Integral Test

In this section, we show how to use the integral test to decide whether a series of the form $\sum_{n=a}^{\infty} \frac{1}{n^p}$ (where $a \geq 1$) converges or diverges by comparing it to an improper integral.

Integral Test Suppose $f(x)$ is a positive decreasing continuous function on the interval $[1, \infty)$ with

$$f(n) = a_n.$$

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ converges, that is:

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is convergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is convergent.}$$

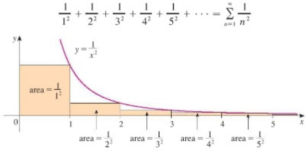
$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is divergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is divergent.}$$

- **Note** The result is still true if the condition that $f(x)$ is decreasing on the interval $[1, \infty)$ is relaxed to "the function $f(x)$ is decreasing on an interval $[M, \infty)$ for some number $M \geq 1$."

Integral Test (Why it works: convergence)

We know from a previous lecture that
$\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leq 1$.

- In the picture we compare the series $\sum_{n=1}^\infty \frac{1}{n^2}$ to the improper integral $\int_1^\infty \frac{1}{x^2} dx$.



- The k th partial sum is $s_k = 1 + \sum_{n=2}^k \frac{1}{n^2} < 1 + \int_1^\infty \frac{1}{x^2} dx = 1 + 1 = 2.$
- Since the sequence $\{s_k\}$ is increasing (because each $a_n > 0$) and bounded, we can conclude that the sequence of partial sums converges and hence the series

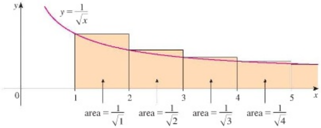
$$\sum_{i=1}^\infty \frac{1}{n^2} \quad \text{converges.}$$

- **NOTE** We are not saying that $\sum_{i=1}^\infty \frac{1}{n^2} = \int_1^\infty \frac{1}{x^2} dx$ here.

Integral Test (Why it works: divergence)

We know that $\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \le 1$.

- In the picture, we compare the series $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ to the improper integral $\int_1^\infty \frac{1}{\sqrt{x}} dx$.



$$\sum_{k=1}^\infty \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

- This time we draw the rectangles so that we get

$$s_n > s_{n-1} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > \int_1^n \frac{1}{\sqrt{x}} dx$$

- Thus we see that $\lim_{n\to\infty} s_n > \lim_{n\to\infty} \int_1^n \frac{1}{\sqrt{x}} dx$.
- However, we know that $\int_1^n \frac{1}{\sqrt{x}} dx$ grows without bound and hence since $\int_1^\infty \frac{1}{\sqrt{x}} dx$ diverges, we can conclude that $\sum_{k=1}^\infty \frac{1}{\sqrt{n}}$ also diverges.

p-series

We know that $\int_1^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges if $p \leq 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.$$

Example Determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{15}}, \qquad \sum_{n=10}^{\infty} \frac{1}{n^{15}}, \qquad \sum_{n=100}^{\infty} \frac{1}{\sqrt[5]{n}},$$

- $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges since $p = 1/3 < 1$.
- $\sum_{n=1}^{\infty} \frac{1}{n^{15}}$ converges since $p = 15 > 1$.
- $\sum_{n=10}^{\infty} \frac{1}{n^{15}}$ also converges since a finite number of terms have no effect whether a series converges or diverges.
- $\sum_{n=100}^{\infty} \frac{1}{\sqrt[5]{n}}$ conv/diverges if and only if $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ conv/div. This diverges since $p = 1/5 < 1$.

Comparison Test

In this section, as we did with improper integrals, we see how to compare a series (with Positive terms) to a well known series to determine if it converges or diverges.

- We will of course make use of our knowledge of p-series and geometric series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.$$

$$\sum_{n=1}^{\infty} ar^{n-1} \quad \text{converges if} \quad |r| < 1, \quad \text{diverges if} \quad |r| \geq 1.$$

- **Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms.**

  (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, than $\sum a_n$ is also convergent.

