Absolute convergence

Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
If the terms of the series a_n are positive, absolute convergence is the same as convergence.

Example Are the following series absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

- To check if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent, we need to check if the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.
- Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3 > 1$, it converges and therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent.
- To check if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is absolutely convergent, we need to check if the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.
- Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$, it diverges and therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is not absolutely convergent.

Conditional convergence

Definition A series $\sum a_n$ is called **conditionally convergent** if the series is convergent but not absolutely convergent.

Which of the series in the above example is conditionally convergent?

- Since the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent, it is **not conditionally convergent.**
- Since the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent (used the alternating series test last day to show this), but the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ **is conditionally convergent.**

Absolute conv. implies conv.

Theorem If a series is absolutely convergent, then it is convergent, that is if $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.
(A proof is given in your notes)
Example Are the following series convergent (test for absolute convergence)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}.$$

- Since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent, we can conclude that this series is convergent.
- To check if the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}$ is absolutely convergent, we consider the series of absolute values $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^4}\right|$.
- Since $0 \leq |\sin(n)| \leq 1$, we have $0 \leq \left|\frac{\sin(n)}{n^4}\right| \leq \frac{1}{n^4}$.
- Therefore the series $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^4}\right|$ converges by comparison with the converging p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
- Therefore the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}$ is convergent since it is absolutely convergent.

The Ratio Test

This test is useful for determining absolute convergence.
Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).
Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

- If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $L > 1$ or ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $L = 1$, then the Ratio test is inconclusive and we cannot determine if the series converges or diverges using this test.

This test is especially useful where factorials and powers of a constant appear in terms of a series. (Note that when the ratio test is inconclusive for an alternating series, the alternating series test may work.)

Example 1 Test the following series for convergence

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n!}$$

- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}\big/(n+1)!}{2^n\big/n!}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$
- Therefore, the series converges.

Example 2

Ratio Test Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).
Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.
If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
If $L > 1$ or ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
If $L = 1$, then the Ratio test is inconclusive.
Example 2 Test the following series for convergence

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{5^n}\right)$$

- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\big/5^{n+1}}{n\big/5^n}\right| = \lim_{n\to\infty} \frac{n+1}{5n} = \frac{1}{5}\lim_{n\to\infty}(1 + 1/n) = \frac{1}{5} < 1.$
- Therefore, the series converges.

Example 3

Example 3 Test the following series for convergence $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}\right| = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} =$
  $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x.$
- $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+1/x)} = e^{\lim_{x\to\infty} x\ln(1+1/x)}.$

  $\lim_{x\to\infty} x\ln(1+1/x) = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = (L'Hop) \lim_{x\to\infty} \frac{\frac{-1/x^2}{(1+1/x)}}{-1/x^2} =$
  $\lim_{x\to\infty} \frac{1}{(1+1/x)} = 1.$
- Therefore $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e^1 = e > 1$ and the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

Example 4

Example 4 Test the following series for convergence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

- We know already that this series converges absolutely and therefore it converges. (we could also use the alternating series test to deduce this).
- Lets see what happens when we apply the ratio test here.
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1\big/(n+1)^2}{1\big/n^2}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 =$ $\lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^2 = 1.$
- Therefore the ratio test is inconclusive here.

The Root Test

Root Test Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root test is inconclusive and we cannot determine if the series converges or diverges using this test.

Example 5 Test the following series for convergence $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n}{n+1}\right)^n$

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = \lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{2}{1+1/n} = 2 > 1$
- Therefore by the n th root test, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n}{n+1}\right)^n$ diverges.

Example 6

Root Test For $\sum_{n=1}^{\infty} a_n$. $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
If $L > 1$ or ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
If $L = 1$, then the Root test is inconclusive.

Example 6 Test the following series for convergence $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+1/n} = 1/2 < 1$
- Therefore by the n th root test, the series $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$ converges.

Example 7

Root Test For $\sum_{n=1}^{\infty} a_n$. $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
If $L > 1$ or ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
If $L = 1$, then the Root test is inconclusive.

Example 7 Test the following series for convergence $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$.

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} =$ $(L'Hop) \lim_{x\to\infty} \frac{1/x}{1} = 0 < 1$
- Therefore by the n th root test, the series $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$ converges.

Rearranging sums

If we rearrange the terms in a finite sum, the sum remains the same. This is not always the case for infinite sums (infinite series). It can be shown that:

- If a series $\sum a_n$ is an absolutely convergent series with $\sum a_n = s$, then any rearrangement of $\sum a_n$ is convergent with sum s.
- It a series $\sum a_n$ is a conditionally convergent series, then for any real number r, there is a rearrangement of $\sum a_n$ which has sum r.
- **Example** The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is absolutely convergent with $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3}$ and hence any rearrangement of the terms has sum $\frac{2}{3}$.

Rearranging sums

- It a series $\sum a_n$ is a conditionally convergent series, then for any real number r, there is a rearrangement of $\sum a_n$ which has sum r.
- **Example Alternating Harmonic series** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent, it can be shown that its sum is $\ln 2$,
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{n} + \cdots = \ln 2.$$
- Now we rearrange the terms taking the positive terms in blocks of one followed by negative terms in blocks of 2
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} \cdots =$$
$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots =$$
- $$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{n} + \dots\right) = \frac{1}{2} \ln 2.$$
- Obviously, we could continue in this way to get the series to sum to any number of the form $(\ln 2)/2^n$.