Lecture 19: Sequences

A **Sequence** is a list of numbers written in order.

$$\{a_1,\ a_2,\ a_3,\ \dots\}$$

The sequence may be infinite. The **n th term** of the sequence is the n th number on the list. On the list above

$$a_1 = \text{1st term},\ \ a_2 = 2 \text{ nd term},\ \ a_3 = 3 \text{ rd term},\ \ \text{etc....}$$

Example In the sequence $\{1, 2, 3, 4, 5, 6, \dots\}$, we have $a_1 = 1, a_2 = 2, \dots$. The n^{th} term is given by $a_n = n$.

Some sequences have **patterns**, some do not.
Example If I roll a 20 sided die repeatedly, I generate a sequence of numbers, which have no pattern.

Example The sequences

$$\{1, 2, 3, 4, 5, 6, \dots\}$$

and

$$\{1,\ -1,\ 1,\ -1,\ 1,\ \dots\}$$

have patterns.

Sometimes we can give a **formula for the n th term of a sequence, $a_n = f(n)$.**

Example For the sequence

$$\{1, 2, 3, 4, 5, 6, \dots\},$$

we can give a formula for the n th term. $a_n = n$.

Example Assuming the following sequences follow the pattern shown, give a formula for the n-th term:

$$\{1,\ -1,\ 1,\ -1,\ 1,\ \dots\}$$

$$\{-1/2,\ 1/3,\ -1/4,\ 1/5,\ -1/6,\ \dots\}$$

Factorials are commonly used in sequences

$$0! = 1,\quad 1! = 1,\quad 2! = 2 \cdot 1,\quad 3! = 3 \cdot 2 \cdot 1,\quad \dots,\quad n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1.$$

Example Find a formula for the n th term in the following sequence

$$\left\{ \frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \dots \quad , a_n = \qquad , \right\}$$

Below we show **3 different ways to represent a sequence**:

$$\left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \dots,\ \frac{n}{n+1},\ \dots\right\} \qquad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1}.$$

$$\left\{\frac{-3}{3},\ \frac{5}{9},\ \frac{-7}{27},\ \dots,\ (-1)^n\frac{(2n+1)}{3^n},\ \dots\right\} \qquad \left\{(-1)^n\frac{(2n+1)}{3^n}\right\}_{n=1}^{\infty} \qquad a_n = (-1)^n\frac{(2n+1)}{3^n}.$$

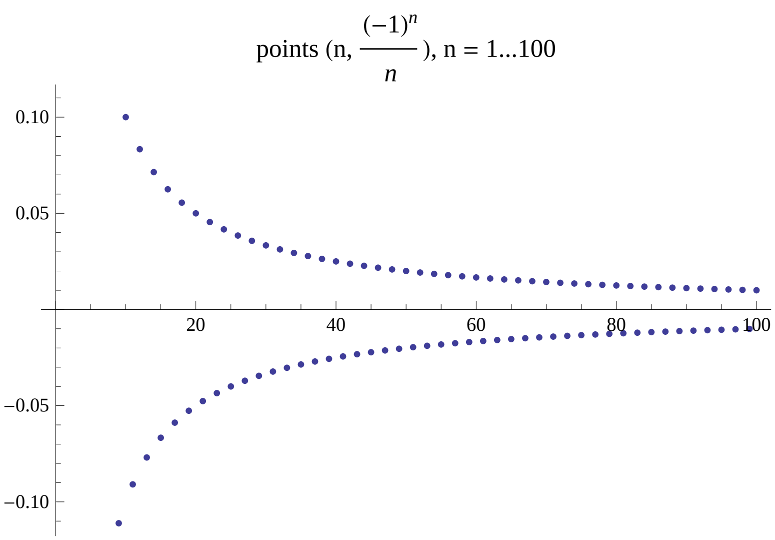
$$\left\{\frac{e}{1},\ \frac{e^2}{2},\ \frac{e^3}{6},\ \ldots,\ \frac{e^n}{n!},\ \ldots\right\} \qquad \left\{\frac{e^n}{n!}\right\}_{n=1}^{\infty} \qquad a_n = \frac{e^n}{n!}.$$

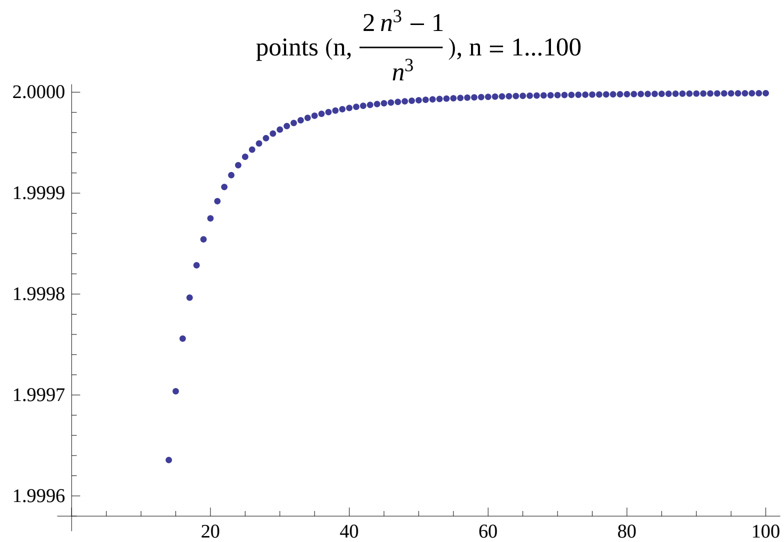
Graph of a Sequence

A sequence is a function from the positive integers to the real numbers, with $f(n) = a_n$. We can draw a graph of this function as a set of points in the plane. The points on the graph are

$$(1, a_1), \quad (2, a_2), \quad (3, a_3), \quad \ldots \quad , (n, a_n), \quad \ldots$$

