y = \ln\|1 + e^x\| + C$

Example For the differential equation

$$\frac{dy}{dx} = \frac{e^x}{1 + e^x},$$

we can find the general solution using methods of integration. (we will solve the others using the methods of separable equations and Linear First order equations.)

The graph below shows a sketch of some solutions from the family of solutions :



Note that only one of these solution curves passes through the point $(0, \ln 2)$, i.e. satisfies the requirement $y(0) = \ln 2$.

An **Initial Value Problem** asks for a specific solution to a differential equation satisfying an **initial condition** of the form $y(t_0) = y_0$.

Example Problem: Using the general solution given above, find a solution to the initial value problem $y' = x - y$ with the property that $y(0) = 0$.

(At the end of this lecture, we give an approximate numerical solution to this problem using Euler's method.)

Equilibrium solutions An *equilibrium solution* to a differential equation is a solution of the form $y(t) = c$, where c is a constant. So $y'(x) = 0$, no matter what value x is.

Example Find the equilibrium solutions to $y' = xy$.

Solution: If $y' = 0$ for this equation, then $xy = 0$ so either $x = 0$ or $y = 0$. This tells us that $y(x) = 0$ is an equilibrium solution. The other equation $x = 0$ doesn't generate an equilibrium solution, because it doesn't correspond to something of the form $y(x) = c$.

Direction Fields

If we have a differential equation of the type

$$y' = F(x, y)$$

where $F(x, y)$ is an expression in x and y only, then the slope of a solution curve at a point (x, y) is $F(x, y)$. We can use the formula to calculate the slopes of the graphs of the solutions of the differential equation that pass through particular points on the plane. We can draw a picture of these slopes by drawing a small line (or arrow) indicating the direction of the curve at each point we have considered.

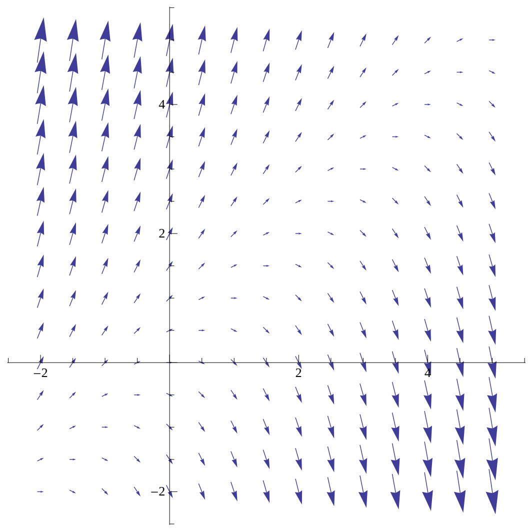
Example Consider the equation $y' = y - x$.
The graph of any solution to this differential equation passing through the point $(x, y) = (2, 1)$ has slope ________ .

The graph of any solution to this differential equation passing through the point $(x, y) = (0, 1)$has slope _______ .

The graph of any solution to this differential equation passing through the point $(x, y) = (-1, 1)$has slope _______.

etc....

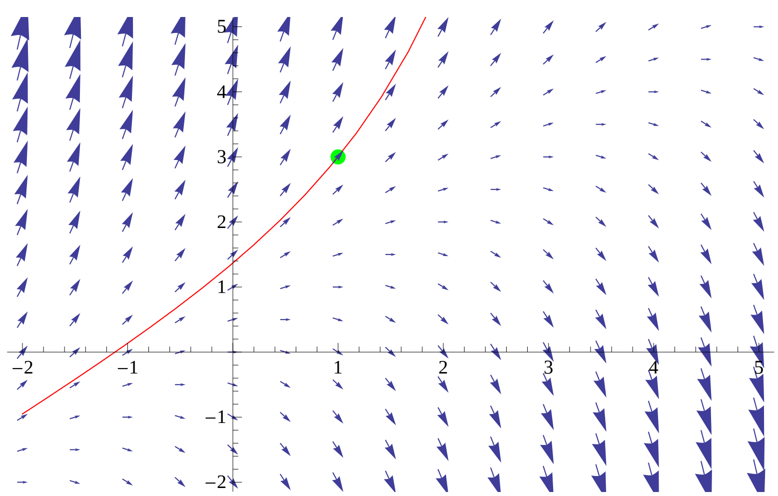
We can get some idea of what the graphs of the solutions to differential equation look like by drawing a **Direction Field** where we draw a short line segment (or arrow) with slope $y - x$ at each point (x, y) on the plane to indicate the direction of a solution running through that point. The picture below shows a computer generated direction field for the equation $y' = y - x$.



For any Differential equation of the form $y' = F(x, y)$ we can make a **direction field** by drawing an arrow with slope $F(x, y)$ at many points in the plane. The more points we include, the better the picture we get of the behavior of the solutions.

We can use this picture to give a rough sketch of a solution to an initial value problem.

Example Below is a sketch of a solution to the differential equation $y' = y - x$, where $y(1) = 3$.



we see that a solution to the initial value problem $y' = y - x$, $y(1) = 3$ passes through the point (1.3) and follows the direction of the arrows.