  (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Example 1

Example 1 Use the comparison test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}$$

- First we check that $a_n > 0$ −> true since $\frac{2^{-1/n}}{n^3} > 0$ for $n \geq 1$.
- We have $2^{1/n} = \sqrt[n]{2} > 1$ for $n \geq 1$. Therefore $2^{-1/n} = \frac{1}{\sqrt[n]{2}} < 1$ for $n \geq 1$.
- Therefore $\frac{2^{-1/n}}{n^3} < \frac{1}{n^3}$ for $n > 1$.
- Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p > 1$, it converges.
- Comparing the above series with $\sum_{n=1}^{\infty} \frac{1}{n^3}$, we can conclude that $\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}$ also converges and $\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$

Example 2

Example 2 Use the comparison test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2^{1/n}}{n}$$

- First we check that $a_n > 0$ −> true since $\frac{2^{1/n}}{n} > 0$ for $n \geq 1$.
- We have $2^{1/n} = \sqrt[n]{2} > 1$ for $n \geq 1$.
- Therefore $\frac{2^{1/n}}{n} > \frac{1}{n}$ for $n > 1$.
- Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$ (a.k.a. the harmonic series), it diverges.
- Therefore, by comparison, we can conclude that $\sum_{n=1}^{\infty} \frac{2^{1/n}}{n}$ also diverges.

Example 3

Example 3 Use the comparison test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

- First we check that $a_n > 0$ −> true since $\frac{1}{n^2+1} > 0$ for $n \geq 1$.
- We have $n^2 + 1 > n^2$ for $n \geq 1$.
- Therefore $\frac{1}{n^2+1} < \frac{1}{n^2}$ for $n > 1$.
- Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$, it converges.
- Therefore, by comparison, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges and $\sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Example 4

Example 4 Use the comparison test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^{-2}}{2^n}$$

- First we check that $a_n > 0$ –> true since $\frac{n^{-2}}{2^n} = \frac{1}{n^2 2^n} > 0$ for $n \geq 1$.
- We have $\frac{1}{n^2 2^n} < \frac{1}{n^2}$ for $n \geq 1$.
- Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$, it converges.
- Therefore, by comparison, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$ also converges and $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Example 5

Example 5 Use the comparison test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

- First we check that $a_n > 0$ −> true since $\frac{\ln n}{n} > \frac{1}{n} > 0$ for $n \geq e$. Note that this allows us to use the test since a finite number of terms have no bearing on convergence or divergence.
- We have $\frac{\ln n}{n} > \frac{1}{n}$ for $n > 3$.
- Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges.

Example 6

Example 6 Use the comparison test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

- First we check that $a_n > 0$ −> true since $\frac{1}{n!} > 0$ for $n \geq 1$.
- We have $n! = n(n-1)(n-2) \cdot \dots \cdot 2 \cdot 1 > 2 \cdot 2 \cdot 2 \cdot \dots \cdot 2 \cdot 1 = 2^{n-1}$. Therefore $\frac{1}{n!} < \frac{1}{2^{n-1}}$.
- Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

Limit Comparison Test

Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms**. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a **finite number** and $c > 0$, then either both series converge or both diverge. (Note $c \neq 0$ or ∞.)

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

(Note that our previous comparison test is difficult to apply in this and most of the examples below.)

- First we check that $a_n > 0$ –> true since $a_n = \frac{1}{n^2-1} > 0$ for $n \geq 2$. (after we study absolute convergence, we see how to get around this restriction.)
- We will compare this series to $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which converges, since it is a p-series with $p = 2$. $b_n = \frac{1}{n^2}$.
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{1}{1-(1/n^2)} = 1$
- Since $c = 1 > 0$, we can conclude that both series converge.

Example

Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms**. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a **finite number** and $c > 0$, then either both series converge or both diverge. (Note $c \neq 0$ or ∞.)