Example Below, we show the graphs of the sequences $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$ and $\left\{\frac{2n^3-1}{n^3}\right\}_{n=1}^{\infty}$.





We can see from these pictures that the graphs get closer to a horizontal asymptote as $n \to \infty$, $y = 0$ for the sequence on the left and $y = 2$ for the sequence on the right. Algebraically this means that as $n \to \infty$, we have $\frac{(-1)^n}{n} \to 0$ and $\frac{2n^3-1}{n^3} \to 2$.

Limit of a Sequence

Definition A sequence $\{a_n\}$ has **limit** L if we can make the terms a_n as close as we like to L by taking n sufficiently large. We denote this by

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ \text{ as } \ n \to \infty.$$

If $\lim_{n\to\infty} a_n$ exists (is finite), we say the sequence **converges** or is convergent. Otherwise, we say the sequence **diverges**.

Graphically: If $\lim_{n\to\infty} a_n = L$, the graph of the sequence $\{a_n\}_{n=1}^{\infty}$ has a unique horizontal asymptote $y = L$.

Formal Definition A sequence $\{a_n\}$ has limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ \text{ as } \ n \to \infty$$

if for every $\epsilon > 0$ there is and integer N with the property that

$$\text{if} \ \ n > N \quad \text{then} \quad |a_n - L| < \epsilon.$$

Determining if a sequence is convergent.

Using our previous knowledge of limits :

Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = L$.

Example Determine if the following sequences converge or diverge:

$$\left\{\frac{2^n - 1}{2^n}\right\}_{n=1}^{\infty}, \qquad \left\{\frac{2n^3 - 1}{n^3}\right\}_{n=1}^{\infty}$$

We can use L'Hospital's rule to determine the limit of $f(x)$ if we have an indeterminate form.

Example Is the following sequence convergent?

$$\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$$

Diverging to ∞. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M, there is an integer N with the property

$$\text{if } \ n > N, \qquad \text{then} \qquad a_n > M.$$

In this case we say the sequence $\{a_n\}$ **diverges to infinity**.

Note: If $\lim_{x\to\infty} f(x) = \infty$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = \infty$.

Example Show that the sequence $\{r^n\}_{n=1}^{\infty}$, $r \geq 0$, converges if $0 \leq r \leq 1$ and diverges to infinity if $r > 1$.

The usual **Rules of Limits** apply:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any constant then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if} \quad p > 0 \quad \text{and} \quad a_n > 0$$

In fact if $\lim_{n\to\infty} a_n = L$ and $f(x)$ is a continuous function at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example Determine if the following sequence converges or diverges and if it converges find the limit.

$$\left\{\sqrt[3]{\frac{2n+1}{n}} - \frac{1}{n}\right\}_{n=1}^{\infty}.$$

Note We cannot always find a function $f(x)$ with $f(n) = a_n$.
The **Squeeze Theorem** or Sandwich Theorem can also be applied :

$$\boxed{\text{If} a_n \le b_n \le c_n \quad \text{for} \quad n \ge n_0 \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, \quad \text{then} \quad \lim_{n\to\infty} b_n = L.}$$

Example Find the limit of the following sequence

$$\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty},$$

Alternating Sequences

For any sequence, we have $-|a_n| \leq a_n \leq |a_n|$. We can use the squeeze theorem to see that

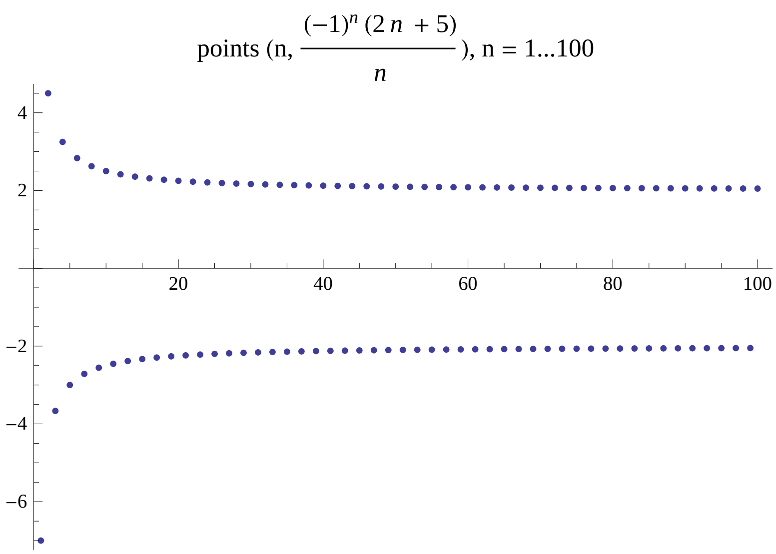
$$\boxed{\text{if} \quad \lim_{n\to\infty} |a_n| = 0, \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.}$$

