Lecture 17 : Separable Equations

A **Separable Differential Equation** is a first order differential equation of the form

$$\frac{dy}{dx} = f(y)g(x)$$

.

In this case we have

$$\int \frac{1}{f(y)}dy = \int g(x)dx$$

.

We can see this by differentiating both sides with respect to x.

When we perform the above integration, we get an equation relating x and y which defines y implicitly as a function of x. Sometimes we can solve for y explicitly in terms of x.

Example Solve the following differential equation:

$$y' = \frac{\sqrt{x}}{3y}$$

Example Solve the following differential equation:

$$xy + y' = 100x$$

Example Solve the following initial value problem:

$$y' = \frac{y(\sin x)}{y^2 + 1}, \quad y(0) = 1$$

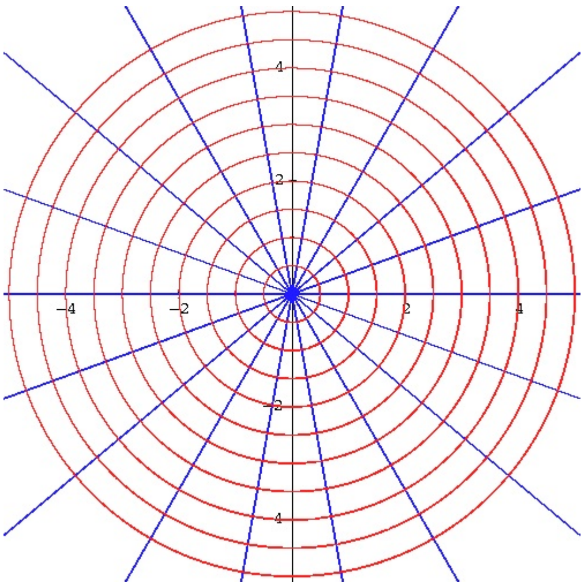
Example Solve the following Initial value problem:

$$y' = ye^x, \quad y(0) = 2$$

Orthogonal Trajectories.

An **Orthogonal Trajectory** of a family of curves is a curve that intersects each curve in the family of curves at right angles. Two curves intersect at right angles if their tangents at that point intersect at right angles. That is if the product of their slopes at the point of intersection is -1.

Example The family of curves $x^2 + y^2 = a^2$ is a family of concentric circles centered at the origin. The family of lines of form $y = kx$, are a family of orthogonal trajectories. Why?

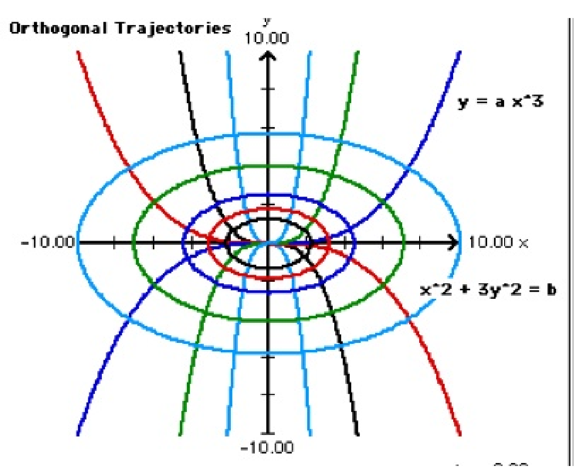


To find the orthogonal trajectories to a family of curves,

- We find a differential equation satisfied by all of the curves solving for any constants in the description in terms of x and y.
- We then use the fact that the products of the derivatives of orthogonal curves is -1 to find a new differential equation whose solution is the family of orthogonal trajectories.

Example Find the orthogonal trajectories to the family of curves $y = ax^3$.

$$\frac{dy}{dx} = 3ax^2.$$



Mixture Problems

In these problems a chemical in a liquid solution (or gas) runs into a container holding the liquid. The liquid in the container may already have a specified amount of the chemical dissolved in it. We assume the mixture is kept uniform by stirring and flows out of the container at a known rate. The differential equation describing the process is based on the formula

$$\begin{array}{c}\text{Rate of Change}\\ \text{of the amount}\\ \text{in the container}\end{array} = \left|\begin{array}{c}\text{Rate at which}\\ \text{chemical}\\ \text{arrives}\end{array}\right| - \left|\begin{array}{c}\text{Rate at which}\\ \text{chemical}\\ \text{departs}\end{array}\right|$$

Example A vat at Guinness' brewery with 500 gallons of beer contains 3% alcohol (15 gallons). Beer with 5% alcohol per unit of volume is pumped into the vat at a rate of 5 gallons/ min and the mixture is pumped out at the same rate. What is the percentage of alcohol after one hour?

Newton's Law of Cooling.

$$y' = -0.1(y - 90)$$

$$\frac{dy}{dx} = -0.1(y - 90)$$

Separating variables, we get

$$\frac{1}{y - 90} dy = -0.1 dx$$

Integrating both sides, we get

$$\ln |y - 90| = -0.1x + C$$

Applying the exponential function to both sides, we get

$$|y - 90| = Ke^{-0.1x}$$

Since the value of K may be positive or negative, we get

$$y - 90 = Ke^{-0.1x}$$

and

$$y = 90 + Ke^{-0.1x}.$$

Exponential Growth and Decay

Example Solve the following Initial value problem:

$$y' = 2y, \quad y(0) = 10$$

Separating variables, we get

$$\frac{1}{y} dy = 2dx$$

Integrating both sides, we get

$$\ln |y| = 2x + C$$

applying the exponential function to both sides, we get

$$|y| = e^{2x}e^{C} = Ke^{2x}$$

since K can be positive or negative, we can discard absolute value signs and

$$y = Ke^{2x}.$$

Using the initial value condition, we get

$$10 = Ke^{0} = K$$

Therefore

$$y = 10e^{2x}.$$