Lecture 23: Alternating Series

The integral test and the comparison test given in previous lectures, apply only to series with positive terms.

A series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$, where $b_n > 0$ for all n, is called **an alternating series**, because the terms alternate between positive and negative values.

Example

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \dots$$

We can use the divergence test to show that the second series above diverges, since

$$\lim_{n\to\infty}(-1)^{n+1}\frac{n}{2n+1} \quad \text{does not exist}$$

We have the following test for such alternating series:
Alternating Series test If the alternating series

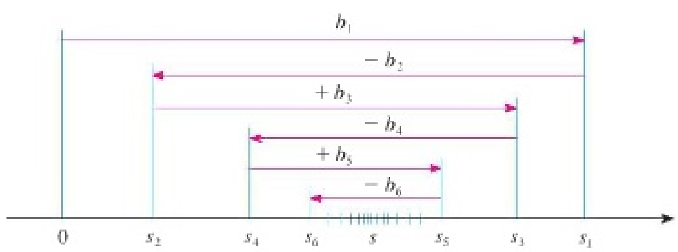
$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots \qquad b_n > 0$$

satisfies

$$(i) \quad b_{n+1} \le b_n \quad \text{for all } n$$
$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

we see from the graph below that because the values of b_n are decreasing, the partial sums of the series cluster about some point in the interval $[0, b_1]$.



A proof is given at the end of the notes.

Notes

- A similar theorem applies to the series $\sum_{i=1}^{\infty}(-1)^n b_n$.

- Also we really only need $b_{n+1} \leq b_n$ for all $n > N$ for some N, since a finite number of terms do not change whether a series converges or not.
- Recall that if we have a differentiable function $f(x)$, with $f(n) = b_n$, then we can use its derivative to check if terms are decreasing.

Example Test the following series for convergence

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}, \qquad \sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}, \qquad \sum_{n=1}^{\infty}(-1)^n\frac{2n^2}{n^2+1}, \qquad \sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n^2}, \qquad \sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{\pi}{n}\right)$$

Note that an alternating series may converge whilst the sum of the absolute values diverges. In particular the alternating harmonic series above converges.

Estimating the Error

Suppose $\sum_{i=1}^{\infty}(-1)^{n-1}b_n$, $b_n > 0$, converges to s. Recall that we can use the partial sum $s_n = b_1 - b_2 + \cdots + (-1)^{n-1}b_n$ to estimate the sum of the series, s. If the series satisfies the conditions for the Alternating series test, we have the following simple estimate of the size of **the error in our approximation** $|R_n| = |s - s_n|$.
(R_n here stands for the remainder when we subtract the n th partial sum from the sum of the series.)

Alternating Series Estimation Theorem If $s = \sum(-1)^{n-1}b_n,\quad b_n > 0$ is the sum of an alternating series that satisfies

$$(i)\quad b_{n+1} < b_n \qquad \text{for all } n$$

$$(ii)\quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \le b_{n+1}.$$

A proof is included at the end of the notes.

Example Find a partial sum approximation the sum of the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ where the error of approximation is less than $.01 = 10^{-2}$.

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Proof of the Alternating Series Test

$$s_2 = b_1 - b_2 \geq 0 \qquad \text{since} \;\; b_2 < b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geq s_2 \qquad \text{since} \;\; b_4 < b_3$$

$$\vdots$$

$$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2}$$

Hence the sequence of even partial sums is increasing:

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \ldots$$

Also we have

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1.$$

Hence the sequence of even partial sums is increasing and bounded and thus converges.. Therefore $\lim_{n\to\infty} s_n = s$ for some s.

This takes care of the even partial sums, now we deal with the odd partial sums. We have $s_{2n+1} = s_{2n} + b_{2n+1}$, hence $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(s_{2n}) + \lim_{n\to\infty} b_{2n+1}) = \lim_{n\to\infty}(s_{2n}) = s$, since by assumption (ii), $\lim_{n\to\infty} b_{2n+1} = 0$.

Thus the limits of the entire sequence of partial sums is s and the series converges.

Note that in the proof above we see that if $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$, with then

$$s_{2n} \leq s \leq s_{2n+1}$$

because $s_{2n+1} = s_{2n} + b_{2n+1}$ and $s = s_{2n} + b_{2n+1} - (b_{2n+2} - b_{2n+3}) - .... < s_{2n+1}$. Similarly in the proof above we see that

$$s_{2n-1} \geq s \geq s_{2n}.$$

Proof of Alternating Series Estimation Theorem From our note above, we have that the sum of the series, s, lies between any two consecutive sums, and hence

$$|R_n| = |s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}.$$