In fact any sequence with infinitely many positive and negative values converges if and only if $\lim_{n\to\infty} |a_n| = 0$

$$\textbf{Let } \{a_n\} = \{(-1)^n a_n'\} \textbf{ where } a_n' > 0$$

- If $\lim_{n\to\infty} a_n' = L \neq 0$, then $\lim_{n\to\infty} (-1)^n a_n'$ does not exist.
- If $\lim_{n\to\infty} a_n' = \infty$, then $\lim_{n\to\infty} (-1)^n a_n'$ does not exist.
- If $\lim_{n\to\infty} a_n'$ does not exist, then $\lim_{n\to\infty} (-1)^n a_n'$ does not exist.

Below, we show a picture of a sequence where, as in the first case above, $\lim_{n\to\infty} a_n' = L \neq 0$.



Theorem If $\{a_n\}$ is an alternating sequence of the form $(-1)^n a_n'$ where $a_n' > 0$, then the alternating sequence converges if and only if $\lim_{n\to\infty} |a_n| = 0$ or (for the sequence described above) $\lim_{n\to\infty} a_n' \to 0$.
(also true for sequences of form $(-1)^{n+1} a_n'$ or any sequence with infinitely many positive and negative terms)

Example Determine if the following sequences converge:

$$\left\{(-1)^n \frac{2n+1}{n^2}\right\}_{n=1}^{\infty}, \qquad \left\{(-1)^n \frac{2n+1}{n}\right\}_{n=1}^{\infty}$$

Monotone Sequences

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, or

$$a_1 < a_2 < a_3 < \dots.$$

A sequence $\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$, or

$$a_1 > a_2 > a_3 > \dots.$$

A sequence $\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M for which

$$a_n \leq M \quad \text{for all} \quad n \geq 1.$$

A sequence $\{a_n\}$ is **bounded below** if there is a number m for which

$$a_n \geq m \quad \text{for all} \quad n \geq 1.$$

A sequence that is bounded above and below is called **Bounded**.

Theorem Every bounded monotonic sequence is convergent.

(This theorem will be very useful later in determining if series are convergent.)

To check for monotonicity
If we have a differentiable function $f(x)$ with $f(n) = a_n$, then the sequence $\{a_n\}$ is increasing if $f'(x) > o$ and the sequence $\{a_n\}$ is decreasing if $f'(x) < o$.

Example Show that the following sequence is monotone and bounded and hence converges.

$$\{\tan^{-1}(n)\}_{n=1}^{\infty}$$

Extra Examples

Example Determine if the following sequences converge or diverge:

$$\left\{\frac{1}{n^5}\right\}_{n=1}^{\infty},$$

$\lim_{n\to\infty}\frac{1}{n^5} = \lim_{x\to\infty}\frac{1}{x^5} = 0$. Therefore the sequence converges to 0.

Example Is the following sequence convergent?

$$\left\{\sqrt[n]{n}\right\}_{n=1}^{\infty}.$$

$\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} = e^{\lim_{x\to\infty}\frac{\ln(x)}{x}}$.

Using L'Hospital's rule, we get $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$.

Therefore $\lim_{n\to\infty}\sqrt[n]{n} = e^0 = 1$ and the sequence converges.

Example Determine if the following sequence converges or diverges and if it converges find the limit.

$$\left\{\cos\left(\frac{n}{2^n}\right)\right\}_{n=1}^{\infty}$$

$\lim_{n\to\infty}\frac{n}{2^n} = 0$(see lecture notes)

Using the rules of limits, we have $\lim_{n\to\infty}\cos\left(\frac{n}{2^n}\right) = \cos\left(\lim_{n\to\infty}\frac{n}{2^n}\right) = \cos(0) = 1$. Therefore the sequence converges to 1.

Example Show that the sequence $\{r^n\}_{n=1}^{\infty}$, converges if $-1 < r \leq 1$ and diverges to infinity if $r > 1$.

This was demonstrated in class for $r > 0$. The case $r = 0$ is obvious.

The case where $r < 0$ gives an alternating series $\{r^n\}_{n=1}^{\infty}$. This converges if and only if $\lim_{n\to\infty}|r|^n = 0$. this happens only when $|r| < 1$, giving the desired result.

Example Show that the following sequence is decreasing and bounded and hence convergent

$$a_1 = 3, \quad a_{n+1} = \frac{a_n}{2}.$$

The terms of this sequence are positive, since the first term is 3 and each term is half of the previous term. Therefore the sequence is bounded, since $0 < a_n < 3$ for all n.

$a_{n+1} = \frac{1}{2}a_n < a_n$, therefore the sequence is decreasing and bounded and thus it converges.