Lecture 18 : Linear Differential Equations

A **First Order Linear Differential Equation** is a first order differential equation which can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x), Q(x)$ are continuous functions of x on a given interval.

The above form of the equation is called the **Standard Form** of the equation.

Example Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y.$$

To solve an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we multiply by a function of of x called an **Integrating Factor**. This function is

$$\boxed{I(x) = e^{\int P(x)dx}.}$$

(we use a particular antiderivative of $P(x)$ in this equation.)
$I(x)$ has the property that

$$\frac{dI(x)}{dx} = P(x)I(x).$$

Multiplying across by $I(x)$, we get an equation of the form

$$I(x)y' + I(x)P(x)y = I(x)Q(x).$$

The left hand side of the above equation is the derivative of the product $I(x)y$. Therefore we can rewrite our equation as

$$\frac{d[I(x)y]}{dx} = I(x)Q(x).$$

Integrating both sides with respect to x, we get

$$\int \frac{d[I(x)y]}{dx}dx = \int I(x)Q(x)dx$$

or

$$I(x)y = \int I(x)Q(x)dx + C$$

giving us a solution of the form

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

Example Solve the differential equation

$$x\frac{dy}{dx} = x^2 + 3y.$$

Example Solve the initial value problem

$$y' + xy = x, \quad y(0) = -6.$$

Exam 2 review problems

1. Use Euler's method with step size $h = 1$ to find an approximate value for $y(3)$, where y is the solution to the initial value problem

$$y' = \sqrt{y + 3x}\,, \quad y(1) = 1.$$

(a) 3 $\quad (b)$ 5 $\quad (c)$ $\sqrt{8}$ $\quad (d)$ 6 $\quad (e)$ $5 - \sqrt{20}$

2. Solve the initial the value problem $y' = 3x^2y^2$, $y(-1) = -\frac{1}{3}$. Which number below is $y(0)$?

(a) $-\frac{1}{6}$ $\quad (b)$ $-\frac{1}{4}$ $\quad (c)$ $-\frac{1}{7}$ $\quad (d)$ 0 $\quad (e)$ $\frac{1}{5}$

3. Evaluate the integral

$$\int \frac{x}{\sqrt{x^2+25}}\, dx$$

(a) $\dfrac{5}{x\sqrt{x^2+25}} + C$ (b) $\dfrac{5}{\sqrt{x^2+25}} + C$ (c) $\ln|\sqrt{x^2+25}/5 + x/5| + C$

(d) $x\sqrt{x^2+25} + C$ (e) $\sqrt{x^2+25} + C$

4. The midpoint rule with $n = 4$ is used to approximate the integral

$$\int_1^3 \ln(x)\, dx.$$

What is the maximum error of this approximation? Recall that the error formula for the midpoint rule is given by

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

(a) $\dfrac{1}{192}$ (b) $\dfrac{1}{48}$ (c) $\dfrac{1}{432}$ (d) $\dfrac{-1}{48}$ (e) $\dfrac{-1}{432}$