Sketch a solution to the equation with $y(2) = 0$ on the vector field above.

Euler's Method (Following The Arrows)

Euler's method makes precise the idea of following the arrows in the direction field to get an approximate solution to a differential equation of the form $y' = F(x, y)$ satisfying the initial condition $y(x_0) = y_0$.
For such an initial value problem we can use a computer to generate a table of approximate numerical values of y for values of x in an interval $[x_0, b]$. This is called a **numerical solution** to the problem.

Example Estimate $y(4)$ where $y(x)$ is a solution to the differential equation $y' = y - x$ which satisfies the initial condition $y(2) = 0$, on the interval $2 \leq x \leq 4$.

Euler's method approximates the path of the solution curve with a series of line segments following the directions of the arrows in the direction fields.

1. First we choose the **Step Size** of our approximation, which will be the change in the value of x on each line segment. In general a smaller step size means shorter line segments and a better approximation.

2. The **first point** on our approximating curve is determined by the initial condition $y(x_0) = y_0$. The corresponding point on the curve is
$$(x_0, y_0).$$

3. To get the **next (defining) point** on the curve, we follow the arrow in the direction field which starts at (x_0, y_0) (with slope $F(x_0, y_0)$) and which ends at $x_1 = x_0 + h$. (where h is the step size). We can write down algebraic formulas for the endpoint of this arrow (x_1, y_1). We know that $\boxed{x_1 = x_0 + h}$. We have the slope of the arrow is $F(x_0, y_0) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h}$. Therefore
$$y_1 - y_0 = hF(x_0, y_0) \quad \text{or} \quad \boxed{y_1 = y_0 + hF(x_0, y_0)}.$$

4. We can now draw the first segment of our approximating curve as the line segment between the points (x_0, y_0) and (x_1, y_1).

5. To get the **next (defining) point** on the curve, , we follow the arrow in the direction field which starts at (x_1, y_1) (with slope $F(x_1, y_1)$) and which ends at $x_2 = x_1 + h$. In other words, we repeat the process starting at (x_1, y_1). By the same argument, we get the following equations for the point (x_2, y_2):
$$x_2 = x_1 + h, \quad \text{and} \quad y_2 = y_1 + hF(x_1, y_1).$$

6. The second line segment of our approximating curve is the line between (x_1, y_1) and (x_2, y_2).

7. We repeat the process until $x_n = a$, if we wish to approximate $y(a)$. Note that we should choose the step size, h, so that $\frac{a - x_0}{h}$ is an integer n.

In summary, to use this approximation;

- We first decide on the step size h. (If we want to estimate $y(x_0 + L)$ where y is a solution to the IVP $y' = F(x, y)$, $y(x_0) = y_0$, and we wish to use n steps, then the step size should be L/n.)

- Our series of approximations is then given by

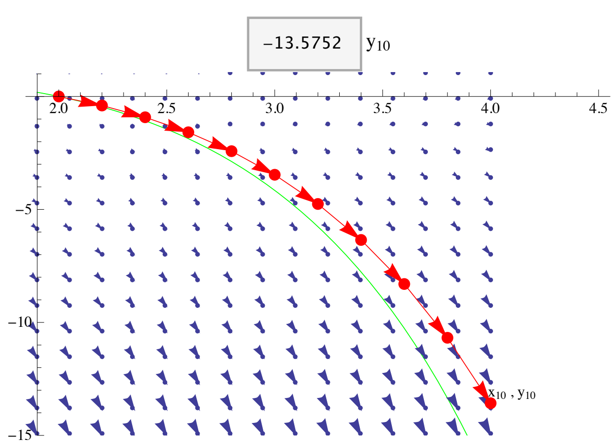
$$\text{Initial point} = (x_0, y_0).$$

$$y_1 = y_0 + hF(x_0, y_0) \quad \text{new point on approximate curve} = (x_1, y_1) = (x_0 + h, y_1)$$
$$y_2 = y_1 + hF(x_1, y_1) \quad \text{new point on approximate curve} = (x_2, y_2) = (x_0 + 2h, y_2)$$
$$y_3 = y_2 + hF(x_2, y_2) \quad \text{new point on approximate curve} = (x_3, y_3) = (x_0 + 3h, y_3)$$
$$\vdots$$
$$y_i = y_{i-1} + hF(x_{i-1}, y_{i-1}) \quad \text{corresponding point on approximate curve} = (x_i, y_i) = (x_0 + ih, y_i)$$
$$\vdots$$

Example Use Euler's method with step size $h = 0.2$ to find an approximation for $y(4)$, where y is a solution to the initial value problem

$$y' = y - x, \quad y(2) = 0.$$

i	$x_i = x_0 + ih$	$y_i = y_{i-1} + h(y_{i-1} - x_{i-1})$	Δx_i	slope	Δy_i
0	2	0	0.2		
1	2.2	-0.4			
2					
3					
4					
5					
6					
7					
8					
9					
10					



In the above picture, we show the approximate solution in red alongside the real solution to the Initial value problem in green. In general a smaller step size should give a more accurate approximation.

