

## Lecture 20 : Series

So far our definition of a sum of numbers applies only to adding a finite set of numbers. We can extend this to a definition of a sum of an infinite set of numbers in much the same way as we extended our notion of the definite integral to an improper integral over an infinite interval.

**Example**

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

We call this infinite sum a **series**.

**Definition** Given a series  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ , we let  $s_n$  denote its  $n$  **th partial sum**

$$s_n = a_1 + a_2 + \dots + a_n.$$

If the sequence  $\{s_n\}$  is convergent and  $\lim_{n \rightarrow \infty} s_n = S$ , then we say that the series  $\sum_{n=1}^{\infty} a_n$  is **convergent** and we let

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_n = \lim_{n \rightarrow \infty} s_n = S.$$

The number  $S$  is called the sum of the series. Otherwise the series is called **divergent**.

**Example** Find the partial sums  $s_1, s_2, s_3, \dots, s_n$  of the series  $\sum_{n=1}^{\infty} \frac{1}{2^n}$ . Find the sum of this series. Does the series converge?

**Example** Recall that  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ . Does the series

$$\sum_{n=1}^{\infty} n$$

converge?

## Geometric Series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if  $|r| < 1$  and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1.$$

If  $|r| \geq 1$ , the geometric series is divergent.

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**Proof** If  $r = 1$ ,

$$\sum_{n=1}^{\infty} ar^{n-1} = a + a + a + a + \dots, \quad s_n = an, \quad \lim_{n \rightarrow \infty} s_n = \infty$$

and the series diverges.

If  $r = -1$ ,

$$\sum_{n=1}^{\infty} ar^{n-1} = a - a + a - a + \dots, \quad s_n = a \text{ if } n \text{ is odd and } s_n = 0 \text{ if } n \text{ is even,} \quad \lim_{n \rightarrow \infty} s_n = \text{does not exist}$$

and the series diverges.

If  $|r| \neq 1$ , we have

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + \dots + ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n \end{aligned}$$

Thus we get

$$s_n - rs_n = a - ar^n \quad \text{or} \quad s_n(1-r) = a(1-r^n) \quad \text{or} \quad s_n = \frac{a(1-r^n)}{(1-r)}$$

If  $-1 < r < 1$ , we saw in the section on sequences, that  $\lim_{n \rightarrow \infty} r^n = 0$  and thus

$$\lim_{n \rightarrow \infty} s_n = \frac{a}{(1-r)}$$

giving us the desired result.

If  $|r| > 1$ , then  $\lim_{n \rightarrow \infty} r^n$  does not exist and hence  $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{(1-r)}$  does not exist. Thus the series does not converge.

**Example** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 10}{4^{n-1}} = -10 + \frac{10}{4} - \frac{10}{16} + \dots$$

**Example** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$$

**Example** Find the sum of the series

$$\sum_{n=4}^{\infty} \frac{2^{n-1}}{3^n}$$

**Example** For which values of  $x$  does the series  $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$  converge?

**Example** Write the numbers  $0.66666666\cdots = 0.\bar{6}$  and  $1.521212121\cdots = 1.5\bar{2}1$  as fractions. What about  $0.999999\cdots$ ?

### Telescoping Series

These are series of the form similar to  $\sum f(n) - f(n+1)$ . Because of the large amount of cancellation, they are relatively easy to sum.

**Example** Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 7k + 12} = \sum_{k=1}^{\infty} \frac{1}{(k+3)} - \frac{1}{(k+4)}$$

converges.

**Example** Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k} - \frac{1}{2^{k+1}}$$

converges.

### Harmonic Series

The following series, known as the harmonic series, diverges:

$$\sum_{k=1}^{\infty} \frac{1}{n}$$

We can see this if we look at a subsequence of partial sums:  $\{s_{2^n}\}$ .

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] > 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] = 2$$

$$s_8 = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] > s_4 + \left[\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right] > 2 + \frac{1}{2} = \frac{5}{2}$$

Similarly we get

$$s_{2^n} > \frac{n+2}{2}$$

and  $\lim_{n \rightarrow \infty} s_n > \lim_{n \rightarrow \infty} \frac{n+2}{2} = \infty$ . Hence the harmonic series diverges. (You will see an easier proof in the next section.)

**Note** that convergence or divergence is unaffected by adding or deleting a finite number of terms at the beginning of the series.

**Example**

$$\sum_{n=10}^{\infty} \frac{1}{n} \text{ is divergent}$$

and

$$\sum_{k=50}^{\infty} \frac{1}{2^k} \text{ is convergent.}$$

### Divergence Test

**Theorem** If a series  $\sum_{i=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Warning** The converse is not true, we may have a series where  $\lim_{n \rightarrow \infty} a_n = 0$  and the series is divergent. For example, the harmonic series.

**Proof** Suppose the series  $\sum_{i=1}^{\infty} a_n$  is convergent with sum  $S$ . Since  $a_n = s_n - s_{n-1}$  and

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} s_{n-1} = S$$

we have  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = S - S = 0$ .

This gives us a **Test for Divergence**:

If  $\lim_{n \rightarrow \infty} a_n$  does not exist or if  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{i=1}^{\infty} a_n$  is divergent.

If  $\lim_{n \rightarrow \infty} a_n = 0$  the test is inconclusive.

**Example** Test the following series for divergence with the above test:

$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{2n^2} \quad \sum_{n=1}^{\infty} \frac{n^2 + 1}{2n^3} \quad \sum_{n=1}^{\infty} \frac{n^2 + 1}{2n}$$

**Note** that if  $\lim a_n = 0$ , this test is inconclusive and the series may diverge or converge.

### Properties of Series

The following properties of series follow from the corresponding laws of limits:

Suppose  $\sum a_n$  and  $\sum b_n$  are convergent series, then the series  $\sum(a_n + b_n)$ ,  $\sum(a_n - b_n)$  and  $\sum ca_n$  also converge. We have

$$\sum ca_n = c \sum a_n, \quad \sum(a_n + b_n) = \sum a_n + \sum b_n, \quad \sum(a_n - b_n) = \sum a_n - \sum b_n.$$

**Example** Sum the following series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 7k + 12} - \frac{2}{3^k}.$$

Because both of these series converge we can break it into the difference of two series to sum it.

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2 + 7k + 12} - \frac{2}{3^k} &= \sum_{k=1}^{\infty} \frac{1}{k^2 + 7k + 12} - \sum_{k=1}^{\infty} \frac{2}{3^k} \\ &= 1/4 - 1 \end{aligned}$$

from our previous calculations.

**Puzzle: Ant on a rubber band.** An ant starts at one end of a one meter rubber band, placed conveniently at  $x = 0$  on the  $x$  axis. Initially the other end of the rubber band is at  $x = 1$ . Each second the ant walks 1 cm. At the end of each second Mike, who likes teasing ants, stretches the rubber band by one meter. (Note the point at which the ant is at moves when the band is stretched.). Will the ant ever reach the end of the rubber band?

Hint calculate the proportion of the distance covered by the ant after 1 second, 2 seconds 3 seconds, .....,  $n$  seconds, and derive your answer from the sum of the series.