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1}$$

- First we check that $a_n > 0$ –> true since $a_n = \frac{n^2+2n+1}{n^4+n^2+2n+1} > 0$ for $n \geq 1$.
- For a rational function, the rule of thumb is to compare the series to the series $\sum \frac{n^p}{n^q}$, where p is the degree of the numerator and q is the degree of the denominator.
- We will compare this series to $\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges, since it is a p-series with $p = 2$. $b_n = \frac{1}{n^2}$.
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} (\frac{n^2+2n+1}{n^4+n^2+2n+1})/(1/n^2) = \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{n^4+n^2+2n+1} = \lim_{n\to\infty} \frac{1+2/n+1/n^2}{1+1/n^2+2/n^3+1/n^4} = 1.$
- Since $c = 1 > 0$, we can conclude that both series converge.

Example

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^3+1}}$$

- First we check that $a_n > 0$ –> true since $a_n = \frac{2n+1}{\sqrt{n^3+1}} > 0$ for $n \geq 1$.
- We will compare this series to $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{n}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges, since it is a p-series with $p = 1/2$. $\boxed{b_n = \frac{1}{\sqrt{n}}}$.
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{2n+1}{\sqrt{n^3+1}}\right) \Big/ (1/\sqrt{n}) = \lim_{n\to\infty} \frac{+2n^{3/2}+\sqrt{n}}{\sqrt{n^3+1}} =$ $\lim_{n\to\infty} \frac{(2n^{3/2}+\sqrt{n}) \big/ n^{3/2}}{\sqrt{n^3+1} \big/ n^{3/2}} = \lim_{n\to\infty} \frac{(2+1/n)}{\sqrt{(n^3+1)/n^3}} = \lim_{n\to\infty} \frac{(2+1/n)}{\sqrt{(1+1/n^3)}} = 2.$
- Since $c = 2 > 0$, we can conclude that both series diverge.

Example

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{e}{2^n - 1}$$

- First we check that $a_n > 0$ –> true since $a_n = \frac{e}{2^n-1} > 0$ for $n \geq 1$.
- We will compare this series to $\sum_{n=1}^{\infty} \frac{1}{2^n}$ which converges, since it is a geometric series with $r = 1/2 < 1$. $\boxed{b_n = \frac{1}{2^n}}$.
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}(\frac{e}{2^n-1})\Big/(1/2^n) = \lim_{n\to\infty} \frac{e}{1-1/2^n} = e$.
- Since $c = e > 0$, we can conclude that both series converge.

Example

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2}$$

- First we check that $a_n > 0$ −> true since $a_n = \frac{2^{1/n}}{n^2} > 0$ for $n \geq 1$.
- We will compare this series to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges, since it is a p-series with $p = 2 > 1$. $\boxed{b_n = \frac{1}{n^2}}$.
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} (\frac{2^{1/n}}{n^2}) \Big/ (1/n^2) = \lim_{n\to\infty} 2^{1/n} = \lim_{n\to\infty} e^{\frac{\ln 2}{n}} = 1$.
- Since $c = 1 > 0$, we can conclude that both series converge.

Example

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{3}3^{-n}$$

- First we check that $a_n > 0$ –> true since $a_n = \left(1+\frac{1}{n}\right)^{3}3^{-n} > 0$ for $n \geq 1$.
- We will compare this series to $\sum_{n=1}^{\infty}\frac{1}{3^n}$ which converges, since it is a geometric series with $r = 1/3 < 1$. $\boxed{b_n = \frac{1}{3^n}}$.
- $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}(\left(1+\frac{1}{n}\right)^{3}3^{-n})\Big/(1/3^n) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3} = 1.$
- Since $c = 1 > 0$, we can conclude that both series converge.

Example

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right)$$

- First we check that $a_n > 0$ –> true since $a_n = \sin\left(\frac{\pi}{n}\right) > 0$ for $n > 1$.
- We will compare this series to $\sum_{n=1}^{\infty} \frac{\pi}{n} = \pi \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges, since it is a constant times a p-series with $p = 1$. $\boxed{b_n = \frac{\pi}{n}}$.
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}(\sin\left(\frac{\pi}{n}\right))\Big/(\frac{\pi}{n}) = \lim_{x\to 0} \frac{\sin x}{x} = 1$.
- Since $c = 1 > 0$, we can conclude that both series diverge.