Lecture 24 :Absolute Convergence, Ratio and Root test.

Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
If the terms of the series a_n are positive, absolute convergence is the same as convergence.

Example Are the following series absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Definition A series $\sum a_n$ is called **conditionally convergent** if the series is convergent but not absolutely convergent.

Which of the series in the above example is conditionally convergent?

Theorem If a series is absolutely convergent, then it is convergent, that is if $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

Proof Let us assume that $\sum |a_n|$ is convergent. Since

$$0 \leq a_n + |a_n| \leq 2|a_n|,$$

we have that $\sum (a_n + |a_n|)$ is convergent by the comparison test. By the laws of limits

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is convergent, since it is the difference of two convergent series.

Example Are the following series convergent (test for absolute convergence)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}.$$

The Ratio Test

This test is useful for determining absolute convergence.

Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).
Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

- If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $L > 1$ or ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $L = 1$, then the Ratio test is inconclusive and we cannot determine if the series converges or diverges using this test.

This test is especially useful where factorials and powers of a constant appear in terms of a series.

Example Test the following series for convergence

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n!}, \qquad \sum_{n=1}^{\infty}\frac{n^n}{n!}, \qquad \sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{5^n}\right) \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

Note that when the ratio test is inconclusive for an alternating series, the alternating series test may work.

The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root test is inconclusive and we cannot determine if the series converges or diverges using this test.

Example Test the following series for convergence:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n}{n+1}\right)^n, \qquad \sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n, \qquad \sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n.$$

Rearranging sums

If we rearrange the terms in a finite sum, the sum remains the same. This is not always the case for infinite sums (infinite series). It can be shown that:

- If a series $\sum a_n$ is an absolutely convergent series with $\sum a_n = s$, then any rearrangement of $\sum a_n$ is convergent with sum s.
- It a series $\sum a_n$ is a conditionally convergent series, then for any real number r, there is a rearrangement of $\sum a_n$ which has sum r.

Example The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is absolutely convergent with $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3}$ and hence any rearrangement of the terms has sum $\frac{2}{3}$.

Example Alternating Harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent, it can be shown that its sum is $\ln 2$,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{n} + \cdots = \ln 2.$$

Now we rearrange the terms taking the positive terms in blocks of one followed by negative terms in blocks of 2

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} \cdots =$$

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots =$$

$$\left(\frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{14}\right) \cdots =$$

$$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{n} + \dots\right) = \frac{1}{2} \ln 2.$$

Obviously, we could continue in this way to get the series to sum to any number of the form $(\ln 2)/2^n$.