Alternating Series

The integral test and the comparison test given in previous lectures, apply only to series with positive terms.

- A series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} b_n$, where $b_n > 0$ for all n, is called **an alternating series**, because the terms alternate between positive and negative values.
- **Example**

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \dots$$

- We can use the divergence test to show that the second series above diverges, since

$$\lim_{n\to\infty}(-1)^{n+1}\frac{n}{2n+1} \quad \text{does not exist}$$

Alternating Series test

We have the following test for such alternating series:
Alternating Series test If the alternating series

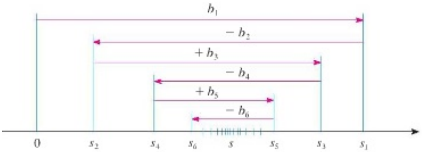
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots \qquad b_n > 0$$

satisfies

$$(i) \quad b_{n+1} \le b_n \quad \text{for all } n$$
$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

- we see from the graph that because the values of b_n are decreasing, the partial sums of the series cluster about some point in the interval $[0, b_1]$.



- A proof is given at the end of the notes.

Notes

Alternating Series test If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots \qquad , b_n > 0$$

satisfies

$$(i) \quad b_{n+1} \leq b_n \quad \text{for all } n$$
$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

- A similar theorem applies to the series $\sum_{i=1}^{\infty}(-1)^n b_n$.
- Also we really only need $b_{n+1} \leq b_n$ for all $n > N$ for some N, since a finite number of terms do not change whether a series converges or not.
- Recall that if we have a differentiable function $f(x)$, with $f(n) = b_n$, then we can use its derivative to check if terms are decreasing.

Example 1

Alternating Series test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$ $b_n > 0$ satisfies

$$(i) \quad b_{n+1} \leq b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

Example 1 Test the following series for convergence

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$$

- We have $b_n = \frac{1}{n}$.
- $\lim_{n\to\infty} \frac{1}{n} = 0$.
- $b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n}$ for all $n \geq 1$.
- Therefore, we can conclude that the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges.
- Note that an alternating series may converge whilst the sum of the absolute values diverges. In particular the alternating harmonic series above converges.

Example 2

Alternating Series test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$ $b_n > 0$ satisfies

$$(i) \quad b_{n+1} \leq b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

Example 2 Test the following series for convergence $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$

- We have $b_n = \frac{n}{n^2+1}$.
- $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = 0$.
- To check if the terms b_n decrease as n increases, we use a derivative. Let $f(x) = \frac{x}{x^2+1}$. We have $f(n) = b_n$.
- $f'(x) = \frac{(x^2+1)-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$.
- Since this function is decreasing as x increases, for $x > 1$, we must have $b_{n+1} < b_n$ for $n \geq 1$.
- Therefore, we can conclude that the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ converges.

Example 3

Alternating Series test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$ $b_n > 0$ satisfies

$$(i)\quad b_{n+1} \le b_n \quad \text{for all } n$$

$$(ii)\quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

Example 3 Test the following series for convergence: $\sum_{n=1}^{\infty}(-1)^n \frac{2n^2}{n^2+1}$

- We have $b_n = \frac{2n^2}{n^2+1}$.
- Here we can use the divergence test (you should always check if this applies first)
- We have $\lim_{n\to\infty} \frac{2n^2}{n^2+1} = \lim_{n\to\infty} \frac{2}{1+1/n^2} = 2 \neq 0$.
- Therefore $\lim_{n\to\infty}(-1)^n \frac{2n^2}{n^2+1}$ does not exist and we can conclude that the series

  $$\sum_{n=1}^{\infty}(-1)^n \frac{2n^2}{n^2+1}$$

  diverges.

Example 4

Alternating Series test If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$ $b_n > 0$ satisfies

$$(i) \quad b_{n+1} \leq b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then the series converges.

Example 4 Test the following series for convergence: $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$

- We have $b_n = \frac{1}{n!}$.
- Since $0 \leq b_n = \frac{1}{n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1} \leq \frac{1}{n}$, we must have $\lim_{n\to\infty} \frac{1}{n!} = 0$.
- $b_{n+1} = \frac{1}{(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1} = \frac{1}{(n+1)} \cdot \frac{1}{n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1} = \frac{1}{n+1} \cdot b_n < b_n$ if $n > 1$.
- Therefore by the Alternating series test, we can conclude that the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$ converges.

Example 5

Example 5 Test the following series for convergence: $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n^2}$

- We have $b_n = \frac{\ln n}{n^2}$.
- $\lim_{n\to\infty} \frac{\ln n}{n^2} = \lim_{x\to\infty} \frac{\ln x}{x^2} = (L'Hop) \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$
- To check if b_n is decreasing as n increases, we calculate the derivative of $f(x) = \frac{\ln x}{x^2}$.
- $f'(x) = \frac{(x^2)(1/x)-2x\ln x}{x^2} = \frac{x-2x\ln x}{x^2} = \frac{x(1-2\ln x)}{x^2} < 0$ if $1 - 2\ln x < 0$ or $\ln x > 1/2$. This happens if $x > \sqrt{e}$, which certainly happens if $x \geq 2$.
- This is enough to show that $b_{n+1} < b_n$ if $n \geq 2$ and hence $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n^2}$ converges.

Example 6

Example 6 Test the following series for convergence: $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$

- We have $b_n = \cos\left(\frac{\pi}{n}\right)$. $b_n > 0$ for $n \geq 2$.
- $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \lim_{x\to\infty} \cos\left(\frac{\pi}{x}\right) = 1 \neq 0$.
- Therefore $\lim_{n\to\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$ diverges by the divergence test.

Error of Estimation

Estimating the Error

Suppose $\sum_{i=1}^{\infty}(-1)^{n-1}b_n,\ b_n > 0$, converges to s. Recall that we can use the partial sum $s_n = b_1 - b_2 + \cdots + (-1)^{n-1}b_n$ to estimate the sum of the series, s. If the series satisfies the conditions for the Alternating series test, we have the following simple estimate of the size of **the error in our approximation** $|R_n| = |s - s_n|$.
(R_n here stands for the remainder when we subtract the n th partial sum from the sum of the series.)

Alternating Series Estimation Theorem If $s = \sum(-1)^{n-1}b_n, \quad b_n > 0$ is the sum of an alternating series that satisfies

$$(i) \quad b_{n+1} < b_n \qquad \text{for all } n$$

$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

A proof is included at the end of the notes.

Example

Alternating Series Estimation Theorem If $s = \sum(-1)^{n-1} b_n$, $b_n > 0$ is the sum of an alternating series that satisfies

$$(i) \quad b_{n+1} < b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \le b_{n+1}.$$

Example Find a partial sum approximation the sum of the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ where the error of approximation is less than $.01 = 10^{-2}$.

- We have $b_n = \frac{1}{n}$. $b_n > 0$ for $n \ge 1$ and we have already seen that the conditions of the alternating series test are satisfied in a previous example.
- Therefore the n th remainder, $|R_n| = |s - s_n| \le b_{n+1} = \frac{1}{n+1}$.
- Therefore, if we find a value of n for which $\frac{1}{n+1} \le \frac{1}{10^2}$, we will have the error of approximation $R_n \le \frac{1}{10^2}$.
- $\frac{1}{n+1} \le \frac{1}{10^2}$ if $10^2 \le n+1$ or $n \ge 101$.
- Checking with Mathematica, we get the actual error $R_{101} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n} - \sum_{n=1}^{101}(-1)^n \frac{1}{n} = 0.00492599$ which is indeed less than .01.