Differential Equations

A **Differential Equation** is an equation relating an unknown function and one or more of its derivatives.

Examples Population growth : $\frac{dP}{dt} = kP$, or $\frac{dP}{dt} = kP(1 - \frac{P}{K})$.
Motion of a spring with a mass m attached: $m\frac{d^2x}{dt^2} = -kx$.
Body of mass m falling under the action of gravity g encounters air resistance.
The velocity of the falling body at time t satisfies the equation :
$m\frac{dv(t)}{dt} = mg - k[v(t)]^2$.
General Examples

$$y' = x - y, \qquad y' = yx, \qquad y' + xy = x^2.$$

The **Order** of a differential equation is the order of the highest derivative that occurs in the equation.

Example

- The differential equation $2\frac{d^2x}{dt^2} = -10x$ has order ____
- The differential equation $\frac{dv(t)}{dt} = 32 - 10[v(t)]^2$ has order ____

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The **Order** of a differential equation is the order of the highest derivative that occurs in the equation.

Example

- The differential equation $2\frac{d^2x}{dt^2} = -10x$ has order 2
- The differential equation $\frac{dv(t)}{dt} = 32 - 10[v(t)]^2$ has order 1

Solutions to Differential Equations

A function $y = f(x)$ is **a solution of a differential equation** if the equation is satisfied when $y = f(x)$ and its appropriate derivatives are substituted into the equation.

Example Match the following differential equations with their solutions:

Equation	**Solution**
$\frac{dy}{dt} = 2y$	$y = x - 1$
$y' = x - y$	$y = \ln\|1 + e^x\|$
$y' = \frac{e^x}{1+e^x}$	$y(t) = 10e^{2t}$
	$y = x - 1 + \frac{1}{e^x}$

- $\frac{dy}{dt} = 2y \rightarrow y(t) = 10e^{2t}$, $\quad y' = x - y \rightarrow y = x - 1 + \frac{1}{e^x}$, $y' = x - y \rightarrow y = x - 1$, $\quad y' = \frac{e^x}{1+e^x} \rightarrow y = \ln|1 + e^x|$.

Solutions to Differential Equations

When asked to **Solve** a differential equation we aim to find all possible solutions. Our solution will be a family of functions. A **General Solution** is a solution involving constants which can be specialized to give any particular solution.

Example The general solutions to the differential equations given above are

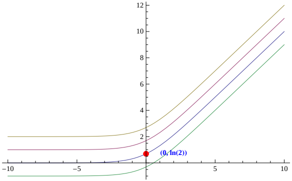
Equation	**General Solution**
$\frac{dP}{dt} = 2P$	$P(t) = Ke^{2t}$
$y' = x - y$	$y = x - 1 + \frac{C}{e^x}$
$y' = \frac{e^x}{1+e^x}$	$y = \ln\|1 + e^x\| + C$

Example

- For the differential equation $\frac{dy}{dx} = \frac{e^x}{1+e^x}$, we can find the general solution using methods of integration. (we will solve the others using the methods of seperable equations and Linear First order equations.)

Initial Value Problems

The graph below shows a sketch of some solutions from the family of solutions to the differential equation $\frac{dy}{dx} = \frac{e^x}{1+e^x}$, :



Note that only one of these solution curves passes through the point $(0, \ln 2)$, i.e. satisfies the requirement $y(0) = \ln 2$.

An **Initial Value Problem** asks for a specific solution to a differential equation satisfying an **initial condition** of the form $y(t_0) = y_0$.

Example Problem: Using the general solution given above ($y = x - 1 + \frac{C}{e^x}$), find a solution to the initial value problem $y' = x - y$ with the property that $y(0) = 0$.

- We have $Y(0) = 0 - 1 + \frac{C}{e^0} = C - 1$. We set $C - 1 = 0 \rightarrow C = 1$, $y(t) = x - 1 + \frac{1}{e^x}$.
  (At the end of your lecture notes, we give an approximate numerical solution to this problem using Euler's method.)

Direction Fields

If we have a differential equation of the type

$$y' = F(x, y)$$

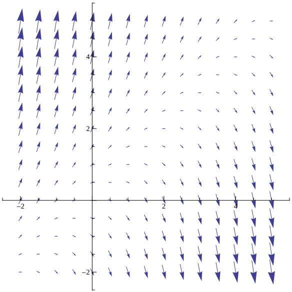
where $F(x, y)$ is an expression in x and y only, then the slope of a solution curve at a point (x, y) is $F(x, y)$. We can use the formula to calculate the slopes of the graphs of the solutions of the differential equation that pass through particular points on the plane. We can draw a picture of these slopes by drawing a small line (or arrow)indicating the direction of the curve at each point we have considered.