Extra Example Use Euler's method with step size $h = 0.2$ to find an approximation for $y(2)$, where y is a solution to the initial value problem

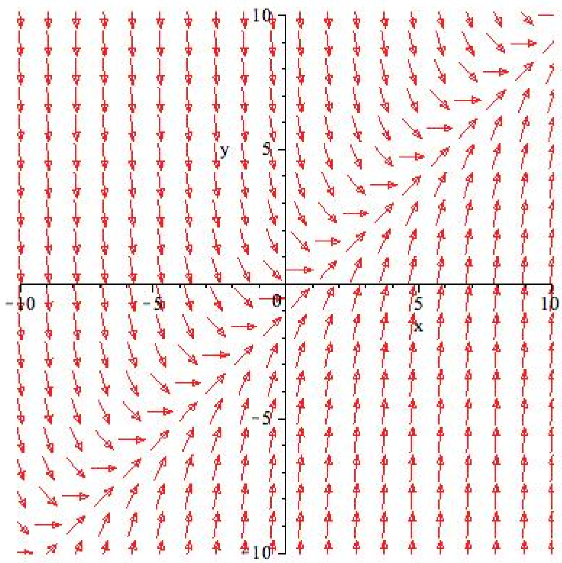
$$y' = x - y, \quad y(0) = 0.$$

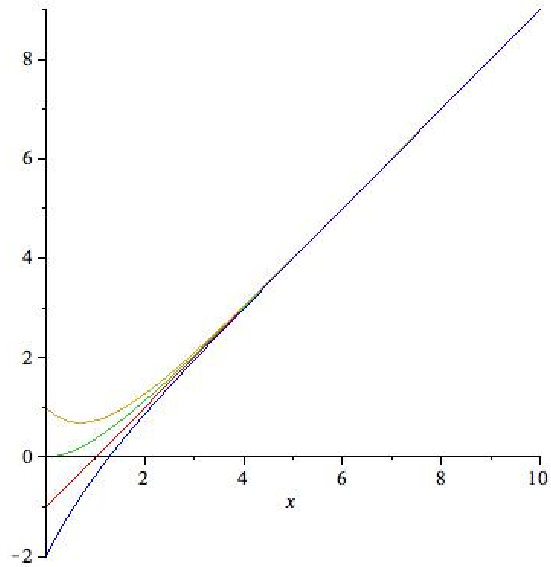
i	$x_i = x_0 + ih$	$y_i = y_{i-1} + h(x_{i-1} - y_{i-1})$
0	0	0
1	0.2	
2		
3		
4		
5		
6		
7		
8		
9		
10		

We can compare our numerical solution to the actual values of y along the curve when $x = x_0, x_1, \ldots, x_n = 2$, since we know that the solution is $y = x - 1 + \frac{1}{e^x}$.

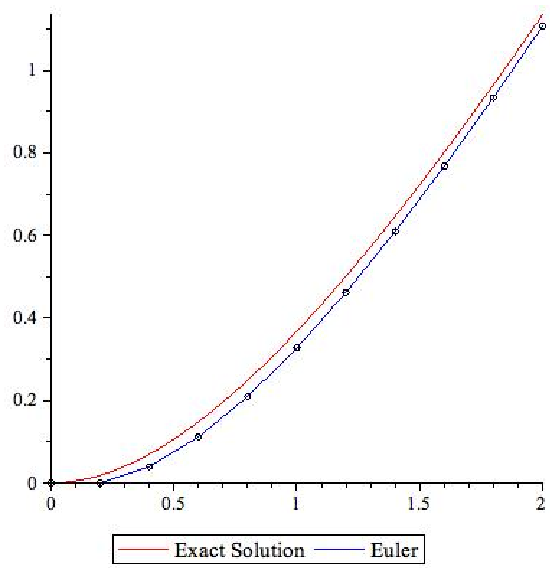
i	x_i	$y_i = y_{i-1} + h(x_{i-1} - y_{i-1})$	$x_i - 1 + \frac{1}{e^{x_i}}$	error $= \quad x_i - 1 + \frac{1}{e^{x_i}} - y_i$
0	0	0	0	0
1	0.2	0	0.0187	0.0187
2	0.4	0.04	0.0703	.0303
3	0.6	0.1120	0.1488	0.0368
4	0.8	0.2096	0.2493	0.0397
5	1.0	0.327	0.3679	.0402
6	1.2	0.4621	0.5012	0.03905
7	1.4	0.6097	0.6466	0.03688
8	1.6	0.7678	0.8019	0.03412
9	1.8	0.9342	0.9653	0.03108
10	2.0	1.107	1.1353	0.02796

Here is a picture of some solutions and a picture of the direction field for the differential equation $y' = x - y$.





Here is a picture of our numerical approximation in blue alongside the real solution in red.



Sketching/matching slope fields. How do you match a differential equation with its slope field? The basic principle is to think about where the slopes are positive, negative, and zero.
Example Give a rough sketch of the slope field for $y' = (x^2 - 1)y$.