Determine whether the following series converge or diverge, and state the test you used to arrive at your conclusion.

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

Solution: Alternating series. The terms $b_n = \frac{1}{\sqrt{n}}$ are decreasing, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, so the AST says this converges.

2.
$$\sum_{n=1}^{\infty} \frac{e^n}{10^n}$$

Solution: This is a geometric series with ratio $r = \frac{e}{10} < 1$. A geometric series with $|r| < 1$ converges, so this converges. For the variant

$$\sum_{n=1}^{\infty} \frac{e^n}{2^{2n}},$$

note that this is still a geometric series, with $r = e/4 < 1$, so is convergent:

$$\sum_{n=1}^{\infty} \frac{e^n}{2^{2n}} = \sum_{n=1}^{\infty} \frac{e^n}{4^n}.$$

3.
$$\sum_{n=1}^{\infty} \frac{n+2}{n^3+2n+1}$$

Solution: Try limit comparison test with $b_n = \frac{1}{n^2}$. You'll see

$$\lim_{n\to\infty} \frac{n+2}{n^3+2n+1} \div \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+2n+1} = 1.$$

Since $c = 1 > 0$ and $\sum \frac{1}{n^2}$ converges, the limit comparison test says that the series converges.

4.
$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

Solution: Ratio test. Compute

$$\lim_{n\to\infty} \frac{(n+1)!}{e^{(n+1)^2}} \div \frac{n!}{e^{n^2}} = \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = 0.$$

Since the ratio is 0 and $0 < 1$, the ratio test says this converges.

5.
$$\sum_{n=1}^{\infty}(\sqrt[n]{3}-1)^n$$

Solution: Root test. Compute

$$\lim_{n\to\infty}\sqrt[n]{(\sqrt[n]{3}-1)^n}=\lim_{n\to\infty}\sqrt[n]{3}-1=0,$$

since $\lim_{n\to\infty}\sqrt[n]{3}=1$. As the limit is 0 and $0<1$, the root test says this converges.

6.
$$\sum_{n=1}^{\infty}\frac{n2^n}{n!}$$

Solution: Ratio test. Compute

$$\lim_{n\to\infty}\frac{(n+1)2^{n+1}}{(n+1)!}\div\frac{n2^n}{n!}=\lim_{n\to\infty}\frac{2}{n}=0.$$

As the limit is 0 and $0<1$, the ratio test says that this converges.

7.
$$\sum_{n=1}^{\infty}\frac{n+2}{2n+1}$$

Solution: Divergence test. Compute

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n+2}{2n+1}=\frac{1}{2}.$$

As this limit is not equal to zero, the divergence test says that the series diverges.

8.
$$\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$$

Solution: Use the comparison test, replacing $\ln n$ with $\sqrt{n}>\ln n$. Thus

$$a_n=\frac{\ln n}{n}<b_n=\frac{\sqrt{n}}{n^2}=\frac{1}{n^{3/2}}.$$

Since this is a p-series with $p=3/2>1$, it converges, and hence the original series converges as well.

9.
$$\sum_{n=1}^{\infty}\frac{n+e^n}{n^2+10^n}$$

Solution: Limit comparison test, comparing to the convergent geometric series $b_n = \frac{e^n}{10^n}$. Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+e^n}{n^2+10^n} \div \frac{e^n}{10^n} = \lim_{n\to\infty} \frac{\left(\frac{n}{e^n}\right)+1}{\left(\frac{n^2}{10^n}\right)+1} = 1.$$

Since $c = 1 > 0$ and the comparison series converges, LCT says that the original series converges too.

Alternatively, you can split this up into two series:

$$\sum_{n=1}^{\infty} \frac{n+e^n}{n^2+10^n} = \sum_{n=1}^{\infty} \frac{n}{n^2+10^n} + \sum_{n=1}^{\infty} \frac{e^n}{n^2+10^n}.$$

You can do comparison test with each piece, eliminating the n^2 in the denominator:

$$\frac{n}{n^2+10^n} \leq \frac{n}{10^n} \qquad \text{and} \qquad \frac{e^n}{n^2+10^n} \leq \frac{e^n}{10^n}.$$

The second of these is a convergent geometric series (see problem 2). The first of these is seen to be convergent, e.g. by ratio test.

10.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$$

Solution: Root test. Compute

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+2}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+2}\right)^n.$$

To evaluate the limit, rewrite n as $(n+2)-2$ and simplify:

$$\lim_{n\to\infty} \left(\frac{n}{n+2}\right)^n = \lim_{n\to\infty} \left(\frac{(n+2)-2}{n+2}\right)^n = \lim_{n\to\infty} \left(1-\frac{2}{n+2}\right)^n.$$

Remembering that

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e,$$

and more generally

$$\lim_{n\to\infty} \left(1+\frac{c}{n}\right)^n = e^c,$$

we can recognize this limit as giving e^{-2}. As this is less than 1, the Root test gives that the series is convergent.

11.
$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right)$$

Solution: Limit comparison test with the convergent series $\sum \frac{1}{n^2}$. Compute

$$\lim_{n\to\infty} \frac{\arctan\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}}.$$

Set $x = \frac{1}{n^2}$, and see

$$\lim_{n\to\infty} \frac{\arctan\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{x\to 0} \frac{\arctan(x)}{x} = 1.$$

(The limit can be computed by L'Hopital, or recognized as the derivative of $\arctan(x)$ at $x = 0$). Since $c = 1 > 0$, LCT says that since $\sum \frac{1}{n^2}$ converges, so does the original series.