Example Consider the equation $y' = y - x$

- The graph of any solution to this differential equation passing through the point $(x, y) = (2, 1)$ has slope
- $y' = y - x = 1 - 2 = -1$.
- The graph of any solution to this differential equation passing through the point $(x, y) = (0, 1)$has slope
- $y' = y - x = 1 - 0 = 1$.
- The graph of any solution to this differential equation passing through the point $(x, y) = (-1, 1)$has slope
- $y' = y - x = 1 - (-1) = 2$.

Direction Fields

We can get some idea of what the graphs of the solutions to differential equation look like by drawing a **Direction Field** where we draw a short line segment (or arrow) with slope $y - x$ at each point (x, y) on the plane to indicate the direction of a solution running through that point. The picture below shows a computer generated direction field for the equation $y' = y - x$.

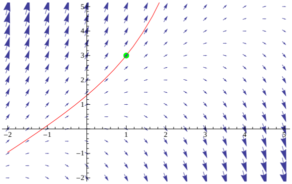


For any Differential equation of the form $y' = F(x, y)$ we can make a **direction field** by drawing an arrow with slope $F(x, y)$ at many points in the plane. The more points we include, the better the picture we get of the behavior of the solutions.

Direction Fields

We can use this picture to give a rough sketch of a solution to an initial value problem.

Example Below is a sketch of a solution to the differential equation $y' = y - x$, where $y(1) = 3$.



we see that a solution to the initial value problem $y' = y - x$, $y(1) = 3$ passes through the point (1.3) and follows the direction of the arrows.

Sketch a solution to the equation with $y(2) = 0$ on the vector field above.

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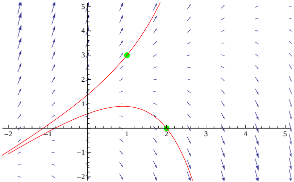
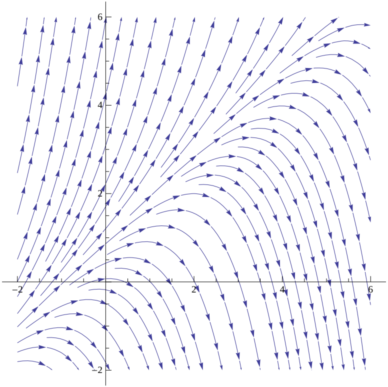


we see that a solution to the initial value problem $y' = y - x$, $y(1) = 3$ passes through the point (1.3) and follows the direction of the arrows.

Sketch a solution to the equation with $y(2) = 0$ on the vector field above.

Direction Fields

In this way we can get some idea of what the family of solutions to the differential equation $y' = y - x$ look like.



Euler's Method

Euler's method makes precise the idea of following the arrows in the direction field to get an approximate solution to a differential equation of the form $y' = F(x, y)$ satisfying the initial condition $y(x_0) = y_0$.
For such an initial value problem we can use a computer to generate a table of approximate numerical values of y for values of x in an interval $[x_0, b]$. This is called a **numerical solution** to the problem.

Example Estimate $y(4)$ where $y(x)$ is a solution to the differential equation $y' = y - x$ which satisfies the initial condition $y(2) = 0$, on the interval $2 \leq x \leq 4$.

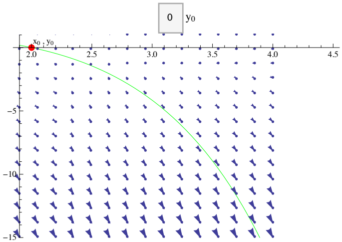
Euler's method approximates the path of the solution curve with a series of line segments following the directions of the arrows in the direction fields.

- First we choose the **Step Size** of our approximation, which will be the change in the value of x on each line segment. In general a smaller step size means shorter line segments and a better approximation. We will use $h = 0.2$ as the step size for our example above.

Euler's Method

Example Estimate $y(4)$ where $y(x)$ is a solution to the differential equation $y' = y - x$ which satisfies the initial condition $y(2) = 0$, on the interval $2 \leq x \leq 4$. Use a step size of $h = 0.2$.

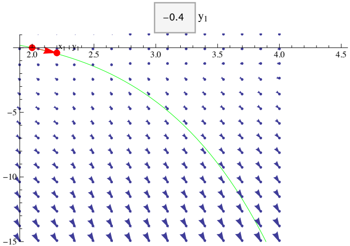
The **first point** on our approximating curve is determined by the initial condition $y(x_0) = y_0$. The corresponding point on the curve is (x_0, y_0).



- In the case of the above example, the initial value gives us that the first point on our approximating curve is $(2, 0)$
- The green curve shown here is the actual solution to the differential equation which passes through the point $(2, 0)$. It is the curve that we are trying to estimate.

Euler's Method

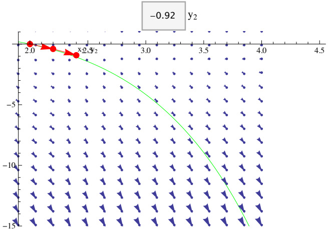
To get the **next (defining) point** on the curve, we follow the arrow in the direction field which starts at (x_0, y_0) (with slope $F(x_0, y_0)$) until we get to a point where $x_1 = x_0 + h$. (recall h is the step size).



- We can write down algebraic formulas for the endpoint of this arrow (x_1, y_1). We know that $x_1 = x_0 + h$. We have the slope of the arrow is $F(x_0, y_0) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h}$.
- Therefore $y_1 - y_0 = hF(x_0, y_0)$ or $y_1 = y_0 + hF(x_0, y_0)$.
- In our example $x_1 = 2 + .2 = 2.2$ and $y_1 = 0 + (.2)(0 - 2) = -.4$.

Euler's Method

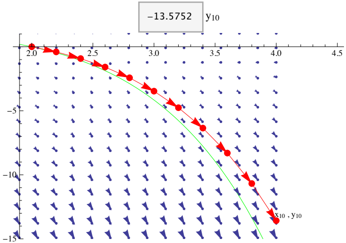
We draw the first segment of our approximating curve as the line segment between the points (x_0, y_0) and (x_1, y_1).



- To get the **next (defining) point** on the curve, , we follow the arrow in the direction field which starts at (x_1, y_1) (with slope $F(x_1, y_1)$) and which ends at $x_2 = x_1 + h$. In other words, we repeat the process starting at (x_1, y_1). By the same argument, we get the following equations for the point (x_2, y_2):
  $x_2 = x_1 + h$, and $y_2 = y_1 + hF(x_1, y_1)$.
- The second line segment of our approximating curve is the line between (x_1, y_1) and (x_2, y_2).

Euler's Method

We repeat the process until $x_n = a$, if we wish to approximate $y(a)$. Note that we should choose the step size, h, so that $\frac{a-x_0}{h}$ is an integer n.



- In our approximation, we wanted to estimate $y(4)$. We started at the initial point with $x = 2$. With a step size of $h = 0.2$, we get to our approximation in $\frac{4-2}{h} = \frac{4-2}{.2} = \frac{2}{.2} = 10$ steps.
- Note how our approximate solution(in red) compares to the true solution (in green). To improve accuracy, one can make the step size smaller.

Euler's Method

In Summary, to use this approximation;

- We first decide on the step size h. (If we want to estimate $y(x_0 + L)$ where y is a solution to the IVP $y' = F(x, y)$, $y(x_0) = y_0$, and we wish to use n steps, then the step size should be L/n.)
- Our series of approximations is then given by
- Initial point $= (x_0, y_0)$.
- $y_1 = y_0 + hF(x_0, y_0)$ new point on approximate curve $= (x_1, y_1) = (x_0 + h, y_1)$.
- $y_2 = y_1 + hF(x_1, y_1)$ new point on approximate curve $= (x_2, y_2) = (x_0 + 2h, y_2)$.
- $y_3 = y_2 + hF(x_2, y_2)$ new point on approximate curve $= (x_3, y_3) = (x_0 + 3h, y_3)$.

$$\vdots$$

- $y_i = y_{i-1} + hF(x_{i-1}, y_{i-1})$ corresponding point on approximate curve $= (x_i, y_i) = (x_0 + ih, y_i)$

$$\vdots$$

Euler's Method

Example Use Euler's method with step size $h = 0.2$ to find an approximation for $y(4)$, where y is a solution to the initial value problem

$$y' = y - x, \quad y(2) = 0.$$

i	$x_i = x_0 + ih$	$y_i = y_{i-1} + h(y_{i-1} - x_{i-1})$
0	2	0
1	2.2	-0.4
2	2.4	-0.92
3		
4		
5		
6		
7		
8		
9		
10		

Euler's Method

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$$y' = y - x, \quad y(2) = 0.$$

i	$x_i = x_0 + ih$	$y_i = y_{i-1} + h(y_{i-1} - x_{i-1})$
0	2	0
1	2.2	-0.4
2	2.4	-0.92
3	2.6	-1.584
4	2.8	-2.4208
5	3	-3.46496
6	3.2	-4.75795
7	3.4	-6.34954
8	3.6	-8.29945
9	3.8	-10.6793
10	4	-13.5752